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THE CESÀRO OPERATOR IN THE FRÉCHET SPACES ℓ^{p+} AND L^{p-}

ANGELA A. ALBANESE

Dipartimento di Matematica e Fisica "E. De Giorgi"
Università del Salento- C.P.193, I-73100 Lecce, Italy
e-mail: angela.albanese@unisalento.it

JOSÉ BONET

Instituto Universitario de Matemática Pura y Aplicada IUMPA,
Universitat Politècnica de València
E-46071 València, Spain
e-mail: jbonet@mat.upv.es

and WERNER J. RICKER

Math.-Geogr. Fakultät, Katholische Universität Eichstätt-Ingolstadt
D-85072 Eichstätt, Germany
e-mail: werner.ricker@ku.de

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Abstract. The classical spaces ℓ^{p+}, $1 \leq p < \infty$, and L^{p-}, $1 < p \leq \infty$, are *countably normed*, reflexive Fréchet spaces in which the Cesàro operator C acts continuously. A detailed investigation is made of various operator theoretic properties of C (e.g., spectrum, point spectrum, mean ergodicity) as well as certain aspects concerning the dynamics of C (e.g., hypercyclic, supercyclic, chaos). This complements the results of [**3**, **4**], where C was studied in the spaces $\mathbb{C}^{\mathbb{N}}$, $L^p_{\text{loc}}(\mathbb{R}^+)$ for $1 < p < \infty$ and $C(\mathbb{R}^+)$, which belong to a very different collection of Fréchet spaces, called *quojections*; these are automatically Banach spaces whenever they admit a continuous norm.

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1. Introduction. The Cesàro operator, whether acting on sequences or on functions, is based on an averaging process. Many features of this classical operator (e.g., continuity, spectrum, dynamics, mean ergodicity etc.) have been intensively studied in a large variety of Banach spaces. Such investigations have also been extended into the setting of *Fréchet* spaces, [**11**]. In [**3**], the Cesàro operator is analysed in the Fréchet sequence space $\omega := \mathbb{C}^{\mathbb{N}}$ and in [**4**] it is studied in the Fréchet function spaces $L^p_{\text{loc}}(\mathbb{R}^+)$, $1 < p < \infty$, and in $C(\mathbb{R}^+)$ when equipped with its compact convergence topology in $\mathbb{R}^+ := [0, \infty)$. Each of the spaces ω, $C(\mathbb{R}^+)$ and $L^p_{\text{loc}}(\mathbb{R}^+)$ is a *quojection* Fréchet space. In such spaces, special features arise which are not available for Fréchet spaces in general. Our aim is to analyse the Cesàro operator in the classical reflexive Fréchet sequence spaces ℓ^{p+}, $1 \leq p < \infty$, and in the reflexive Fréchet function spaces $L^{p-} := L^{p-}([0, 1])$, $1 < p \leq \infty$. These are (non–Montel) *countably normed* Fréchet spaces (i.e., which can be written as the intersection of a decreasing sequence of

Banach spaces with continuous inclusions) and hence, they are "far away" from being quojections. For more features of ℓ^{p+} and L^{p-} see **[19]** and **[8]**, respectively. It is time to be more precise.

The discrete Cesàro operator C is defined on the linear space $\omega := \mathbb{C}^{\mathbb{N}}$ (consisting of all scalar sequences) by

$$\mathsf{C}(x) := \left(x_1, \frac{x_1 + x_2}{2}, \ldots, \frac{x_1 + x_2 + \cdots + x_i}{i}, \ldots\right), \quad x = (x_j)_{j=1}^{\infty} \in \omega. \tag{1.1}$$

It is a linear (algebraic) isomorphism of ω onto itself with $\mathsf{C}^{-1}\colon \omega \to \omega$ given by

$$\mathsf{C}^{-1}(y) := (jy_j - (j-1)y_{j-1})_{j=1}^{\infty}, \quad y = (y_j)_{j=1}^{\infty} \in \omega, \tag{1.2}$$

where we set $y_0 := 0$. The discrete Cesàro operator C is said to act in a vector subspace $X \subseteq \omega$ if it maps X into itself. If X has a locally convex Hausdorff topology, then the continuity of $\mathsf{C}\colon X \to X$ also needs to be addressed.

Let $p \in [1, \infty)$. Recall that $\ell^{p+} = \cap_{r>p} \ell^r$ is a Fréchet space with respect to the coarser locally convex topology on ℓ^{p+} for which the inclusion map $\ell^{p+} \hookrightarrow \ell^r$ is continuous for all $r > p$. So, if $p_n \downarrow p$ (so that $p_n > p$ for all $n \in \mathbb{N}$), then $\ell^{p+} = \cap_{n=1}^{\infty} \ell^{p_n}$ and its Fréchet topology is generated by the sequence of *norms*

$$\|x\|_n := \left(\sum_{i=1}^{\infty} |x_i|^{p_n}\right)^{1/p_n}, \quad x \in \ell^{p+},\ n \in \mathbb{N}. \tag{1.3}$$

Clearly the Banach space $\ell^p \subseteq \ell^{p+}$ continuously and $\ell^{p+} \subseteq \mathbb{C}^{\mathbb{N}}$. It turns out (see Section 2) that C acts continuously in each Fréchet space ℓ^{p+}, $1 \leq p < \infty$, which we denote by $\mathsf{C}^{(p+)}\colon \ell^{p+} \to \ell^{p+}$.

Analogously, for $1 < p \leq \infty$ the space $L^{p-} = \cap_{1<r<p} L^r$, which contains $L^p := L^p([0,1])$ continuously, is a Fréchet space with respect to the coarser locally convex topology on L^{p-} for which the inclusion map $L^{p-} \hookrightarrow L^r$ is continuous for each $1 < r < p$. So, if $1 < p_n \uparrow p$ (so that $1 < p_n < p$ for all $n \in \mathbb{N}$), then $L^{p-} = \cap_{n=1}^{\infty} L^{p_n}$ and its Fréchet topology is generated by the sequence of *norms*

$$\|f\|_n := \left(\int_0^1 |f(t)|^{p_n}\, dt\right)^{1/p_n}, \quad f \in L^{p-},\ n \in \mathbb{N}. \tag{1.4}$$

The Cesàro operator C is defined pointwise by

$$\mathsf{C}f(x) := \frac{1}{x} \int_0^x f(t)\, dt, \quad x \in (0, 1], \tag{1.5}$$

for each $f \in L^1$. It turns out (see Section 3) that C acts continuously in each Fréchet space L^{p-}, $1 < p \leq \infty$, which we denote by $\mathsf{C}^{(p-)}\colon L^{p-} \to L^{p-}$.

An analysis of the operator $\mathsf{C}^{(p+)}$ (resp., $\mathsf{C}^{(p-)}$) is carried out in Section 2 (resp., Section 3). To explain this in more detail, we require some further notation and definitions. Let X be a locally convex Hausdorff space (briefly, lcHs) and Γ_X be a system of continuous semi-norms determining the topology of X. The strong operator topology τ_s in the space $\mathcal{L}(X)$ of all continuous linear operators from X into itself is determined by the semi-norms $q_x(S) := q(Sx)$, for $S \in \mathcal{L}(X)$, for each $x \in X$ and $q \in \Gamma_X$, in which case we write $\mathcal{L}_s(X)$. Denote by $\mathcal{B}(X)$ the collection of all bounded subsets of X. The topology τ_b of uniform convergence on bounded sets is defined in $\mathcal{L}(X)$ via the semi-norms $q_B(S) := \sup_{x\in B} q(Sx)$, for $S \in \mathcal{L}(X)$, for each $B \in \mathcal{B}(X)$ and $q \in \Gamma_X$; in this case we write $\mathcal{L}_b(X)$. For X a Banach space, τ_b is the operator norm topology in $\mathcal{L}(X)$. If Γ_X can be taken countable and X is complete, then X is called a Fréchet space. The identity operator on a lcHs X is denoted by I. Finally, the *dual operator* of $T \in \mathcal{L}(X)$ is denoted by $T'\colon X' \to X'$, where $X' = \mathcal{L}(X, \mathbb{C})$ is the topological dual space of X. The strong topology in X (resp. X') is denoted by $\beta(X, X')$ (resp. $\beta(X', X)$) and we write X_β (resp. X'_β). If X is a Fréchet space, then $X_\beta = X$. As a general reference for lcHs' see [**18**].

We say that $T \in \mathcal{L}(X)$, with X a lcHs, is *power bounded* if $\{T^n\}_{n=1}^\infty$ is an equicontinuous subset of $\mathcal{L}(X)$. For a Banach space X, this means precisely that $\sup_{n\in\mathbb{N}} \|T^n\|_{op} < \infty$. Given $T \in \mathcal{L}(X)$, we can consider its sequence of averages

$$T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m, \qquad n \in \mathbb{N}, \tag{1.6}$$

called the Cesàro means of T. Then T is called *mean ergodic* (resp., *uniformly mean ergodic*) if $\{T_{[n]}\}_{n=1}^\infty$ is a convergent sequence in $\mathcal{L}_s(X)$ (resp., in $\mathcal{L}_b(X)$). It follows from (1.6) that $\frac{T^n}{n} = T_{[n]} - \frac{n-1}{n} T_{[n-1]}$, for $n \geq 2$. Hence, τ_s-$\lim_{n\to\infty} \frac{T^n}{n} = 0$ whenever T is mean ergodic. A relevant text is [**13**].

Concerning the dynamics of a continuous linear operator T defined on a separable lcHs X, recall that T is *hypercyclic* if there exists $x \in X$ whose orbit $\{T^n x\colon n \in \mathbb{N}_0\}$ is dense in X. If, for some $x \in X$, the projective orbit $\{\lambda T^n x\colon \lambda \in \mathbb{C},\ n \in \mathbb{N}_0\}$ is dense in X, then T is called *supercyclic*. Finally, T is called *chaotic* if it is hypercyclic and the set of its periodic points $\{u \in X\colon \exists\, n \in \mathbb{N} \text{ with } T^n u = u\}$ is dense in X. As general references, we refer to [**5**, **11**].

For a Fréchet space X and $T \in \mathcal{L}(X)$, the *resolvent set* $\rho(T)$ of T consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$. The set $\sigma(T) := \mathbb{C} \setminus \rho(T)$ is called the *spectrum* of T. The *point spectrum* $\sigma_{pt}(T)$ of T consists of all $\lambda \in \mathbb{C}$ such that $(\lambda I - T)$ is not injective. If we need to stress the space X, then we also write $\sigma(T; X)$, $\sigma_{pt}(T; X)$ and $\rho(T; X)$. Whenever $\lambda, \mu \in \rho(T)$ we have the *resolvent identity* $R(\lambda, T) - R(\mu, T) = (\mu - \lambda)R(\lambda, T)R(\mu, T)$. Unlike for Banach spaces, it may happen that $\rho(T) = \emptyset$ or that $\rho(T)$ is not open in $\mathbb{C}$. This is why some authors prefer the subset $\rho^*(T)$ of $\rho(T)$ consisting of all $\lambda \in \mathbb{C}$ for which there exists $\delta > 0$ such that each $\mu \in B(\lambda, \delta) := \{z \in \mathbb{C}\colon |z - \lambda| < \delta\}$ belongs to $\rho(T)$ and the set $\{R(\mu, T)\colon\ \mu \in B(\lambda, \delta)\}$ is equicontinuous in $\mathcal{L}(X)$. If X is a Fréchet space, then it is enough that this set is bounded in $\mathcal{L}_s(X)$. The advantage of $\rho^*(T)$, whenever it is non–empty, is that it is open and the resolvent map $R\colon \lambda \mapsto R(\lambda, T)$ is holomorphic from $\rho^*(T)$ into $\mathcal{L}_b(X)$, [**2**, Proposition 3.4]. Define $\sigma^*(T) := \mathbb{C} \setminus \rho^*(T)$, which is a closed set containing $\sigma(T)$. If $T \in \mathcal{L}(X)$ with X a Banach space, then $\sigma(T) = \sigma^*(T)$. In [**2**, Remark 3.5(vi), p.265] an example of a continuous linear operator T on a Fréchet space X is presented such that $\overline{\sigma(T)} \subset \sigma^*(T)$ properly. For the Cesàro operator this turns out not to be the case.

The mean ergodic properties and the dynamics of the Cesàro operators $\mathsf{C}^{(p+)}$, $1 \leq p < \infty$ (resp., $\mathsf{C}^{(p-)}$, $1 < p \leq \infty$) are presented in Section 2 (resp., Section 3) as is the precise connection between the two notions of spectra $\sigma(\cdot)$ and $\sigma^*(\cdot)$. Of interest is the fact that both of the spectra $\sigma(\mathsf{C}^{(1+)})$ and $\sigma^*(\mathsf{C}^{(1+)})$ are unbounded subsets of $\mathbb{C}$, whereas the spectra of $\mathsf{C}^{(p+)}$, $1 < p < \infty$, and $\mathsf{C}^{(p-)}$, $1 < p \leq \infty$, are bounded subsets of $\mathbb{C}$. For purposes of comparison, the final Section 4 is devoted to an analysis of the two

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notions of spectra for the Cesàro operator acting in the quojection spaces ω, $C(\mathbb{R}^+)$ and $L^p_{\mathrm{loc}}(\mathbb{R}^+)$, $1 < p < \infty$. It turns out that $\sigma(\cdot)$ and $\sigma^*(\cdot)$ *coincide* for $\mathsf{C}\colon C(\mathbb{R}^+) \to C(\mathbb{R}^+)$ and for $\mathsf{C}\colon L^p_{\mathrm{loc}}(\mathbb{R}^+) \to L^p_{\mathrm{loc}}(\mathbb{R}^+)$, which suggests that this may always be true whenever C acts in an appropriate quojection Fréchet space. That this is not so is illustrated by $\mathsf{C}\colon \omega \to \omega$, for which it is shown that $\sigma(\mathsf{C}) \subset \sigma^*(\mathsf{C})$ properly.

2. The Cesàro operator on the space ℓ^{p+}, $1 \le p < \infty$. Fix $1 < p < \infty$. It is known that the discrete Cesàro operator maps the Banach space ℓ^p continuously into itself, which we denote by $\mathsf{C}^{(p)}\colon \ell^p \to \ell^p$, and that $\|\mathsf{C}^{(p)}\|_{op} = q$, where $\frac{1}{p} + \frac{1}{q} = 1$, [**12**, Theorem 326, p. 239]. Consequently, the Cesàro operator maps the Fréchet space ℓ^{p+}, $1 < p < \infty$, continuously into itself. In fact, for a sequence $p_n \downarrow p$ (so that $p_n > p$ for all $n \in \mathbb{N}$), consider the *norms* (1.3) and, for each $n \in \mathbb{N}$, let $\mathsf{C}_n := \mathsf{C}^{(p_n)} \in \mathcal{L}(\ell^{p_n})$. If we denote by $i_n\colon \ell^{p+} \hookrightarrow \ell^{p_n}$ and $i_{n,n+1}\colon \ell^{p_{n+1}} \hookrightarrow \ell^{p_n}$ the canonical inclusion maps (which clearly have dense range), then $i_n \circ \mathsf{C}^{(p+)} = \mathsf{C}_n \circ i_n$ and also $i_{n,n+1} \circ \mathsf{C}_{n+1} = \mathsf{C}_n \circ i_{n,n+1}$ for all $n \in \mathbb{N}$. Hence, for every $n \in \mathbb{N}$, we have

$$\|\mathsf{C}^{(p+)}x\|_n = \|i_n\mathsf{C}^{(p+)}x\|_n = \|\mathsf{C}_n i_n x\|_n = \|\mathsf{C}_n x\|_n \le q\|x\|_n, \quad x \in \ell^{p+}.$$

According to [**14**, p. 123], the dual operator $(\mathsf{C}^{(p)})'\colon \ell^q \to \ell^q$ is given by

$$(\mathsf{C}^{(p)})'(x) = \left(\sum_{h=i}^{\infty} \frac{x_h}{h}\right)_{i=1}^{\infty}, \quad x = (x_k)_{k=1}^{\infty} \in \ell^q,\ 1 < p < \infty. \tag{2.1}$$

The following result will be useful to study the spectrum of $\mathsf{C}^{(p+)}$.

LEMMA 2.1. *Let $X = \bigcap_{n\in\mathbb{N}} X_n$ be a Fréchet space which is the intersection of a sequence of Banach spaces $(X_n, ||.||_n)$, $n \in \mathbb{N}$, satisfying $X_{n+1} \subset X_n$ with $||x||_n \le ||x||_{n+1}$ for each $n \in \mathbb{N}$ and each $x \in X_{n+1}$. Let $T \in \mathcal{L}(X)$ satisfy the following condition:*

(A) *For each n there exists $T_n \in \mathcal{L}(X_n)$ such that the restriction of T_n to X (resp. of T_n to X_{n+1}) coincides with T (resp. with T_{n+1}).*

Then $\sigma(T; X) \subseteq \bigcup_{n\in\mathbb{N}} \sigma(T_n; X_n)$ and $R(\lambda, T)$ coincides with the restriction of $R(\lambda, T_n)$ to X for each $n \in \mathbb{N}$ and each $\lambda \in \bigcap_{n\in\mathbb{N}} \rho(T_n; X_n)$.

Moreover, if $\bigcup_{n\in\mathbb{N}} \sigma(T_n; X_n) \subseteq \overline{\sigma(T; X)}$, then

$$\sigma^*(T; X) = \overline{\sigma(T; X)}.$$

Proof. Let $\lambda \in \cap_{n=1}^{\infty} \rho(T_n; X_n)$. To show that $(\lambda I - T)\colon X \to X$ is injective, suppose that $(\lambda I - T)x = 0$ for some $x \in X$. Then, condition (A) yields $(\lambda I - T_1)x = 0$ in X_1. Since $\lambda \in \rho(T_1; X_1)$, this implies that $x = 0$.

To check that $(\lambda I - T)\colon X \to X$ is surjective, fix $y \in X$. For each n there is $x_n \in X_n$ satisfying $(\lambda I - T_n)x_n = y$ in X_n. By condition (A), for each $n \in \mathbb{N}$ the restriction of T_n to X_{n+1} is T_{n+1}. Hence, $y = (\lambda I - T_n)x_n = (\lambda I - T_n)x_{n+1}$ with the equality holding in X_n. Since $\lambda \in \rho(T_n; X_n)$, this yields $x_n = x_{n+1}$ for each $n \in \mathbb{N}$ and so $x_1 \in X$ with $(\lambda I - T)x_1 = y$. Consequently, $\lambda \in \rho(T; X)$.

Since $\sigma(T; X) \subseteq \sigma^*(T; X)$ with $\sigma^*(T; X)$ closed, we always have $\overline{\sigma(T; X)} \subseteq \sigma^*(T; X)$. Suppose now that $\bigcup_{n\in\mathbb{N}} \sigma(T_n; X_n) \subseteq \overline{\sigma(T; X)}$. Let $\lambda \in \mathbb{C} \setminus \overline{\sigma(T; X)}$ in which case there exists $\varepsilon > 0$ such that $B(\lambda, \varepsilon) \cap \overline{\sigma(T; X)} = \emptyset$. By our assumption, we also have $B(\lambda, \varepsilon) \subseteq \rho(T_n; X_n)$ for each $n \in \mathbb{N}$. Suppose there exists $x \in X$ such that

$\{R(\mu, T)x\colon \mu \in B(\lambda, \varepsilon)\}$ is an unbounded subset of X. Then there is $n_0 \in \mathbb{N}$ such that the set $\{R(\mu, T_{n_0})x\colon \mu \in B(\lambda, \varepsilon)\}$ is unbounded in X_{n_0} (as $X \subseteq X_{n_0}$ and $R(\mu, T)$ is the restriction of $R(\mu, T_{n_0})$ to X for $\mu \in B(\lambda, \varepsilon)$). This is a contradiction as $B(\lambda, \varepsilon) \subseteq \rho(T_{n_0}; X_{n_0})$ with X_{n_0} a Banach space. □

THEOREM 2.2. *Let* $1 < p < \infty$ *and* q *satisfy* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

(i) $\sigma(\mathsf{C}^{(p+)}) = \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{q}{2}\right| < \frac{q}{2}\right\} \cup \{0\}$.

(ii) $\sigma_{pt}(\mathsf{C}^{(p+)}) = \emptyset$ *and* $\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{q}{2}\right| < \frac{q}{2}\right\} \subseteq \sigma_{pt}((\mathsf{C}^{(p+)})')$.

(iii) $\sigma^*(\mathsf{C}^{(p+)}) = \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{q}{2}\right| \leq \frac{q}{2}\right\} = \overline{\sigma(\mathsf{C}^{(p+)})}$.

Moreover, for every non-zero $\lambda \in \sigma(\mathsf{C}^{(p+)})$ *the subspace* $(\lambda I - C^{(p+)})(\ell^{p+})$ *is closed in* ℓ^{p+} *with* $\operatorname{codim}(\lambda I - C^{(p+)})(\ell^{p+}) = 1$.

Proof. Fix $p_n \downarrow p$. Then the conjugate indices satisfy $q_n \uparrow q$ (with $q_n < q$ for all $n \in \mathbb{N}$). Moreover, for every $n \in \mathbb{N}$, it is known that

$$\sigma(\mathsf{C}_n) = \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{q_n}{2}\right| \leq \frac{q_n}{2}\right\} \quad \text{and} \quad \sigma_{pt}(\mathsf{C}_n) = \emptyset, \tag{2.2}$$

and, if $\lambda \in \mathbb{C}$ satisfies $\left|\lambda - \frac{q_n}{2}\right| < \frac{q_n}{2}$, then $(\lambda I - \mathsf{C}_n)(\ell^{p_n})$ is closed in ℓ^{p_n} with $\operatorname{codim}(\lambda I - \mathsf{C}_n)(\ell^{p_n}) = 1$; see [**14**, Theorem 1] and [**10**, Theorems 1& 2], respectively. Clearly $\sigma_{pt}(\mathsf{C}^{(p+)}) \subseteq \sigma_{pt}(\mathsf{C}_n)$, for all $n \in \mathbb{N}$, and so the first statement in part (ii) follows at once. Since $q_n < q$ for all $n \in \mathbb{N}$, it is clear via (2.2) that

$$\sigma(\mathsf{C}_n) \subset \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{q}{2}\right| < \frac{q}{2}\right\} \cup \{0\}, \quad n \in \mathbb{N}.$$

According to Lemma 2.1, with $X_n := \ell^{p_n}$ and $T_n := \mathsf{C}_n$, for $n \in \mathbb{N}$, we have that

$$\sigma(\mathsf{C}^{(p+)}) \subset \bigcup_{n \in \mathbb{N}} \sigma(\mathsf{C}_n) = \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{q}{2}\right| < \frac{q}{2}\right\} \cup \{0\},$$

and that $R(\lambda, \mathsf{C}^{(p+)})$ coincides with the restriction of $R(\lambda, \mathsf{C}_n)$ to ℓ^{p+} for each $n \in \mathbb{N}$ and each $\lambda \in \bigcap_{n \in \mathbb{N}} \rho(\mathsf{C}_n)$.

Since $\mathsf{C}\colon \omega \to \omega$ is a bicontinuous (algebraic) isomorphism, it is clear that $\mathsf{C}^{(p+)}$ is injective. Moreover, $\mathsf{C}^{(p+)}$ has dense range in ℓ^{p+}, which follows from the identities $e_r = r\mathsf{C}^{(p+)}(e_r - e_{r+1})$, for $r \in \mathbb{N}$ (as $\mathsf{C}^{(p+)}e_r = \sum_{i=r}^{\infty} \frac{1}{i} e_i$, for $r \in \mathbb{N}$). Here, $e_r \in \omega$ is the element with 1 in the rth coordinate and 0 elsewhere, for each $r \in \mathbb{N}$, in which case $\{e_r\}_{r=1}^{\infty} \subseteq \ell^s$ for all $1 \leq s \leq \infty$. But, $\mathsf{C}^{(p+)}$ is not surjective in ℓ^{p+}. Indeed, $y := \sum_{i=1}^{\infty} \frac{1}{(2i-1)} e_{2i-1} \in \ell^{p+}$. However, by (1.2), the vector $C^{-1}(y) = (1, -1, 1, -1, 1, -1, \ldots) \in \ell^{\infty} \setminus c_0$. This establishes that $0 \in \sigma(\mathsf{C}^{(p+)})$.

Fix $\lambda \in \mathbb{C}$ with $\left|\lambda - \frac{q}{2}\right| < \frac{q}{2}$. Since $q_n \uparrow q$, it follows that $\left|\lambda - \frac{q_n}{2}\right| < \frac{q_n}{2}$ for all $n \geq n_0$ and some $n_0 \in \mathbb{N}$. So, as noted above, for every $n \geq n_0$ the operator $(\lambda I - \mathsf{C}_n)$ is injective with range $(\lambda I - \mathsf{C}_n)(\ell^{p_n})$ closed in ℓ^{p_n} and satisfies $\operatorname{codim}(\lambda I - \mathsf{C}_n)(\ell^{p_n}) = 1$. This yields that $(\lambda I - \mathsf{C}^{(p+)})(\ell^{p+})$ is also a proper closed subspace of ℓ^{p+}. Indeed, let $\{y_j\}_{j=1}^{\infty} \subseteq (\lambda I - C^{(p+)})(\ell^{p+})$ be a sequence which converges to some y in ℓ^{p+}. For each $j \in \mathbb{N}$, let $x_j \in \ell^{p+}$ satisfy $y_j = (\lambda I - \mathsf{C}^{(p+)})x_j$. So, for every $j \in \mathbb{N}$, it follows that

$$y_j = i_n y_j = i_n(\lambda I - \mathsf{C}^{(p+)})x_j = (\lambda I - \mathsf{C}_n)i_n x_j = (\lambda I - \mathsf{C}_n)x_j \in (\lambda I - \mathsf{C}_n)(\ell^{p_n}),$$

with $y_j \to y$ in ℓ^{p_n} for each $n \geq n_0$. The closedness of $(\lambda I - \mathsf{C}_n)(\ell^{p_n})$ in ℓ^{p_n} implies that $y \in (\lambda I - \mathsf{C}_n)(\ell^{p_n})$, i.e., $y = (\lambda I - \mathsf{C}_n)z_n$ for some $z_n \in \ell^{p_n}$ and all $n \geq n_0$. As $\ell^{p_{n+1}} \subseteq \ell^{p_n}$, we have $z_{n+1} \in \ell^{p_n}$ and so, for $n \geq n_0$, it follows that

$$\begin{aligned}(\lambda I - \mathsf{C}_n)z_{n+1} &= (\lambda I - \mathsf{C}_n)i_{n,n+1}z_{n+1} = i_{n,n+1}(\lambda I - \mathsf{C}_{n+1})z_{n+1}\\ &= i_{n,n+1}y = y = (\lambda I - \mathsf{C}_n)z_n.\end{aligned}$$

The injectivity of the maps $(\lambda I - \mathsf{C}_n)$ for $n \geq n_0$, then yields that $z_{n+1} = z_n$ for all $n \geq n_0$. Setting $z = z_{n_0}$, it follows that $z = z_n \in \ell^{p_n}$ for all $n \geq n_0$ and so $z \in \ell^{p+}$ with $(\lambda I - \mathsf{C}^{(p+)})z = y$. Thus, $(\lambda I - \mathsf{C}^{(p+)})(\ell^{p+})$ is a closed subspace of ℓ^{p+}. Suppose that $(\lambda I - \mathsf{C}^{(p+)})(\ell^{p+}) = \ell^{p+}$. Then

$$\ell^{p+} = (\lambda I - \mathsf{C}^{(p+)})(\ell^{p+}) \subseteq (\lambda I - \mathsf{C}_{n_0})(\ell^{p_{n_0}}),$$

with $(\lambda I - \mathsf{C}_{n_0})(\ell^{p_{n_0}})$ closed in $\ell^{p_{n_0}}$. The density of ℓ^{p+} in $\ell^{p_{n_0}}$ implies that $(\lambda I - \mathsf{C}_{n_0})(\ell^{p_{n_0}}) = \ell^{p_{n_0}}$; a contradiction. So, the closed subspace $(\lambda I - \mathsf{C}^{(p+)})(\ell^{p+})$ of ℓ^{p^+} is *proper*. In particular, $\lambda \in \sigma(\mathsf{C}^{(p+)})$. This establishes part (i).

Next, we prove that $\operatorname{codim}(\lambda I - \mathsf{C}^{(p+)})(\ell^{p+}) = 1$, still assuming that $\left|\lambda - \frac{q}{2}\right| < \frac{q}{2}$ and hence, $\left|\lambda - \frac{q_{n_0}}{2}\right| < \frac{q_{n_0}}{2}$ for some $n_0 \in \mathbb{N}$. Observe that the dual operator $(\mathsf{C}^{(p+)})' \colon \ell^{q-} \to \ell^{q-}$ (with $\ell^{q-} := \cup_{n=1}^{\infty} \ell^{q_n}$ being the strong dual $(\ell^{p+})'_\beta$ of ℓ^{p+}) is given by the same formula as in (2.1). So, if $(\mathsf{C}^{(p+)})'u = \lambda u$ for some $u \in \ell^{q-}$ with $u \neq 0$, then $u_{i+1} = u_1 \prod_{h=1}^{i}\left(1 - \frac{1}{\lambda h}\right)$ for all $i \in \mathbb{N}$, [**14**, p. 125]. This shows that each eigenvalue of $(\mathsf{C}^{(p+)})'$ (if it exists) is necessarily simple, i.e., $\dim \operatorname{Ker}(\lambda I - (\mathsf{C}^{(p+)})') = 1$. But, $\left|\lambda - \frac{q_{n_0}}{2}\right| < \frac{q_{n_0}}{2}$ implies, via [**14**, Theorem 1(b)], that there exists a non-zero vector $u \in \ell^{q_{n_0}} \subseteq \ell^{q-}$ such that $(\mathsf{C}_{n_0})'u = \lambda u$ and so $(\mathsf{C}^{(p+)})'u = \lambda u$. Accordingly, since $\dim \operatorname{Ker}(\lambda I - (\mathsf{C}^{(p+)})') = 1$ and the dual of the quotient $(\ell^{p+}/(\lambda I - \mathsf{C}^{(p+)})(\ell^{p+}))' \simeq \operatorname{Ker}(\lambda I - (\mathsf{C}^{(p+)})')$ (algebraically; actually, also isomorphically), it follows that $\operatorname{codim}(\lambda I - \mathsf{C}^{(p+)})(\ell^{p+}) = 1$. Along the way it has also been verified that $\lambda \in \sigma_{pt}((\mathsf{C}^{(p+)})')$, i.e., $\left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{q}{2}\right| < \frac{q}{2}\right\} \subseteq \sigma_{pt}((\mathsf{C}^{(p+)})')$. So part (ii) is completely verified.

Finally, part (iii) follows from Lemma 2.1 as it was shown above in the proof of part (i) that $\sigma(\mathsf{C}^{(p+)}) = \bigcup_{n\in\mathbb{N}} \sigma(\mathsf{C}_n)$. □

THEOREM 2.3. *The Cesàro operator* $\mathsf{C}^{(p+)} \colon \ell^{p+} \to \ell^{p+}$, $1 < p < \infty$, *is not mean ergodic, not power bounded and not supercyclic.*

Proof. By Theorem 2.2(ii) the number $\frac{(1+q)}{2} > 1$ belongs to $\sigma_{pt}((\mathsf{C}^{(p+)})')$ and so there exists a non-zero vector $u \in \ell^{q-}$ satisfying $(\mathsf{C}^{(p+)})'u = \frac{(1+q)}{2}u$. Choose any $x \in \ell^{p+}$ such that $\langle x, u\rangle \neq 0$. Then

$$\left\langle \frac{1}{n}(\mathsf{C}^{(p+)})^n x, u \right\rangle = \frac{1}{n}\langle x, ((\mathsf{C}^{(p+)})')^n u\rangle = \frac{1}{n}\left(\frac{(1+q)}{2}\right)^n \langle x, u\rangle, \quad n \in \mathbb{N},$$

and so the set $\left\{\frac{1}{n}(\mathsf{C}^{(p+)})^n x \colon n \in \mathbb{N}\right\}$ is unbounded in ℓ^{p+}. In particular, the sequence $\left\{\frac{1}{n}(\mathsf{C}^{(p+)})^n\right\}_{n=1}^{\infty}$ does *not* converge to 0 in $\mathcal{L}_s(\ell^{p+})$, thereby violating a necessary condition for $\mathsf{C}^{(p+)}$ to be mean ergodic; see Section 1. Since the power boundedness of $\mathsf{C}^{(p+)}$ would imply that $\frac{1}{n}(\mathsf{C}^{(p+)})^n \to 0$ in $\mathcal{L}_s(\ell^{p+})$ for $n \to \infty$, it also follows that $\mathsf{C}^{(p+)}$ is not power bounded.

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Suppose that $\mathsf{C}^{(p+)}$ is supercyclic. As ℓ^{p+} is dense in the Fréchet space ω (see Section 4) and $\mathsf{C}^{(p+)}$ coincides with the restriction of $\mathsf{C}: \omega \to \omega$ to ℓ^{p+}, it follows that $\mathsf{C}: \omega \to \omega$ is supercyclic. This contradicts Proposition 4.3 below. □

Since the Cesàro operator fails to map ℓ^1 into ℓ^1 (e.g., $\mathsf{C}e_1 = \left(\frac{1}{n}\right)_{n=1}^{\infty} \notin \ell^1$) it is to be expected that the situation is different for $p = 1$. Let $p_n \downarrow 1$ and equip $\ell^{1+} = \cap_{r>1}^{\infty} \ell^r$ with the lc-topology generated by the norms (1.3). Then ℓ^{1+} is a reflexive Fréchet space with strong dual $\ell^{\infty-} := \cup_{q\geq 1} \ell^q$. The same argument given prior to Lemma 2.1 shows that the Cesàro operator $\mathsf{C}^{(1+)}: \ell^{1+} \to \ell^{1+}$ is continuous. Unlike for $p > 1$, the spectra $\sigma(\mathsf{C}^{(1+)})$ and $\sigma^*(\mathsf{C}^{(1+)})$ are now *unbounded* subsets of $\mathbb{C}$. The following result should be compared with Theorem 2.2.

THEOREM 2.4. *For $p = 1$ the following assertions hold.*

(i) $\sigma(\mathsf{C}^{(1+)}) = \{\lambda \in \mathbb{C} \colon \operatorname{Re} \lambda > 0\} \cup \{0\}$.

(ii) $\sigma_{pt}(\mathsf{C}^{(1+)}) = \emptyset$ *and* $\{\lambda \in \mathbb{C} \colon \operatorname{Re} \lambda > 0\} \subseteq \sigma_{pt}((\mathsf{C}^{(1+)})')$.

(iii) $\sigma^*(\mathsf{C}^{(1+)}) = \overline{\sigma(\mathsf{C}^{(1+)})}$.

Moreover, for every non-zero $\lambda \in \sigma(\mathsf{C}^{(1+)})$ the subspace $(\lambda I - C^{(1+)})(\ell^{1+})$ is closed in ℓ^{1+} with $\operatorname{codim}(\lambda I - C^{(1+)})(\ell^{1+}) = 1$.

Proof. Fix $p_n \downarrow 1$. Then the conjugate indices satisfy $q_n \uparrow \infty$ (with $1 < q_n < \infty$ for all $n \in \mathbb{N}$). Moreover, for every $n \in \mathbb{N}$, the identities (2.2) hold. So, via Lemma 2.1 with $X_n := \ell^{p_n}$ and $T_n := \mathsf{C}_n$, for $n \in \mathbb{N}$, we have that

$$\sigma(\mathsf{C}^{(1+)}) \subset \bigcup_{n\in\mathbb{N}} \sigma(\mathsf{C}_n) \subseteq \{\lambda \in \mathbb{C} \colon \operatorname{Re} \lambda > 0\} \cup \{0\},$$

and that $R(\lambda, \mathsf{C}^{(1+)})$ coincides with the restriction of $R(\lambda, \mathsf{C}_n)$ to ℓ^{1+} for each $n \in \mathbb{N}$ and each $\lambda \in \bigcap_{n\in\mathbb{N}} \rho(\mathsf{C}_n)$.

To prove the reverse containment let $\alpha \in \{\lambda \in \mathbb{C} \colon \operatorname{Re} \lambda > 0\}$. Then there exists $n \in \mathbb{N}$ such that $\left|\alpha - \frac{q_n}{2}\right| < \frac{q_n}{2}$. Hence, there exists $u \in \ell^{q_n} \setminus \{0\}$ (and so $u \in (\ell^{1+})' = \ell^{\infty-}$) satisfying $(\mathsf{C}_n)'u = \alpha u$ and hence, $(\mathsf{C}^{(1+)})'u = \alpha u$. Then, for each $x \in \ell^{1+}$, we have

$$\langle (\mathsf{C}^{(1+)} - \alpha I)x, u\rangle = \langle x, ((\mathsf{C}^{(1+)})' - \alpha I)u\rangle = 0.$$

Hence, $\langle y, u\rangle = 0$ for every y in the range of $(\mathsf{C}^{(1+)} - \alpha I)$ with $u \neq 0$ and so $(\mathsf{C}^{(1+)} - \alpha I)$ cannot be surjective. This shows that $\{\lambda \in \mathbb{C} \colon \operatorname{Re} \lambda > 0\} \subseteq \sigma(\mathsf{C}^{(1+)})$.

Adapting the proof of Theorem 2.2 it can be shown that $\mathsf{C}^{(1+)}$ is injective, not surjective and $\mathsf{C}^{(1+)}$ has dense range in ℓ^{1+}. Part (i) is thereby established.

Fix $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > 0$, i.e., $\lambda \in \sigma(\mathsf{C}^{(1+)}) \setminus \{0\}$. Since $q_n \uparrow 0$, there exists $n_0 \in \mathbb{N}$ such that $\left|\lambda - \frac{q_n}{2}\right| < \frac{q_n}{2}$ for all $n \geq n_0$. Then, arguing as in the proof of Theorem 2.2, it can be shown that the subspace $(\lambda I - C^{(1+)})(\ell^{1+})$ is closed in ℓ^{1+} with $\operatorname{codim}(\lambda I - C^{(1+)})(\ell^{1+}) = 1$. Actually, as in the proof of Theorem 2.2, it is established along the way that also $\lambda \in \sigma_{pt}((\mathsf{C}^{(1+)})')$.

Finally, part (iii) follows from Lemma 2.1 as it was shown above in the proof of part (i) that $\sigma(\mathsf{C}^{(1+)}) = \bigcup_{n\in\mathbb{N}} \sigma(\mathsf{C}_n)$. □

THEOREM 2.5. *The Cesàro operator $\mathsf{C}^{(1+)}: \ell^{1+} \to \ell^{1+}$ is not mean ergodic, not power bounded and not supercyclic.*

Proof. Via the inclusion in Theorem 2.4(ii) it follows that $2 \in \sigma_{pt}((\mathsf{C}^{(1+)})')$ and so there exists $u \in \ell^{\infty-} \setminus \{0\}$ satisfying $(\mathsf{C}^{(1+)})'u = 2u$. Choose any $x \in \ell^{1+}$ such that

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$\langle x, u\rangle \neq 0$. Then

$$\langle \frac{1}{n}(\mathsf{C}^{(1+)})^n x, u\rangle = \frac{1}{n}\langle x, ((\mathsf{C}^{(1+)})')^n u\rangle = \frac{1}{n}2^n\langle x, u\rangle, \quad n \in \mathbb{N},$$

and so $\left\{\frac{1}{n}(\mathsf{C}^{(1+)})^n x \colon n \in \mathbb{N}\right\}$ is an unbounded subset of ℓ^{1+}. To complete the proof it now suffices to argue as in the proof of Theorem 2.3. □

REMARK 2.6. (i) We point out that the range

$$(I - \mathsf{C}^{(p+)})(\ell^{p+}) = \overline{\operatorname{span}}\{e_r\}_{r\geq 2} = \{x \in \ell^{p+} \colon x_1 = 0\}.$$

Clearly $(I - \mathsf{C}^{(p+)})(\ell^{p+}) \subseteq \{x \in \ell^{p+} \colon x_1 = 0\}$. Since $\{e_r\}_{r=1}^{\infty}$ is a basis of ℓ^{p+}, it is routine to check that $\overline{\operatorname{span}}\{e_r\}_{r\geq 2} = \{x \in \ell^{p+} \colon x_1 = 0\}$. In view of this observation and the fact that $(I - \mathsf{C}^{(p+)})(\ell^{p+})$ is closed in ℓ^{p+} (by Theorem 2.2), it remains to show that $e_r \in (I - \mathsf{C}^{(p+)})(\ell^{p+})$, for $r \geq 2$. But, this follows from the identities $e_{r+1} = (I - \mathsf{C}^{(p+)})y_r$, for $r \in \mathbb{N}$, with

$$y_r := e_{r+1} - \frac{1}{r}\sum_{k=1}^{r} e_k \in \ell^{p+}, \quad r \in \mathbb{N}.$$

A similar argument shows that also

$$(I - \mathsf{C}^{(1+)})(\ell^{1+}) = \overline{\operatorname{span}}\{e_r\}_{r\geq 2} = \{x \in \ell^{1+} \colon x_1 = 0\}.$$

(ii) We have seen that $(I - \mathsf{C}^{(p+)})(\ell^{p+})$ is a (proper) closed subspace of ℓ^{p+}, but C^{p+} is not even mean ergodic. This fact should be compared with the equivalence of uniform mean ergodicity of $T \in \mathcal{L}(X)$ with the closedness of the subspace $(I - T)(X)$ when X is a (pre)quojection Fréchet space and $(1/n)T^n \to 0$ in $\mathcal{L}_b(X)$ for $n \to \infty$, [**3**, Theorem 3.5]. Of course, ℓ^{p+} is not a (pre)quojection.

3. The Cesàro operator on the space L^{p-}, $1 < p \leq \infty$. We now consider the "continuous" Cesàro operator C defined pointwise by (1.5). Hardy's inequality, [**12**, p. 240], ensures that C maps each Banach space L^p, $1 < p \leq \infty$, continuously into itself. We denote it by $\mathsf{C}^{(p)} \colon L^p \to L^p$, in which case its operator norm satisfies $\|\mathsf{C}^{(p)}\|_{op} = q$ if $1 < p < \infty$ (with $\frac{1}{p} + \frac{1}{q} = 1$) and $\|\mathsf{C}^{(\infty)}\|_{op} = 1$. Accordingly, the Cesàro operator maps the Fréchet space L^{p-} continuously into itself. In fact, if $1 < p_n \uparrow p$ (so that $1 < p_n < p$ for all $n \in \mathbb{N}$), then $L^{p-} = \cap_{n=1}^{\infty} L^{p_n}$ and its Fréchet topology is generated by the sequence of norms (1.4). For each $n \in \mathbb{N}$, let $\mathsf{C}_n := \mathsf{C}^{(p_n)}$. If we denote by $i_n \colon L^{p-} \hookrightarrow L^{p_n}$ and $i_{n,n+1} \colon L^{p_{n+1}} \hookrightarrow L^{p_n}$ the canonical inclusion maps (which clearly have dense range), then $i_n \circ \mathsf{C}^{(p-)} = \mathsf{C}_n \circ i_n$ and also $i_{n,n+1} \circ \mathsf{C}_{n+1} = \mathsf{C}_n \circ i_{n,n+1}$ for all $n \in \mathbb{N}$. Accordingly, for every $n \in \mathbb{N}$, we have (with $\frac{1}{p_n} + \frac{1}{q_n} = 1$) that

$$\|\mathsf{C}^{(p-)}f\|_n = \|i_n\mathsf{C}^{(p-)}f\|_n = \|\mathsf{C}_n i_n f\|_n = \|\mathsf{C}_n f\|_n \leq q_n\|f\|_n, \quad f \in L^{p-}.$$

THEOREM 3.1. *Let $1 < p \leq \infty$ and q satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then*

(i) $\sigma(\mathsf{C}^{(p-)}) = \left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{q}{2}\right| \leq \frac{q}{2}\right\} = \sigma^*(\mathsf{C}^{(p-)})$.

(ii) $\sigma_{pt}(\mathsf{C}^{(p-)}) = \left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{q}{2}\right| \leq \frac{q}{2}\right\} \setminus \{0\}$.

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Proof. Fix $p_n \uparrow p$. If $\frac{1}{p_n} + \frac{1}{q_n} = 1$ for all $n \in \mathbb{N}$, then $q_n \downarrow q$ (so that $q_n > q$ for all $n \in \mathbb{N}$). Moreover, for every $n \in \mathbb{N}$, we have

$$\sigma(\mathsf{C}_n) = \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{q_n}{2}\right| \leq \frac{q_n}{2}\right\} \text{ and } \sigma_{pt}(\mathsf{C}_n) = \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{q_n}{2}\right| < \frac{q_n}{2}\right\},$$

[**15**], [**16**, Theorem 1]. Accordingly, for each $n \in \mathbb{N}$ we have

$$\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{q}{2}\right| \leq \frac{q}{2}\right\} \subseteq \sigma(\mathsf{C}_n) \text{ and } \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{q}{2}\right| < \frac{q}{2}\right\} \subseteq \sigma_{pt}(\mathsf{C}_n).$$

Fix $\lambda \in \mathbb{C}$ with $\left|\lambda - \frac{q}{2}\right| > \frac{q}{2}$. Then $\left|\lambda - \frac{q_n}{2}\right| > \frac{q_n}{2}$ for all $n \geq n_0$ and some $n_0 \in \mathbb{N}$ (as $q_n \downarrow q$) and hence, $\lambda \in \rho(\mathsf{C}_n)$ for all $n \geq n_0$. Since we also have $L^{p-} = \cap_{n \geq n_0} L^{p_n}$, Lemma 2.1 applied to $T := \mathsf{C}^{(p-)} \in \mathcal{L}(L^{p-})$ with $X_n := L^{p_n}$ and $T_n := \mathsf{C}_n$, for $n \geq n_0$, implies that $\bigcap_{n \geq n_0} \left\{z \in \mathbb{C}\colon \left|z - \frac{q_n}{2}\right| > \frac{q_n}{2}\right\} \subseteq \rho(\mathsf{C}^{(p-)})$ and hence, $\lambda \in \rho(\mathsf{C}^{(p-)})$. Accordingly, $\sigma(\mathsf{C}^{(p-)}) \subseteq \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{q}{2}\right| \leq \frac{q}{2}\right\}$.

Now suppose that $\lambda \in \mathbb{C} \setminus \{0\}$ satisfies $\left|\lambda - \frac{q}{2}\right| \leq \frac{q}{2}$ or, equivalently, that $\operatorname{Re}\left(\frac{1}{\lambda}\right) \geq \frac{1}{q}$, in which case $\operatorname{Re}\left(\frac{1}{\lambda}\right) \geq \frac{1}{q} > \frac{1}{q_n}$ for all $n \in \mathbb{N}$. The claim is, for each $n \in \mathbb{N}$, that the function $f_\lambda(x) := x^{\frac{1}{\lambda}-1}$ belongs to L^{p_n} and is an eigenvector of C_n associated to the eigenvalue λ. To see this note that

$$\|f_\lambda\|_n^{p_n} = \int_0^1 |f_\lambda(x)|^{p_n} dx = \int_0^1 x^{p_n\left(\operatorname{Re}\left(\frac{1}{\lambda}\right)-1\right)} dx < \infty, \quad n \in \mathbb{N},$$

as $p_n\left(\operatorname{Re}\left(\frac{1}{\lambda}\right) - 1\right) > p_n\left(\frac{1}{q_n} - 1\right) = -1$. Thus, $f_\lambda \in \cap_{n=1}^\infty L^{p_n} = L^{p-}$. It is routine to check that $\mathsf{C}_n f_\lambda = \lambda f_\lambda$, for $n \in \mathbb{N}$, and hence, $\mathsf{C}^{(p-)} f_\lambda = \lambda f_\lambda$ as $i_n \circ \mathsf{C}^{(p-)} = \mathsf{C}_n \circ i_n$. This shows that $\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{q}{2}\right| \leq \frac{q}{2}\right\} \setminus \{0\} \subseteq \sigma_{pt}(\mathsf{C}^{(p-)})$.

It follows from (1.5) that $\mathsf{C}^{(p-)}$ is injective on L^{p-}. In particular, $0 \notin \sigma_{pt}(\mathsf{C}^{(p-)})$. Moreover, $\mathsf{C}^{(p-)}$ is not surjective, since the range of $\mathsf{C}^{(p-)}$ contains only functions which are continuous on $(0, 1]$. Thus, $0 \in \sigma(\mathsf{C}^{(p-)}) \setminus \sigma_{pt}(\mathsf{C}^{(p-)})$. At this stage part (ii) has been established, as has the first equality in part (i).

It remains to verify the statement in part (i) concerning $\sigma^*(\mathsf{C}^{(p-)})$. From the first equality in (i) and the fact that $\sigma(T) \subseteq \sigma^*(T)$ always holds we have $\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{q}{2}\right| \leq \frac{q}{2}\right\} \subset \sigma^*(\mathsf{C}^{(p-)})$. On the other hand, fix $\lambda \in \mathbb{C}$ such that $\left|\lambda - \frac{q}{2}\right| > \frac{q}{2}$. Then there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that $B(\lambda, \varepsilon) \subseteq \rho(\mathsf{C}_n)$, for $n \geq n_0$. Assume that there exists $f \in L^{p-}$ for which the set $\{R(\mu, \mathsf{C}^{(p-)})f\colon \mu \in B(\lambda, \varepsilon)\}$ is unbounded in L^{p-}. Then there is $n \geq n_0$ such that $\{R(\mu, \mathsf{C}^{(p-)})f\colon \mu \in B(\lambda, \varepsilon)\}$ is an unbounded subset of L^{p_n}. Since $R(\mu, \mathsf{C}^{(p-)})$ coincides with the restriction of $R(\mu, \mathsf{C}_n)$ to L^{p_n}, the set $\{R(\mu, \mathsf{C}_n)f\colon \mu \in B(\lambda, \varepsilon)\}$ is unbounded in L^{p_n}. This contradicts the fact that $\lambda \in \rho(\mathsf{C}_n)$ with L^{p_n} a Banach space. Accordingly, $\{R(\mu, \mathsf{C}^{(p-)})\colon \mu \in B(\lambda, \varepsilon)\}$ is equicontinuous in $\mathcal{L}(L^{p-})$ and so $\lambda \in \sigma^*(\mathsf{C}^{(p-)})$. □

Proposition 3.2. *Let $1 < p \leq \infty$. The Cesàro operator $\mathsf{C}^{(p-)}\colon L^{p-} \to L^{p-}$ is hypercyclic, not power bounded and not mean ergodic. Moreover, $\mathsf{C}^{(p-)}$ is chaotic only if $1 < p < \infty$.*

Proof. Let $1 < p < \infty$. The operator $\mathsf{C}^{(p)}\colon L^p \to L^p$ is known to be hypercyclic and chaotic, [**17**, Theorems 2.3 and 2.6]. Since L^p is separable and dense in L^{p-} and the restriction of $\mathsf{C}^{(p-)}$ to L^p coincides with $\mathsf{C}^{(p)}$, it follows that $\mathsf{C}^{(p-)}\colon L^{p-} \to L^{p-}$ is also hypercyclic and chaotic, [**11**, Propositions 2.24 and 1.31].

Now let $p = \infty$. It is shown in [**11**, Example 12.20] that the Cesàro operator $\mathsf{C}^{(\infty-)}$ is hypercyclic on the separable Fréchet space $L^{\infty-}$; see also [**11**, Corollary 12.19]. But, $\mathsf{C}^{(\infty-)}$ is *not* chaotic because, via Theorem 3.1, we know that $\sigma_{pt}(\mathsf{C}^{(\infty-)}) = \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\} \setminus \{0\}$ which contains only one root of unity, [**11**, Proposition 5.7].

Since $\mathsf{C}^{(p-)}$ is hypercyclic, for $1 < p \leq \infty$, it cannot be power bounded.

Assume that $\mathsf{C}^{(p-)}$ is mean ergodic in L^{p-}. Then

$$L^{p-} = \mathrm{Ker}(I - \mathsf{C}^{(p-)}) \oplus \overline{(I - \mathsf{C}^{(p-)})(L^{p-})},$$

[**1**, Theorem 2.4]. This means precisely that

$$\mathrm{Ker}(I - \mathsf{C}^{(p-)}) \cap \overline{(I - \mathsf{C}^{(p-)})(L^{p-})} = \{0\}$$

and that

$$\mathrm{Ker}(I - \mathsf{C}^{(p-)}) + \overline{(I - \mathsf{C}^{(p-)})(L^{p-})} = L^{p-}.$$

But, $\dim \mathrm{Ker}(I - \mathsf{C}^{(p-)}) = 1$ (as the constant function $\mathbf{1} \in \mathrm{Ker}(I - \mathsf{C}^{(p-)}) \subseteq \mathrm{Ker}(I - \mathsf{C}^{(r)})$, for any $1 < r < p$, with $\dim \mathrm{Ker}(I - \mathsf{C}^{(r)}) = 1$, [**15**, Theorem, p. 28]) and $(I - \mathsf{C}^{(p-)})(L^{p-})$ is dense in L^{p-} as $\mathsf{C}^{(p-)}$ is hypercyclic, [**11**, Lemma 6.3]. So, we have a contradiction, i.e., $\mathsf{C}^{(p-)}$ is not mean ergodic. □

4. The Cesàro operator in other classical Fréchet spaces. The lc-topology of each Fréchet space ℓ^{p+}, $1 \leq p < \infty$, and L^{p-}, $1 < p \leq \infty$, is generated by a sequence of *norms*. This is not so for the classical Fréchet space $C(\mathbb{R}^+)$, equipped with the topology generated by the semi-norms

$$q_j(f) := \max_{x \in [0,j]} |f(x)|, \quad f \in C(\mathbb{R}^+),\ j \in \mathbb{N}, \tag{4.1}$$

nor for the Fréchet space $L^p_{\mathrm{loc}}(\mathbb{R}^+)$, $1 < p < \infty$, consisting of all $\mathbb{C}$-valued, measurable functions f on $\mathbb{R}^+$ such that

$$p_j(f) := \left(\int_0^j |f(x)|^p \, dx\right)^{1/p} < \infty, \quad j \in \mathbb{N}, \tag{4.2}$$

endowed with the topology generated by the semi-norms $\{p_j\}_{j \in \mathbb{N}}$. In fact, $C(\mathbb{R}^+)$ and $L^p_{\mathrm{loc}}(\mathbb{R}^+)$, $1 < p < \infty$, belong to the class of *quojection* Fréchet spaces which, whenever they admit a continuous norm, are necessarily a Banach space, see [**6**, **20**].

The Cesàro operator $\mathsf{C}\colon C(\mathbb{R}^+) \to C(\mathbb{R}^+)$ defined, for every $f \in C(\mathbb{R}^+)$, by $\mathsf{C}f(0) = f(0)$ and $\mathsf{C}f(x) = \frac{1}{x}\int_0^x f(t)\,dt$, for $x > 0$, has been investigated in [**4**], where it is shown that C is power bounded and mean ergodic but, not uniformly mean ergodic and not supercyclic (hence, not hypercyclic). Moreover,

$$\sigma(\mathsf{C}; C(\mathbb{R}^+)) = \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\} \tag{4.3}$$

with $\sigma_{pt}(\mathsf{C}; C(\mathbb{R}^+)) = \sigma(\mathsf{C}; C(\mathbb{R}^+)) \setminus \{0\}$, [**4**, Theorem 3.1]. It remains to clarify the connection between the two notions of spectra.

PROPOSITION 4.1. *For the Cesàro operator* $\mathsf{C}\colon C(\mathbb{R}^+)\to C(\mathbb{R}^+)$ *we have*

$$\sigma(\mathsf{C}; C(\mathbb{R}^+)) = \sigma^*(\mathsf{C}; C(\mathbb{R}^+)).$$

Proof. Let $\lambda\in\mathbb{C}$ satisfy $\left|\lambda-\frac{1}{2}\right|>\frac{1}{2}$ (equivalently, $\mathrm{Re}\left(\frac{1}{\lambda}\right)<1$) and define $\xi:=\frac{1}{\lambda}$. The linear operator P_ξ which maps $f\in C(\mathbb{R}^+)$ to the function

$$P_\xi f\colon x\in\mathbb{R}^+\mapsto\int_0^1 s^{-\xi}f(xs)\,ds\,,$$

is a continuous operator on $C(\mathbb{R}^+)$ such that $\xi I+\xi^2P_\xi$ is the inverse of $(\lambda I-\mathsf{C})$ on $C(\mathbb{R}^+)$. Indeed, applying the dominated convergence theorem to calculate $\lim_{n\to\infty}P_\xi f(x_n)$, whenever $f\in C(\mathbb{R}^+)$ and $x_n\to x$ in $\mathbb{R}^+$ for $n\to\infty$, it follows that $P_\xi f\in C(\mathbb{R}^+)$. Moreover, the substitution $s=e^{-t}$ yields

$$P_\xi f(x)=\int_0^1 s^{-\xi}f(xs)\,ds=\int_0^\infty e^{-(1-\xi)t}f(xe^{-t})\,dt,\quad x\in\mathbb{R}^+,$$

which implies, for each $j\in\mathbb{N}$, that

$$q_j(P_\xi f)\le\int_0^\infty e^{-\mathrm{Re}(1-\xi)t}\max_{x\in[0,j]}|f(xe^{-t})|\,dt\le\frac{1}{\mathrm{Re}(1-\xi)}q_j(f),\ \ f\in C(\mathbb{R}^+).$$

Accordingly, $P_\xi\in\mathcal{L}(C(\mathbb{R}^+))$. That $\xi I+\xi^2P_\xi$ is the inverse of $(\lambda I-\mathsf{C})$ on $C(\mathbb{R}^+)$ follows as in [**15**, p. 29] (or, see the proof of [**7**, Lemma 2(a)]).

So, for every $\lambda\in\mathbb{C}$ such that $\left|\lambda-\frac{1}{2}\right|>\frac{1}{2}$, the operator

$$R(\lambda):=(\lambda I-C)^{-1}=\xi I+\xi^2P_\xi,\quad \xi:=\frac{1}{\lambda},$$

is the resolvent map of C at λ on $C(\mathbb{R}^+)$ and satisfies the estimates

$$q_j(R(\lambda)f)\le\left(|\xi|+\frac{|\xi|^2}{\mathrm{Re}(1-\xi)}\right)q_j(f),\quad j\in\mathbb{N},\ f\in C(\mathbb{R}^+).\tag{4.4}$$

Fix $\lambda_0\in\mathbb{C}$ satisfying $\left|\lambda_0-\frac{1}{2}\right|>\frac{1}{2}$ (i.e., $\lambda\in\rho(\mathsf{C}; C(\mathbb{R}^+))$), and set $\xi_0:=\frac{1}{\lambda_0}$. Via the resolvent equation we have

$$R(\lambda)=R(\lambda_0)+(\lambda_0-\lambda)R(\lambda)R(\lambda_0),\quad \lambda\in\rho(\mathsf{C}; C(\mathbb{R}^+)).$$

Then (4.4) yields, for every $j\in\mathbb{N}$ and $f\in C(\mathbb{R}^+)$, that

$$q_j(R(\lambda)f)\le\left(|\xi_0|+\frac{|\xi_0|^2}{\mathrm{Re}(1-\xi_0)}\right)\left[1+|\lambda_0-\lambda|\left(|\xi|+\frac{|\xi|^2}{\mathrm{Re}(1-\xi)}\right)\right]q_j(f).\tag{4.5}$$

Observe, with $\xi:=\frac{1}{\lambda}$, that $\Phi(\lambda):=1+|\lambda_0-\lambda|\left(|\xi|+\frac{|\xi|^2}{\mathrm{Re}(1-\xi)}\right)$ is defined and continuous on $\mathbb{C}\setminus\left\{\mu\in\mathbb{C}\colon\left|\mu-\frac{1}{2}\right|=\frac{1}{2}\right\}$. Via (4.3) there is $r>0$ such that $D(\lambda_0,r):=\{\lambda\in\mathbb{C}\colon|\lambda-\lambda_0|\le r\}\subseteq\rho(\mathsf{C}; C(\mathbb{R}^+))$. Then (4.5) yields that

$$q_j(R(\lambda)f)\le M\left(|\xi_0|+\frac{|\xi_0|^2}{\mathrm{Re}(1-\xi_0)}\right)q_j(f),\quad j\in\mathbb{N},\ f\in C(\mathbb{R}^+),$$

with $M := \max_{\lambda \in D(\lambda_0, r)} \Phi(\lambda) < \infty$, i.e., $\{R(\lambda) \colon \lambda \in D(\lambda_0, r)\}$ is equicontinuous in $\mathcal{L}(C(\mathbb{R}^+))$. This shows that $\lambda_0 \in \rho^*(\mathsf{C}; C(\mathbb{R}^+))$. By the arbitrariness of λ_0 we have $\rho(\mathsf{C}; C(\mathbb{R}^+)) \subseteq \rho^*(\mathsf{C}; C(\mathbb{R}^+))$, by which the proof is complete. □

We now address the spectra of the Cesàro operator $\mathsf{C} \colon L^p_{\mathrm{loc}}(\mathbb{R}^+) \to L^p_{\mathrm{loc}}(\mathbb{R}^+)$ given by $\mathsf{C}f(x) := \frac{1}{x} \int_0^x f(t)\,dt$, for $x > 0$ and all $f \in L^p_{\mathrm{loc}}(\mathbb{R}^+)$, which is well defined as $L^p([0, x]) \subseteq L^1([0, x])$ for each $x > 0$. By Hardy's inequality, [**12**, p. 240], the linear operator C is continuous on $L^p_{\mathrm{loc}}(\mathbb{R}^+)$. It is known, for each $1 < p < \infty$, that $\mathsf{C} \colon L^p_{\mathrm{loc}}(\mathbb{R}^+) \to L^p_{\mathrm{loc}}(\mathbb{R}^+)$ is not power bounded and not mean ergodic but, it is hypercyclic, chaotic and satisfies

$$\sigma(\mathsf{C}; L^p_{\mathrm{loc}}(\mathbb{R}^+)) = \left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{q}{2}\right| \le \frac{q}{2}\right\} \tag{4.6}$$

with $\sigma_{\mathrm{pt}}(\mathsf{C}; L^p_{\mathrm{loc}}(\mathbb{R}^+)) = \left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{q}{2}\right| < \frac{q}{2}\right\}$, [**4**, Theorem 4.2].

PROPOSITION 4.2. *For the Cesàro operator* $\mathsf{C} \colon L^p_{\mathrm{loc}}(\mathbb{R}^+) \to L^p_{\mathrm{loc}}(\mathbb{R}^+)$ *we have*

$$\sigma(\mathsf{C}; L^p_{\mathrm{loc}}(\mathbb{R}^+)) = \sigma^*(\mathsf{C}; L^p_{\mathrm{loc}}(\mathbb{R}^+)).$$

Proof. Let $\lambda \in \mathbb{C}$ satisfy $\left|\lambda - \frac{q}{2}\right| > \frac{q}{2}$ (equivalently, $\mathrm{Re}\left(\frac{1}{\lambda}\right) < \frac{1}{q}$) and set $\xi := \frac{1}{\lambda}$. The linear operator Q_ξ which maps $f \in L^p_{\mathrm{loc}}(\mathbb{R}^+)$ to the function

$$Q_\xi f \colon x \in \mathbb{R}^+ \mapsto \int_0^1 s^{-\xi} f(xs)\,ds\,,$$

is a continuous operator on $L^p_{\mathrm{loc}}(\mathbb{R}^+)$ such that $\xi I + \xi^2 Q_\xi$ is the inverse of $(\lambda I - \mathsf{C})$ on $L^p_{\mathrm{loc}}(\mathbb{R}^+)$. Indeed, fix $f \in L^p_{\mathrm{loc}}(\mathbb{R}^+)$. For $j \in \mathbb{N}$ set $g := f\chi_{[0,j]}$. Then $g \in L^p(\mathbb{R}^+)$ and hence, by [**7**, Lemma 1(a)], the function $Q_\xi g \in L^p(\mathbb{R}^+)$ and satisfies $\|Q_\xi g\|_p := \left(\int_0^\infty |Q_\xi g(t)|^p\,dt\right)^{1/p} \le \left(\frac{1}{q} - \mathrm{Re}\xi\right)^{-1} \|g\|_p$. Since $f(x) = g(x)$ and $Q_\xi f(x) = Q_\xi g(x)$ whenever $x \in [0, j]$, it follows (with p_j given by (4.2)) that

$$p_j(Q_\xi f) = p_j(Q_\xi g) \le \|Q_\xi g\|_p \le \left(\frac{1}{q} - \mathrm{Re}\xi\right)^{-1} \|g\|_p = \left(\frac{1}{q} - \mathrm{Re}\xi\right)^{-1} p_j(f).$$

Since $j \in \mathbb{N}$ is arbitrary, we have $Q_\xi \in \mathcal{L}(L^p_{\mathrm{loc}}(\mathbb{R}^+))$. That $\xi I + \xi^2 Q_\xi$ is the inverse of $(\lambda I - \mathsf{C})$ on $L^p_{\mathrm{loc}}(\mathbb{R}^+)$ follows as in the proof of [**7**, Lemma 2(a)].

Therefore, for every $\lambda \in \mathbb{C}$ satisfying $\left|\lambda - \frac{q}{2}\right| > \frac{q}{2}$, the operator

$$R(\lambda) := (\lambda I - C)^{-1} = \xi I + \xi^2 Q_\xi, \quad \xi := \frac{1}{\lambda},$$

is the resolvent map of C at λ on $L^p_{\mathrm{loc}}(\mathbb{R}^+)$ and satisfies the estimates

$$p_j(R(\lambda)f) \le \left(|\xi| + |\xi|^2 \left(\frac{1}{q} - \mathrm{Re}\xi\right)^{-1}\right) p_j(f), \quad j \in \mathbb{N},\ f \in L^p_{\mathrm{loc}}(\mathbb{R}^+). \tag{4.7}$$

Fix $\lambda_0 \in \mathbb{C}$ satisfying $\left|\lambda_0 - \frac{q}{2}\right| > \frac{q}{2}$ (i.e., $\lambda \in \rho(\mathsf{C}; L^p_{\rm loc}(\mathbb{R}^+))$), and set $\xi_0 := \frac{1}{\lambda_0}$. Via the resolvent equation we have

$$R(\lambda) = R(\lambda_0) + (\lambda_0 - \lambda)R(\lambda)R(\lambda_0), \quad \lambda \in \rho(\mathsf{C}; L^p_{\rm loc}(\mathbb{R}^+)).$$

It follows from (4.7), for every $j \in \mathbb{N}$ and $f \in L^p_{\rm loc}(\mathbb{R}^+)$, that

$$p_j(R(\lambda)f) \leq \tag{4.8}$$
$$\left(|\xi_0| + |\xi_0|^2\left(\frac{1}{q} - \mathrm{Re}\xi_0\right)^{-1}\right)\left[1 + |\lambda_0 - \lambda|\left(|\xi| + |\xi|^2\left(\frac{1}{q} - \mathrm{Re}\xi\right)^{-1}\right)\right] p_j(f).$$

Observe, with $\xi := \frac{1}{\lambda}$, that $\Psi(\lambda) := 1 + |\lambda_0 - \lambda|\left(|\xi| + |\xi|^2\left(\frac{1}{q} - \mathrm{Re}\xi\right)^{-1}\right)$ is defined and continuous on $\mathbb{C} \setminus \left\{\mu \in \mathbb{C} : \left|\mu - \frac{q}{2}\right| = \frac{q}{2}\right\}$. According to (4.6) there exists $r > 0$ such that $D(\lambda_0, r) \subseteq \rho(\mathsf{C}; L^p_{\rm loc}(\mathbb{R}^+))$. It then follows from (4.8) that

$$p_j(R(\lambda)f) \leq L\left(|\xi_0| + |\xi_0|^2\left(\frac{1}{q} - \mathrm{Re}\xi_0\right)^{-1}\right) p_j(f), \quad j \in \mathbb{N},\ f \in L^p_{\rm loc}(\mathbb{R}^+),$$

with $L := \max_{\lambda \in D(\lambda_0, r)} \Psi(\lambda) < \infty$, i.e., $\{R(\lambda) : \lambda \in D(\lambda_0, r)\}$ is equicontinuous in $\mathcal{L}(L^p_{\rm loc}(\mathbb{R}^+))$. This shows that $\lambda_0 \in \rho^*(\mathsf{C}; L^p_{\rm loc}(\mathbb{R}^+))$. By the arbitrariness of λ_0 we have $\rho(\mathsf{C}; L^p_{\rm loc}(\mathbb{R}^+)) \subseteq \rho^*(\mathsf{C}; L^p_{\rm loc}(\mathbb{R}^+))$, by which the proof is complete. $\square$

Consider now the Cesàro operator $\mathsf{C} : \omega \to \omega$ as given by (1.1). As an increasing sequence of semi-norms defining the Fréchet topology in $\omega = \mathbb{C}^{\mathbb{N}}$ we take $r_k : \omega \to [0, \infty)$, $k \in \mathbb{N}$, where $r_k(x) = \max_{1 \leq j \leq k} |x_j|$, for $x = (x_i)_{i=1}^\infty \in \omega$. Clearly, $\mathsf{C} \in \mathcal{L}(\omega)$. In fact,

$$r_k(\mathsf{C}^n x) \leq r_k(x), \quad x \in \omega,\ k,\ n \in \mathbb{N}. \tag{4.9}$$

Its dual operator $\mathsf{C}' : \varphi \to \varphi$ is continuous on $\varphi := (\omega)'_\beta$ and is given by

$$\mathsf{C}'(x) = \left(\sum_{h=i}^{\infty} \frac{x_h}{h}\right)_{i=1}^{\infty}, \quad x = (x_i)_{i=1}^\infty \in \varphi. \tag{4.10}$$

The linear operator C is a bicontinuous (topological) isomorphism of ω onto itself with $\mathsf{C}^{-1} : \omega \to \omega$ given by (1.2). Denote by $\mathbf{1}$ the constant sequence $(1, 1, \ldots) \in \omega$. The following result, with the exception of the statement about supercyclicity, occurs in [**3**, Proposition 4.1]. The supercyclicity can be deduced from [**9**, Lema 11]; we include a direct proof.

PROPOSITION 4.3. *The Cesàro operator* $\mathsf{C} : \omega \to \omega$ *is power bounded (hence, satisfies* $\frac{\mathsf{C}^n}{n} \to 0$ *in* $\mathcal{L}_b(\omega)$ *as* $n \to \infty$*) and uniformly mean ergodic but, it is not supercyclic. Moreover,* $\mathrm{Ker}(I - \mathsf{C}) = \mathrm{span}\{\mathbf{1}\}$ *and the range* $(I - \mathsf{C})(\omega) = \{x \in \omega : x_1 = 0\} = \overline{\mathrm{span}}\{e_r\}_{r \geq 2}$ *is closed.*

Proof. To show that $\mathsf{C} : \omega \to \omega$ is not supercyclic we proceed by contradiction. So, assume the existence of $x = (x_i)_{i=1}^\infty \in \omega$ such that $\{\lambda \mathsf{C}^i x : \lambda \in \mathbb{C}, i \in \mathbb{N}_0\}$ is dense in ω. Since the 1-st coordinate $(\mathsf{C}^i x)_1 = x_1$, for every $i \in \mathbb{N}$, it follows that $x_1 \neq 0$. On the other hand, there exists a set $\{\mu_k : k \in \mathbb{N}\} \subset \mathbb{C}$ and a strictly increasing sequence

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$(j_k)_k \subseteq \mathbb{N}_0$ such that $\mu_k C^{j_k} x \to e_2$ as $k \to \infty$. Considering the 1-st coordinate and recalling that $x_1 \neq 0$, we may conclude that $\mu_k \to 0$ as $k \to \infty$. Consequently, for all $k \in \mathbb{N}$, the inequality (4.9) implies that $0 \leq r_2(\mu_k C^{j_k} x) \leq |\mu_k| r_2(x) \to 0$ as $k \to \infty$. But, $r_2(\mu_k C^{j_k} x) \to r_2(e_2) = 1$ as $k \to \infty$, which is a contradiction. □

PROPOSITION 4.4. *The spectra of the Cesàro operator* $\mathsf{C}\colon \omega \to \omega$ *are given by*

$$\sigma(\mathsf{C}) = \sigma_{pt}(\mathsf{C}) = \{1/k\colon k \in \mathbb{N}\}$$

and

$$\sigma^*(\mathsf{C}) = \{0\} \cup \sigma(\mathsf{C}) = \overline{\sigma(\mathsf{C})}.$$

Proof. As observed above, $0 \in \rho(\mathsf{C})$. Moreover, $1 \in \sigma_{pt}(\mathsf{C}) \subseteq \sigma(\mathsf{C})$ by Proposition 4.3. For $\lambda \in \mathbb{C} \setminus \{0\}$ the claim is that $(\lambda I - \mathsf{C})$ is injective if and only if $\lambda \notin \{1/k\colon k \in \mathbb{N}\}$.

To establish the claim, fix $\lambda \in \mathbb{C} \setminus \{0\}$ and consider the equation $(\lambda I - \mathsf{C})x = 0$ with $x = (x_n)_{n\in\mathbb{N}} \in \omega$. Then $x_1 = \lambda x_1$ and $(2\lambda - 1)x_2 = x_1$ and $(n\lambda - 1)x_n = \lambda(n-1)x_{n-1}$ for all $n \geq 3$. If $m \in \mathbb{N}$ denotes the smallest positive integer satisfying $x_m \neq 0$, then it follows that $\lambda = \frac{1}{m}$ and so $x_n = \frac{n-1}{n-m} x_{n-1}$ for all $n > m$. This implies that

$$x_n = x_{m+(n-m)} = \frac{(n-1)!}{(m-1)!(n-m)!} x_m, \quad n > m.$$

Then $x = \left(0, \ldots, 0, x_m, m x_m, \frac{m(m+1)}{2} x_m, \ldots\right) \in \omega$ satisfies $\mathsf{C}x = \frac{1}{m} x$ with $x \neq 0$ for any choice of $x_m \neq 0$. This proves the claim.

According to the established claim we have $\sigma_{pt}(\mathsf{C}) = \{1/k\colon k \in \mathbb{N}\} \subseteq \sigma(\mathsf{C}) \subseteq \sigma^*(\mathsf{C})$ with $\sigma^*(\mathsf{C})$ closed, and so $0 \in \sigma^*(\mathsf{C})$.

It remains to show that every $\lambda \notin \{0\} \cup \{1/k\colon k \in \mathbb{N}\}$ belongs to $\rho^*(\mathsf{C})$. To see this, fix $\lambda \notin \{0\} \cup \{1/k\colon k \in \mathbb{N}\}$. The formula for the resolvent operator $(C - \lambda I)^{-1} : \omega \to \omega$ is a matrix which has the entries in it's ith row given by

$$\begin{aligned} a_{i,j} &= \frac{-1}{i\lambda^2 \prod_{h=j}^{i} (1 - \frac{1}{h\lambda})} = \frac{-\lambda^{i-j-1}}{i \prod_{h=j}^{i} (\lambda - \frac{1}{h})}, \quad 1 \leq j < i, \\ a_{i,j} &= 1/(1/i - \lambda), \quad i = j, \end{aligned} \tag{4.11}$$

with all the other entries being 0, [**21**, p. 266]. Select $\delta > 0$ such that the distance ϵ of $B(\lambda, \delta)$ to the compact set $\{0\} \cup \{1/k\colon k \in \mathbb{N}\}$ is strictly positive. Using the form (4.11) of the matrix for the resolvent operator it follows, for each $k \in \mathbb{N}$, that there is $M_k > 0$ such that $r_k((\mathsf{C} - \mu I)^{-1} x) \leq M_k r_k(x)$ for each $\mu \in B(\lambda, \delta)$ and each $x \in \omega$. This implies that $\{(\mathsf{C} - \mu I)^{-1}\colon \mu \in B(\lambda, \delta)\}$ is equicontinuous in $\mathcal{L}(\omega)$ and so $\lambda \in \rho^*(\mathsf{C})$. □

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