Canad. Math. Bull. Vol. **43** (2), 2000 pp. 138–144

# Exponential Laws for the Nachbin Ported Topology

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*Abstract.* We show that for $U$ and $V$ balanced open subsets of (Qno) Fréchet spaces $E$ and $F$ that we have the topological identity

$$\big(\mathcal{H}(U \times V), \tau_\omega\big) = \Big(\mathcal{H}\Big(U; \big(\mathcal{H}(V), \tau_\omega\big)\Big), \tau_\omega\Big).$$

Analogous results for the compact open topology have long been established. We also give an example to show that the (Qno) hypothesis on both $E$ and $F$ is necessary.

For $U$ an open subset of a locally convex space $E$ and $F$ a Banach space we will denote by $\tau_o$ the topology on $\mathcal{H}(U;F)$ of uniform convergence on compact subsets of $U$. We shall say that a semi-norm $p$ on $\mathcal{H}(U;F)$ is ported by the compact subset $K$ of $U$ if for every $V$ open, $K \subset V \subset U$, there exists $c(V) > 0$ such that

$$p(f) \le c(V)\|f\|_V \quad \text{for all } f \in \mathcal{H}(U;F).$$

The $\tau_\omega$ or Nachbin ported topology on $\mathcal{H}(U;F)$ is the topology generated by all semi-norms ported by all compact subsets of $U$. A semi-norm $p$ on $\mathcal{H}(U;F)$ is said to be $\tau_\delta$ continuous if for each increasing countable open cover, $\{V_n\}_{n=1}^{\infty}$, of $U$ there is an positive integer $n_o$ and $C > 0$ so that

$$p(f) \le C\|f\|_{V_{n_o}} \quad \text{for all } f \in \mathcal{H}(U;F).$$

The $\tau_\delta$ topology on $\mathcal{H}(U;F)$ is the topology generated by all $\tau_\delta$-continuous semi-norms. When $F$ is a complete locally convex space we define the $\tau_o$ and $\tau_\omega$ topologies on $\mathcal{H}(U;F)$ by

$$\big(\mathcal{H}(U;F), \tau_o\big) = \varprojlim_{\alpha \in \text{c.s.}(F)} \big(\mathcal{H}(U;F_\alpha), \tau_o\big)$$

and

$$\big(\mathcal{H}(U;F), \tau_\omega\big) = \varprojlim_{\alpha \in \text{c.s.}(F)} \big(\mathcal{H}(U;F_\alpha), \tau_\omega\big).$$

We write $\mathcal{H}(U)$ for $\mathcal{H}(U, \mathbf{C})$. Many authors have investigated necessary conditions for the equivalence of $\tau_o$ and $\tau_\omega$. In [10] it is shown that the condition of quasinormability by operators, (Qno), is a necessary and sufficient condition for their equivalence, irrespective of the range space, on a balanced subset of a Fréchet Schwartz space. We shall now look at exponential laws for the Nachbin ported topology, where we show that (Qno) turns out

Received by the editors July 23, 1998; revised January 22, 1999.
AMS subject classification: Primary: 46G20; secondary: 18D15, 46M05.

to be of importance here also. For $K$ a compact subset of a Fréchet space $E$ we let $\mathcal{H}(K)$ denote the space of all holomorphic germs on $K$. Mujica, [16], defines $G(K)$ as the space of all linear map $\phi\colon \mathcal{H}(K) \to \mathbf{C}$ with the property that the restriction of $\phi$ to each set of holomorphic germ defined and bounded on a neighbourhood of $K$ is $\tau_o$-continuous. He then proceeds to show that

$$G(K)'_i = \big(\mathcal{H}(K), \tau_\omega\big) := \lim_{\substack{\rightarrow\\ K\subset U}} \big(\mathcal{H}(U), \tau_\omega\big).$$

When $K$ is quasinormable $G(K)$ is also quasinormable and we have that $G(K)'_b = \big(\mathcal{H}(K), \tau_\omega\big)$. In [8], [9] and [10] the space $G(K)$ proved useful in obtaining results about holomorphic functions on balanced open subsets of $U$. We shall use this approach once again here. Given a positive integer $n$ we let $P(^nE)$ denote the subspace of $\mathcal{H}(U)$ (and $\mathcal{H}(K)$) of all $n$-homogeneous polynomials on $E$. As with $\mathcal{H}(K)$ it can be shown that for each open subset $U$ of $E$ and each positive integer $n$ there are spaces $G(U)$ and $Q(^nE)$ such that $G(U)'_i = \big(\mathcal{H}(U), \tau_\delta\big)$ and $Q(^nE)'_i = \big(P(^nE), \tau_\omega\big)$. The space $Q(^nE)$ is to isomorphic to $\widehat{\bigotimes}_{s,n,\pi}E$, the space of $n$-fold symmetric tensors on $E$. We refer the reader to [12] for further information on infinite dimensional holomorphy.

In [2] Aron and Schottenloher show that for $U$ and $V$ two open subset of locally convex $k$-spaces $E$ and $F$ we have

$$\big(\mathcal{H}(U\times V), \tau_o\big) = \big(\mathcal{H}(U), \tau_o\big)\epsilon\big(\mathcal{H}(V), \tau_o\big) = \Big(\mathcal{H}\Big(U; \big(\mathcal{H}(V), \tau_o\big)\Big), \tau_o\Big).$$

In this paper we will show that this equality is also true in a limited sense for some Fréchet spaces with $\tau_o$ replaced by $\tau_\omega$. We show, if $U$ and $V$ are balanced open subsets of (Qno) spaces $E$ and $F$, then

$$\big(\mathcal{H}(U\times V), \tau_\omega\big) = \Big(\mathcal{H}\Big(U; \big(\mathcal{H}(V), \tau_\omega\big)\Big), \tau_\omega\Big).$$

This type of result is known as an exponential law (see [3], [4], [5] [11], [19] and [22].)

Let us begin by recalling the definition of (Qno) due to Peris [20]. Let $E$ be a locally convex space and let $\mathcal{U}(E)$ be a fundamental system of absolutely convex neighbourhoods of 0. We shall say that $E$ is quasinormable by operators, (Qno), if given $U$ in $\mathcal{U}(E)$ there is $V$ open in $\mathcal{U}(E)$, $V \subset U$ such that given $\epsilon > 0$ we can find $P \in L(E, E)$ such that

(1) $P(V)$ is bounded in $E$,
(2) $(I - P)(V) \subset \epsilon U$.

Example of space which are (Qno) are Banach spaces, Fréchet Schwartz spaces with the bounded approximation property, Banach valued quasinormable Köthe echelon spaces of order $p$ and $L^p_{\mathrm{Loc}}(\Omega)$ where $\Omega$ is an open subset of $\mathbf{R}^n$.

Following Dineen, [13], if $E$ is (Qno) then we shall say that the $V$ corresponding to $U$ in the above definition is *associated* to $U$.

***Proposition 1*** *Let K be a balanced compact subset of a locally convex space E. Then $G(K)$ is (Qno) if and only if E is (Qno).*

**Proof** Since $E$ is complemented in $G(K)$ one direction is trivial. Now suppose that $E$ is (Qno) and let $W$ be a balanced open neighbourhood of 0 in $G(K)$. By Proposition 5 of [8] we may suppose that $p_W$, the Minkowski functional of $W$, has the form

$$p_W(\phi) = \sum_{n=0}^{\infty} |\phi_n|_{B_U^n},$$

for some $U$ in $\mathcal{U}(E)$, where $\phi = \sum_{n=0}^{\infty} \phi_n$ and $B_U^n = \{P \in P(^nE) : \|P\|_U \leq 1\}$. Using induction and the proofs of Propositions 3.4 and 3.3 of [20] it follows that $Q(^nE)$ is (Qno) and that if $V$ is associated to $U$ then $\bar{\Gamma} \otimes_{n,s} V$ is associated to $\bar{\Gamma} \otimes_{n,s} U$.

Define $q\colon G(K) \to \mathbf{R}^+$ by

$$q(\phi) := \sum_{n=1}^{\infty} n^2 |\phi_n|_{B_V^n}.$$

Then $q$ is a continuous seminorm on $G(K)$ and we will denote its unit ball by $Z$.

Given $\epsilon > 0$ we can choose $n_o$ so that $\sum_{n=n_o}^{\infty} \frac{1}{n^2} \leq \epsilon/2$. For each integer $n$, $0 \leq n \leq n_o$, we can find $P_n \in L\big(Q(^nE), Q(^nE)\big)$ so that

(1) $P_n(\bar{\Gamma} \otimes_{n,\pi} V)$ is bounded in $E$,
(2) $(I - P_n)(\bar{\Gamma} \otimes_{n,s} V) \subset \frac{\epsilon}{2^{n+2}} \bar{\Gamma} \otimes_{n,s} U$.

Now define $P\colon G(K) \to G(K)$ by

$$P\Big(\sum_{n=0}^{\infty} \phi_n\Big) = \sum_{n=0}^{n_o} P_n(\phi_n).$$

Then $P \in L\big(G(K), G(K)\big)$ and $P(Z) \subset \sum_{n=0}^{n_o} \frac{1}{n^2} P_n(\bar{\Gamma} \otimes_{n,s} V)$ is clearly bounded in $G(K)$. Furthermore, if $\phi = \sum_{n=0}^{\infty} \phi_n \in Z$ then

$$\begin{aligned} p_W\big((I-P)(\phi)\big) &= \sum_{n=0}^{n_o} |(I - P_n)(\phi_n)|_{B_U^n} + \sum_{n=n_o}^{\infty} |\phi_n|_{B_U^n} \\ &= \sum_{n=0}^{n_o} \frac{\epsilon}{2^{n+2}} + \frac{\epsilon}{2} \leq \epsilon. \end{aligned}$$

Which shows that $(I - P)(Z) \in \epsilon W$ and therefore $G(K)$ is (Qno). ■

Using an analogous proof we can show:

***Proposition 2*** *Let $U$ be a balanced open subset of a locally convex space $E$. Then $G(U)$ is (Qno) if and only if $E$ is (Qno).*

Before going on to prove our main result let us look at the relationship between various preduals of various spaces of holomorphic functions. For $U$ an open subset of a Banach space $E$ Mujica, [18], introduces the topology $\tau_{bc}$ on $\mathcal{H}^\infty(U)$. Using Theorem 4.4 of [18] $\tau_{bc}$ may be defined as the finest topology on $\mathcal{H}^\infty(U)$ which coincides with $\tau_c$ on each norm

https://doi.org/10.4153/CMB-2000-021-x Published online by Cambridge University Press

bounded subset. The strong dual of $\big(\mathcal{H}^\infty(U), \tau_{bc}\big)$ is denoted by $G^\infty(U)$. This is a Banach space and is a predual of $\big(\mathcal{H}^\infty(U), \|\,.\,\|\big)$.

Let $E$ be a Fréchet space with $\{p_n\}_n$ an increasing family of semi-norms which define its topology. For $K$ a compact subset of $E$ define $U_n$ by

$$U_n = \left\{ x \in E : \inf_{y \in K} p_n(x - y) < \frac{1}{n} \right\}.$$

Then $\{U_n\}_n$ is a fundamental system of neighbourhoods of $K$ and each $U_n$ is open in the Banach space $E_n$, the completion of $\big(E/p_n^{-1}(0), p_n\big)$.

It follows from the definition of $G(K)$ we have that

$$(*) \qquad G(K) = \lim_{\substack{\leftarrow \\ n}} G^\infty(U_n).$$

Using $(*)$ and Theorem 6.1 of [19] we have:

***Proposition 3*** *Let K and L be compact subsets of Fréchet spaces E and F respectively then*

$$G(K \times L) = G(K)\widehat{\otimes}_\pi G(L).$$

Fréchet spaces $E$ and $F$ are said to have the (BB) property if every bounded subset of $E\widehat{\otimes}_\pi F$ is contained in the closed convex hull of the tensor product of a bounded subset of $E$ and a bounded subset of $F$. When this happens $(E\widehat{\otimes}_\pi F)'_b = L_b(E, F'_b)$. From Theorem 2 of [9] we get the following result.

***Corollary 4*** *Let U and V be balanced open subsets of Fréchet spaces E and F respectively then* $\tau_o = \tau_\omega$ *on* $\mathcal{H}(U \times V)$ *if and only if* $\tau_o = \tau_\omega$ *on* $\mathcal{H}(U)$ *and on* $\mathcal{H}(V)$ *and* $\big(G(K), G(L)\big)$ *has (BB) for any K compact balanced in E and L compact balanced in F.*

Using Buchwalter's Duality Theorem, Theorem 16.2.7 of [14], we have:

***Corollary 5*** *Let K and L be compact subsets of Fréchet spaces E and F respectively then* $\big(\mathcal{H}(K \times L), \tau_o\big) = \big(\mathcal{H}(K), \tau_o\big)\epsilon\big(\mathcal{H}(L), \tau_o\big)$.

We are now in a position to prove our main result.

***Theorem 6*** *Let U and V be balanced open subsets of (Qno) Fréchet spaces E and F respectively; then* $\big(\mathcal{H}(U \times V), \tau_\omega\big) = \Big(\mathcal{H}\Big(U; \big(\mathcal{H}(V), \tau_\omega\big)\Big), \tau_\omega\Big)$.

**Proof** Let $K$ and $L$ be compact balanced subsets of $E$ and $F$ respectively then from Proposition 1 and Corollary 4.3 of [20] we see that $\big(G(K), G(L)\big)$ satisfies (BB) and hence

$$\big(\mathcal{H}(K \times L), \tau_\omega\big) = G(K \times L)'_b = \big(G(K)\widehat{\otimes}_\pi G(L)\big)'_b = L_b\Big(G(K); \big(\mathcal{H}(L), \tau_\omega\big)\Big).$$

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Now let us take projective limits over all compact balanced subsets $K$ of $U$ and all $L$ of $V$. We get

$$\big(\mathcal{H}(U \times V), \tau_\omega\big) = \lim_{\substack{\leftarrow \\ K \subset U}} \lim_{\substack{\leftarrow \\ L \subset V}} \big(\mathcal{H}(K \times L), \tau_\omega\big)$$

$$\lim_{\substack{\leftarrow \\ K \subset U}} \lim_{\substack{\leftarrow \\ L \subset V}} L_b\Big(G(K); \big(\mathcal{H}(L), \tau_\omega\big)\Big).$$

Using 39.8.(10) of [15] this is equal to

$$\lim_{\substack{\leftarrow \\ K \subset U}} L_b\Big(G(K); \lim_{\substack{\leftarrow \\ L \subset V}} \big(\mathcal{H}(L), \tau_\omega\big)\Big) = \lim_{\substack{\leftarrow \\ K \subset U}} L_b\Big(G(K); \big(\mathcal{H}(V), \tau_\omega\big)\Big).$$

We can write $\big(\mathcal{H}(V), \tau_\omega\big)$ as

$$\lim_{\substack{\leftarrow \\ \alpha \in \text{c.s.}((\mathcal{H}(V), \tau_\omega))}} X_\alpha,$$

where $X_\alpha$ is the Banach space associated with the seminorm $\alpha$. Using 39.8.(10) of [15] again we have that

$$\lim_{\substack{\leftarrow \\ K \subset U}} L_b\Big(G(K), \big(\mathcal{H}(V), \tau_\omega\big)\Big) = \lim_{\substack{\leftarrow \\ \alpha \in \text{c.s.}((\mathcal{H}(V), \tau_\omega))}} \lim_{\substack{\leftarrow \\ K \subset U}} L_b\big(G(K), X_\alpha\big).$$

Since $G(K)$ is (Qno) it follows from [20, Theorem 4.7] and [7, Proposition 1] that this is equal to

$$\lim_{\substack{\leftarrow \\ \alpha \in \text{c.s.}((\mathcal{H}(V), \tau_\omega))}} \lim_{\substack{\leftarrow \\ K \subset U}} \big(\mathcal{H}(K, X_\alpha), \tau_\omega\big)$$

which of course is

$$\lim_{\substack{\leftarrow \\ \alpha \in \text{c.s.}((\mathcal{H}(V), \tau_\omega))}} \big(\mathcal{H}(U, X_\alpha), \tau_\omega\big).$$

And this is equal to

$$\bigg(\mathcal{H}\Big(U; \big(\mathcal{H}(V), \tau_\omega\big)\Big), \tau_\omega\bigg). \qquad \blacksquare$$

In particular, for $U$ and $V$ balanced open subsets of Banach spaces $E$ and $F$ respectively, we have $\big(\mathcal{H}(U \times V), \tau_\omega\big) = \bigg(\mathcal{H}\Big(U; \big(\mathcal{H}(V), \tau_\omega\big)\Big), \tau_\omega\bigg)$.

Finally, we note that the above Theorem need not be true if we drop the assumption that

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$E$ and $F$ are both Qno spaces. In [21] an example of a Banach space $E$ and a Fréchet Schwartz space $F$ so that the $(E, F)$ property fails (BB) is given. Let us consider these spaces. Given a locally convex space $G$ we shall use $G'_c$ to denote the dual of $G$ endowed with the topology of uniform convergence on compact subsets of $G$. Since $F$ is a Fréchet Schwartz space it will follow from [17] that $\tau_o = \tau_\omega$ on $\mathcal{H}(V)$. The map from $\mathcal{H}(U \times V)$ into $\mathcal{H}\Big(U; \big(\mathcal{H}(V), \tau_\omega\big)\Big)$ which sends $f \to \tilde{f}$ identifies $L_c(E; F'_c)$ with a complemented subspace of $\big(P(^2E \times F), \tau_o\big)$ and hence, using the idea of Lemma 1 of [1] (replacing barrelled with bornological), identifies $L_c(E; F'_c)_{\text{bor}}$ with a complemented subspace of $\big(P(^2E \times F), \tau_\omega\big)$. Since $F'_b$ is complemented in $\big(\mathcal{H}(V), \tau_\omega\big)$ we see that $L_b(E, F'_b)$ is complemented in $L_b\Big(E, \big(\mathcal{H}(V), \tau_\omega\big)\Big)$ and hence in $\Big(\mathcal{H}\Big(U; \big(\mathcal{H}(V), \tau_\omega\big), \tau_\omega\Big)\Big)$. If $\big(\mathcal{H}(U \times V), \tau_\omega\big)$ and $\Big(\mathcal{H}\Big(U; \big(\mathcal{H}(V), \tau_\omega\big), \tau_\omega\Big)\Big)$ are equal then they induce the same topology on $L(E, F'_b)$. Thus we see that $L_b(E, F'_b) = L_c(E, F'_b)_{\text{bor}}$ and so $L_b(E, F'_b)$ is complemented in $\big(P(^2E \times F), \tau_\omega\big)$. Since $\big(P(^2E \times F), \tau_\omega\big)$ is a DF-space it follows that $L_b(E, F'_b)$ is a DF-space. Applying Proposition 2 of [6] this implies that $(E, F)$ has (BB) which is not possible by our choice of $E$ and $F$.

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