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LACUNARY MÜNTZ SYSTEMS

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The classical theorem of Müntz and Szász says that the span of

$$\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}, \quad 0<\lambda_1<\lambda_2<\dots\to\infty$$

is dense in $C[0,1]$ in the uniform norm if and only if $\sum_{i=1}^{\infty} 1/\lambda_i=\infty$. We prove that, if $\{\lambda_i\}$ is lacunary, we can replace the underlying interval $[0,1]$ by any set of positive measure. The key to the proof is the establishment of a bounded Remez-type inequality for lacunary Müntz systems. Namely if $A\subset[0,1]$ and its Lebesgue measure $\mu(A)$ is at least $\varepsilon>0$ then

$$|a_0|\leqq c\left\|\sum_{i=0}^{n} a_i x^{\lambda_i}\right\|_A$$

where c depends only on ε and Λ (not on n and A) and where $\Lambda:=\inf_i \lambda_{i+1}/\lambda_i>1$.

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1. Introduction

A very beautiful theorem of Müntz and Szász says that

$$M:=span\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}, \quad 0<\lambda_1<\lambda_2<\dots\to\infty \tag{1.1}$$

is dense in $C[0,1]$ in the uniform norm if and only if

$$\sum_{i=1}^{\infty}\frac{1}{\lambda_i}=\infty. \tag{1.2}$$

This is very much a theorem about continuous functions on intervals. So it can be proved that exactly the same theorem holds in $C[A]$ provided

$$A\subset[0,\infty)$$

is a closed set with non-empty interior. This result is due to Clarkson and Erdös **[5]**. When $\bar{A}$ has no interior it is by no means obvious what happens. Our intention is to prove the following theorem.

Theorem 1. *Suppose $\lambda_i \geqq \Lambda^i$ $(i=1,2,\ldots)$, where $\Lambda > 1$, and suppose $A \subset [0,\infty)$ is any set with positive Lebesgue measure. Then*

$$M := span\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}$$

fails to be dense in $C[A]$ in the uniform norm.

Indeed, under the above assumptions, if $y \in A$ is a point of Lebesgue density 1 then every function f from the uniform closure of M on A is of the form

$$f(x) = \sum_{i=0}^{\infty} a_i x^{\lambda_i}, \quad x \in [0, y) \cap A$$

where $\lambda_0 = 0$. This in turn rests on the following inequality.

Inequality 1 (Remez-type inequality). *Suppose $\rho > 0$ and $A \subset [\rho, 1)$ is a closed set of measure $\varepsilon > 0$. Suppose $\lambda_0 = 0$ and $\lambda_i \geqq \Lambda^i$ $(i=1,2,\ldots)$, where $\Lambda > 1$. Then*

$$\left\| \sum_{i=0}^{n} a_i x^{\lambda_i} \right\|_{[0,\rho]} \leqq c \left\| \sum_{i=0}^{n} a_i x^{\lambda_i} \right\|_{A}$$

where the constant c depends only on ρ, ε and Λ (and not on n and A).

Here, and in what follows $\|\cdot\|_A$ denotes the uniform norm on A.

In a seminal paper **[5]**, Clarkson and Erdös prove Theorem 1 in the case where $A := [1-\varepsilon, 1]$. The fact that Inequality 1 holds in this interval case is critical to our argument. This follows from **[5]** and is proved in Section 3. (These interval results are more generally applicable to any system where $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$.)

Proofs of the Müntz–Szász Theorem may be found in **[4]**, **[7]** and **[8]** with various generalizations and extensions in **[2, 3, 10, 12, 13,** and **16]**. A discussion of Remez-type inequalities for polynomials is given in **[6]**.

Our proof relies on an examination of (generalized) Chebyshev polynomials from M. In particular we must establish estimates for the size of their zeros. This is done in Section 2. The very close relationship between the location of zeros of the associated Chebyshev polynomials and the possibility of approximation is discussed in **[1]** and **[2]**.

Section 3 contains a proof of the Remez-type inequality for span$\{1, x^{\lambda}, x^{\lambda^2}, \ldots\}$. In the fourth section we offer a comparison theorem which allows us to extend our results to Müntz spaces

$$M := span\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}$$

where for some $\Lambda > 1$,

$$\lambda_i \geqq \Lambda^i, \quad i=1,2,\ldots.$$

Section 5 contains an example which shows that a bounded Remez-type inequality for an infinite Müntz system cannot hold on arbitrary perfect sites of measure 0.

In the final section we characterize the Müntz systems which are dense in $C[A]$ in the uniform norm for every countable closed $A \subset [0,1]$. ($\sum_{i=1}^{\infty} 1/\lambda_i = \infty$ is necessary and sufficient, assuming $\lambda_0 = 0$ and $\inf_i(\lambda_{i+1} - \lambda_i) > 0$.)

2. Zeros of Chebyshev polynomials

The generalized Chebyshev polynomial from

$$M_n := span\{x^{\lambda_0}, \ldots, x^{\lambda_n}\}, \quad 0 \leqq \lambda_0 < \lambda_1 < \cdots \tag{2.1}$$

with respect to a compact set $A \subset [0,\infty)$ is denoted by

$$T_n(x) := T_n\{[\lambda_0, \ldots, \lambda_n] : A\}(x)$$

and is defined to be

$$T_n(x) := c\left(x^{\lambda_n} + \sum_{i=0}^{n-1} a_i x^{\lambda_i}\right)$$

where we chose $\{a_i\}_{i=1}^{n}$ to minimize

$$\left\| x^{\lambda_n} + \sum_{i=0}^{n-1} a_i x^{\lambda_i} \right\|_A$$

and c is chosen so that

$$\|T_n\|_A = 1 \quad \text{and} \quad \lim_{x\to\infty} T_n(x) = +\infty.$$

Then, T_n achieves $\pm\max_{x\in A}|T_n(x)|$ $n+1$ times in A with alternating sign and has exactly n zeros in $(0,\infty)$. Suppose always $0 \leqq \lambda_0 < \lambda_1 < \cdots$.

Lemma 1. *Let $0 \leqq \alpha < \beta$ and $1 \leqq m \leqq n$. Then the positive real zeros of*

$$T_{n-m}\{[0, \lambda_{m+1}, \lambda_{m+2}, \ldots, \lambda_n] : [\alpha, \beta]\}$$

and

$$T_{n-m+1}\{[0, \lambda_m, \lambda_{m+1}, \ldots, \lambda_n] : [\alpha, \beta]\}$$

interlace. In particular, the smallest positive real zero of T_{n-m+1} *is smaller than the smallest positive real zero of* T_{n-m}.

Proof. Consider $T_{n-m+1}-T_{n-m}$. The argument is a straightforward counting of zeros. □

Lemma 2. *Suppose* $\lambda_i \leqq \gamma_i$, $i=1,\ldots,n$, *with strict inequality at least once. Then the smallest positive real zero of* $T_n\{[0,\lambda_1,\ldots,\lambda_n]:[0,\beta]\}$ *is smaller than the smallest positive real zero of* $T_n\{[0,\gamma_1,\ldots,\gamma_n]:[0,\beta]\}$.

Proof. See [3, Proposition 1].

Lemma 3. *If* $0 \leqq m \leqq n$ *and*

$$T_{n-m}\{[0,\lambda_{m+1},\lambda_{m+2},\ldots,\lambda_n]:[0,\beta]\}$$

has all its zeros in $[\alpha,\beta]$ *then*

$$T_n\{[0,\lambda_1,\lambda_2,\ldots,\lambda_n]:[0,\beta]\}$$

has at most m zeros in $(0,\alpha)$.

Proof. This follows from Lemma 1. □

Lemma 4. *Suppose* $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$ *and* $\inf_i(\lambda_{i+1}-\lambda_i)>0$. *Then there exists constant* $c>0$ *independent of n so that the smallest positive real zero of* $T_n\{[0,\lambda_1,\ldots,\lambda_n]:[0,1]\}$ *is greater than c.*

Proof. If $\lambda_1 \geqq 1$, $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$ and $\inf_i(\lambda_{i+1}-\lambda_i)>0$, then we have the Markov-type inequality

$$\max_{0\leqq x\leqq 1-\varepsilon} |p'(x)| \leqq \eta(\varepsilon,\{\lambda_i\}) \max_{0\leqq x\leqq 1} |p(x)|$$

for every $p\in M_n$, $n\in\mathbb{N}$, and $0<\varepsilon<1$, where the constant $\eta(\varepsilon,\{\lambda_i\})$ depends only on ε and the sequence $\{\lambda_i\}$. This was obtained in [1] based on the results of Clarkson and Erdös [5]. Now the lemma follows from the equioscillation of T_n, the Mean Value Theorem, and the above Markov-type inequality. When $0<\lambda_1<1$ the scaling $x\to x^{1/\lambda_1}$ gives the desired result from the already proved case. □

Lemma 5. *Suppose* $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$ *and* λ_{i-1}/λ_i $(i=2,3,\ldots)$ *is nondecreasing. Then there exists a constant c depending only on the sequence* $\{\lambda_i\}$ *so that*

$$T_{n-m}\{[0,\lambda_{m+1},\lambda_{m+2},\ldots,\lambda_n]:[0,1]\}$$

has all its positive real zeros in $[1-c\lambda_{n-m}/\lambda_n, 1]$.

Proof. Note that the assumptions of the lemma imply $\inf_i(\lambda_{i+1}-\lambda_i)>0$. Let $\beta_{m,n}$ be the smallest non-negative real zero of

$$S_{m,n} := T_{n-m}\{[0, \lambda_{m+1}, \lambda_{m+2}, \ldots, \lambda_n] : [0, 1]\}.$$

Then consider, for $n > m \geqq 0$,

$$U_{m,n}(x) := S_{m,n}(x^{\delta_{m,n}})$$

where

$$\delta_{m,n} := \max_{n \geqq i \geqq m+1} \frac{\lambda_{i-1}}{\lambda_i} = \frac{\lambda_{n-1}}{\lambda_n}.$$

Note that

$$U_{m,n} = T_{n-m}\{[0, (\delta_{m,n}\lambda_{m+1}), (\delta_{m,n}\lambda_{m+2}), \ldots, (\delta_{m,n}\lambda_n)] : [0, 1]\}$$

and

$$\lambda_{h-1} \leqq \lambda_h \delta_{m,n} < \lambda_h \quad \text{if} \quad m+1 \leqq h \leqq n.$$

So by Lemma 2 the smallest positive real zero of $U_{m,n}(x)$ (which is just $\beta_{m,n}^{1/\delta_{m,n}}$) is greater than the smallest positive real zero of

$$T_{n-m}\{[0, \lambda_m, \lambda_{m+1}, \ldots, \lambda_{n-1}] : [0, 1]\}.$$

So in particular,

$$\begin{aligned} \beta_{m,n} &> \beta_{m-1,n-1}^{\delta_{m,n}} \\ &\;\vdots \\ &> (\beta_{0,n-m})^{\delta_{m,n} \cdot \delta_{m-1,n-1} \cdots \delta_{1,n-m+1}}. \end{aligned}$$

Here, by Lemma 4,

$$1 > \beta_{0,n-m} > c_1 > 0$$

since $\beta_{0,n-m}$ is just the smallest root of

$$T_{n-m}\{[0, \lambda_1, \ldots, \lambda_{n-m}] : [0, 1]\}.$$

So

$$\begin{aligned}\beta_{m,n} &> c_1^{\delta_{m,n}\cdot\delta_{m-1,n-1}\cdots\delta_{1,n-m+1}}\\ &= c_1^{(\lambda_{n-1}/\lambda_n)\cdot(\lambda_{n-2}/\lambda_{n-1})\cdots(\lambda_{n-m}/\lambda_{n-m+1})}\\ &= c_1^{\lambda_{n-m}/\lambda_n}.\end{aligned}$$

Also, by the Mean Value Theorem, for every $c_1>0$ there exists a $c_2>0$ so that

$$c_1^{\delta} \geqq 1-c_2\delta$$

for $0<\delta<1$. Therefore

$$\beta_{m,n} \geqq 1-\frac{c_2\lambda_{n-m}}{\lambda_n}$$

as required. □

From Lemmas 3 and 5 we deduce:

Lemma 6. *Suppose* $1 \leqq \lambda_1 < \lambda_2, \ldots, \sum_{i=1}^{\infty} 1/\lambda_i < \infty$, λ_{i-1}/λ_i $(i=2,3,\ldots)$ *is nondecreasing, and* $0 \leqq m \leqq n$. *Then there exists a constant c depending only on the sequence* $\{\lambda_i\}$ *so that*

$$T_n := T_n\{[0,\lambda_1,\ldots,\lambda_n]:[0,1]\}$$

has at most m zeros in the interval

$$\left(0, 1-\frac{c\lambda_{n-m}}{\lambda_n}\right)$$

and at least $n-m$ *zeros in*

$$\left[1-\frac{c\lambda_{n-m}}{\lambda_n}, 1\right].$$

3. The Lacunary case

In most of this section let

$$M := span\{1, x^{\lambda}, x^{\lambda^2}\ldots\}, \quad \lambda>1$$

and

$$M_n := \text{span}\{1, x^{\lambda}, \ldots, x^{\lambda^n}\}.$$

Lemma 6 gives the following as a special case.

Lemma 7. $T_n := T_n\{[0, \lambda, \ldots, \lambda^n] : [0, 1]\}$ *has at least* $n-m$ *zeros in*

$$\left[1 - \frac{c}{\lambda^m}, 1\right]$$

and at most m *zeros in* $(0, 1-c/\lambda^m)$ *where* $c>0$ *depends only on* λ.

Let P_n be the Chebyshev polynomial from M_n on a fixed, compact $A \subset [0,1]$ of measure at least $\varepsilon>0$. Since P_n is the Chebyshev polynomial from M_n on $\tilde{A} := \{x \in [0,1] : |P_n(x)| \leqq 1\}$ as well, we may, without loss, assume that A is comprised of at most n disjoint intervals. Choose $\delta > 1$ so that

$$\sum_{k=1}^{\infty} \frac{1}{\delta^k} = \varepsilon.$$

Now partition $[0,1]$ into subintervals

$$I_1 := [0, \beta_1] \qquad \text{with} \quad \mu(I_1 \cap A) = \frac{1}{\delta},$$

$$I_2 := [\beta_1, \beta_2] \qquad \text{with} \quad \mu(I_2 \cap A) = \frac{1}{\delta^2},$$

$$\vdots$$

$$I_{n-1} := [\beta_{n-2}, \beta_{n-1}] \quad \text{with} \quad \mu(I_{n-1} \cap A) = \frac{1}{\delta^{n-1}},$$

$$I_n := [\beta_{n-1}, 1] \qquad \text{with} \quad \mu(I_n \cap A) > \frac{1}{\delta^n},$$

where μ denotes the Lebesgue measure.

Lemma 8. *Suppose* j, $2 \leqq j \leqq n$, *is fixed and* $A \cap I_j$ *contains an interval of length* Δ_j. *Then there are positive constants* c_λ *and* c'_λ *depending only on* λ *so that* $A \cap I_{j-1}$ *contains an interval of length* Δ_{j-1} *where*

$$\Delta_{j-1} > \frac{c_\lambda}{\delta^j |\log \Delta_j|}$$

whenever $0<\Delta_j<c'_\lambda$.

Proof. Let $[a_j, b_j]$ be the interval of length Δ_j in $A\cap I_j$ and choose $m_j\in\mathbb{R}$ so that $\Delta_j=c/\lambda^{m_j}$, where c is as in Lemma 7. Consider the Chebyshev polynomial T_n from M_n on the interval $[0,b_j]$. From Lemma 7, by the scaling $x\to b_jx$ we can deduce that T_n has at least $n-m$ zeros in $[a_j,b_j]$ where m is the smallest nonnegative integer not less than m_j. In particular P_n (the Chebyshev polynomial from M_n on A) has at most $m+2$ zeros in $(0,a_j]$, otherwise $T_n-P_n\in M_n$ would have more than n zeros in $(0,b_j]$ (counting every positive zero without sign change twice). It follows that $A\cap I_{j-1}$ is the union of at most $m+4$ intervals and hence $A\cap I_{j-1}$ contains an interval of length at least

$$\frac{1}{\delta^{j-1}(m+4)}\geqq\frac{1}{\delta^{j-1}\left(\dfrac{\log c-\log\Delta_j}{\log\lambda}+5\right)}\geqq\frac{c_\lambda}{\delta^j|\log\Delta_j|}$$

whenever $0<\Delta_j\leqq c'_\lambda$ with some positive constants c_λ and c'_λ depending only on λ. $\square$

Lemma 9. *Let $\delta>1$ and $c_\lambda>0$ be as in Lemma 8. Consider the (backwards) iteration*

$$\Delta_{k-1}:=\frac{c_\lambda}{\delta^k|\log\Delta_k|}\quad\textit{where}\quad\Delta_n=\frac{1}{n^2\delta^n}.$$

Then there is a constant $c_{\lambda,\delta}$ depending only on λ and δ so that

$$\frac{1}{k^2\delta^k}<\Delta_k<\frac{1}{2}$$

whenever $c_{\lambda,\delta}<k\leqq n$.

Proof. Suppose

$$\frac{1}{2}>\Delta_k>\frac{1}{k^2\delta^k}.$$

Then

$$\frac{1}{2}\Delta_{k-1}=\frac{c_\lambda}{\delta^k|\log\Delta_k|}>\frac{c_\lambda}{\delta^k|\log(k^2\delta^k)|}$$

$$\geqq\frac{c_\lambda/\delta}{k(|\log\delta|+2|\log k|)\delta^{k-1}}$$

$$\geqq \frac{1}{(k-1)^2 \delta^{k-1}}$$

provided $k > c_{\lambda,\delta}$ with a constant $c_{\lambda,\delta}$ depending only on λ and δ. The result now follows. □

We will need the following inequality that may be found in [**14**, p. 54]. We include a distinct new proof.

Inequality 1 in the Interval Case.

Suppose $\lambda_0 = 0$, $\inf_i(\lambda_{i+1} - \lambda_i) > 0$ *and*

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i} < \infty.$$

Then, for every $\varepsilon \in (0,1)$,

$$\left\| \sum_{i=0}^{n} a_i x^{\lambda_i} \right\|_{[0,1]} \leqq c_\varepsilon \left\| \sum_{i=0}^{n} a_i x^{\lambda_i} \right\|_{[1-\varepsilon,1]}$$

where c_ε *depends on the sequence* $\{\lambda_i\}$ *and* ε *but not on* n.

Proof. Let $\varepsilon \in (0,1)$ be fixed. First let $\lambda_1 \geqq 1$. Assume indirectly that there are $p_m \in M$ so that $A_m := \max_{0 \leqq x \leqq 1} |p_m(x)| \to \infty$, while $\|p_m\|_{[1-\varepsilon,1]} = 1$. Let $q_m := A_m^{-1} p_m$. Then, without loss, we may assume that

$$\max_{0 \leqq x \leqq 1} |q_m| = \max_{0 \leqq x \leqq 1-\varepsilon} |q_m|, \quad m = 1, 2, \ldots.$$

As in [**1**, Lemma 2] (see also the proof of Lemma 4), for every $0 < \varepsilon' < 1$ we have

$$\max_{0 \leqq x \leqq 1-\varepsilon'} |q_m'(x)| \leqq \eta(\varepsilon')$$

where $\eta(\varepsilon')$ is a constant depending only on ε'. Therefore $\{q_m\}_{m=1}^{\infty}$ is a sequence of uniformly bounded equicontinuous functions on every closed subinterval of $[0,1)$, hence by the Arzela–Ascoli Theorem there is a subsequence of $\{q_m\}_{m=1}^{\infty}$ (without loss of generality we may assume that this is $\{q_m\}_{m=1}^{\infty}$ itself) which converges to a function F uniformly on $[0, 1-\varepsilon/2]$. Then, by the Clarkson–Erdös Theorem of the Introduction, F is analytic on $(0, 1-\varepsilon/2)$. On the other hand

$$|F(x)| \leqq A_m^{-1}, \quad x \in [1-\varepsilon, 1-\varepsilon/2]$$

and since $A_m \to \infty$, we have

$$F(x)=0, \quad x \in [1-\varepsilon, 1-\varepsilon/2]$$

which implies that $F \equiv 0$ on $[0, 1-\varepsilon/2]$, a contradiction, since $\max_{0 \leqq x \leqq 1-\varepsilon/2} |F(x)|=1$. When $0<\lambda_1<1$ the scaling $x \to x^{1/\lambda_1}$ gives the desired result from the already proved case. □

Inequality 2 **(Remez-type Inequality for Lacunary Systems).** *If* $P \in span\{1, x^{\lambda}, x^{\lambda^2}, \ldots\}$, $\lambda>1$, *and* $A \subset [0,1]$ *is a closed set of measure at least* $\varepsilon>0$, *then*

$$|P(0)|<c_{\varepsilon,\lambda}\|P(x)\|_A$$

where $c_{\varepsilon,\lambda}$ *depends only on* ε *and* λ.

Proof. The extremal polynomial from $M_n = span\{1, x^{\lambda}, x^{\lambda^2}, \ldots, x^{\lambda^n}\}$ is, by a simple perturbation argument, just the Chebyshev polynomial T_n on A. By Lemmas 8 and 9 this polynomial is bounded on an interval of length $c_{\lambda,\varepsilon}>0$ (independently of A and n) in $[0,1]$. The rest now follows from Inequality 1 in the Interval Case. □

4. Comparison theorems

The following comparison theorem holds and shows that the Remez constant gets smaller as the Müntz system gets sparser.

Theorem 2. *Suppose* $\{\lambda_i\}$ *and* $\{\gamma_i\}$ *are increasing sequences of positive real numbers. If*

$$\lambda_i \leqq \gamma_i, \quad \lambda_0=\gamma_0=0$$

and A *is a compact set in* (ρ,∞), *where* $\rho>0$. *Then*

$$\sup_{\{a_i\}} \frac{\|\sum_{i=0}^n a_i x^{\lambda_i}\|_{[0,\rho]}}{\|\sum_{i=0}^n a_i x^{\lambda_i}\|_A} \geqq \sup_{\{b_i\}} \frac{\|\sum_{i=0}^n b_i x^{\gamma_i}\|_{[0,\rho]}}{\|\sum_{i=0}^n b_i x^{\gamma_i}\|_A}$$

Proof. A simple perturbation argument shows that the extremal polynomial for

$$\max_{\{b_i\}} \frac{\|\sum_{i=0}^n b_i x^{\gamma_i}\|_{[0,\delta]}}{\|\sum_{i=0}^n b_i x^{\gamma_i}\|_A}$$

is just the Chebyshev polynomial T_n on A from $span\{1, x^{\gamma_1}, \ldots, x^{\gamma_n}\}$. If not, it would be possible to increase the value of $p(y) := \sum_{i=0}^n a_i y^{\lambda_i}$ for every fixed $0 \leqq y \leqq \rho$ without increasing $\|p(x)\|_A$. Now observe that for T_n

$$\|T_n\|_{[0,\rho]} = |T_n(0)|$$

because T_n and hence T_n' has no zeros on $(0,\rho)$.

Now let $R_n \in span\{1, x^{\lambda_1}, \ldots, x^{\lambda_n}\}$ interpolate T_n at the zeros of T_n, and be normalized so that $R_n(0) = T_n(0)$. Theorem 1 of [15], now gives

$$|R_n(x)| \leqq |T_n(x)|, \quad x \in A$$

and the result follows. □

Proofs of Theorem 1 and Inequality 1. Inequality 1 is immediate from the above theorem and Inequality 2, while Theorem 1 with the remark right after it now follows from Inequality 1 and the results of Clarkson and Erdös given in the Introduction.

5. An example

Theorem 3. *Let $\{\lambda_i\}$ be an arbitrary sequence of distinct positive real numbers. Then there exists a non-empty perfect set $E \subset [0,1]$ and Müntz polynomials $P_m \in span\{x^{\lambda_0}, x^{\lambda_1}, \ldots\}$ such that $\|P_m\|_E \leqq 1$ and $|P_m(0)| \to \infty$ when $m \to \infty$.*

Proof. Let $M_n := \mathrm{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}$. Choose two distinct points $x_{1,1}$ and $x_{1,2}$ in $(0,1)$, By interpolating we can find a $P_1 \in M_2$ such that

$$P_1(x_{1,1}) = P_1(x_{1,2}) = 0 \quad \text{and} \quad P_1(0) = 1.$$

Choose two disjoint closed intervals $E_{1,1}$ and $E_{1,2}$ so that $x_{1,1} \in E_{1,1}$, $x_{1,2} \in E_{1,2}$ and

$$|P_1(x)| \leqq 1 \quad \text{for every} \quad x \in E_{1,1} \cup E_{1,2}.$$

Assume that a sequence of Müntz polynomials $\{P_j\}_{j=1}^m$ and closed intervals $E_{j,k}$ $(1 \leqq j \leqq m,\ 1 \leqq k \leqq 2^j)$ have already been constructed so that the intervals $E_{j,k}$ $(1 \leqq k \leqq 2^j)$ are pairwise disjoint for every fixed j $(1 \leqq j \leqq m)$, $E_{j+1,2i-1} \subset E_{j,i}$ and $E_{j+1,2i} \subset E_{j,i}$ $(1 \leqq j \leqq m-1,\ 1 \leqq i \leqq 2^{j-1})$, $|P_j(x)| \leqq 1$ on each $E_{j,k}$ $(1 \leqq j \leqq m,\ 1 \leqq k \leqq 2^j)$ and $P_j(0) = 2^j$ $(1 \leqq j \leqq m)$. Take two distinct points $x_{m+1,2i-1}$ and $x_{m+1,2i}$ from each $E_{m,i}$ $(1 \leqq i \leqq 2^m)$. By interpolating we can find a $P_{m+1} \in M_{2^{m+1}}$ such that

$$P_{m+1}(x_{m+1,k}) = 0, \quad 1 \leqq k \leqq 2^{m+1} \quad \text{and} \quad P_{m+1}(0) = 2^{m+1}.$$

For every i $(1 \leqq i \leqq 2^m)$ choose two disjoint closed intervals $E_{m+1,2i-1}$ and $E_{m+1,2i}$ such that $x_{m+1,2i-1} \in E_{m+1,2i-1}$, $x_{m+1,2i} \in E_{m+1,2i}$, $E_{m+1,2i-1} \cup E_{m+1,2i} \subset E_{m,i}$ and

$$|P_{m+1}(x)| \leqq 1 \quad \text{for every} \quad x \in \bigcup_{k=1}^{2^{m+1}} E_{m+1,k}.$$

Now let $E:=\bigcap_{m=1}^{\infty}\bigcup_{k=1}^{2^m} E_{m,k}$. Obviously E is perfect, $P_m\in M_{2^m}$ and $P_m(0)=2^m$. Thus the theorem is proved. $\square$

6. Countable sets

Some of the subtleties of Müntz's theorem on subsets are illustrated by the following pair of theorems.

Theorem 4. *Let $\{\lambda_i\}$ be an arbitrary sequence of distinct positive real numbers. Then there exists a closed infinite set (a convergent sequence with its limit) $S\subset[0,1]$ such that span$\{1,x^{\lambda_1},x^{\lambda_2},\dots\}$ is dense in $C[S]$ in the uniform norm.*

Proof. Let $y_1=1/2$ and assume that $\{y_j\}_{j=1}^{n}\subset(0,1)$ has already been constructed. We choose a y_{n+1} such that

(1) $y_n<y_{n+1}<1$,

(2) $1-(n+1)^{-1}<y_{n+1}$,

(3) $|p(x)|\leqq(n+1)^{-1}$ for every $p\in M_n:=\operatorname{span}\{1,x^{\lambda_1},\dots,x^{\lambda_n}\}$ satisfying $|p(y_j)|\leqq 1$, $j=1,2,\dots,n$, and $p(1)=0$, and for every $y_{n+1}\leqq x\leqq 1$.

The existence of such a y_{n+1} follows from the following argument. In M_n we define the norms

$$\|p\|_1:=\sum_{j=0}^{n}|a_j| \quad\text{and}\quad \|p\|_2:=|p(1)|+\sum_{j=1}^{n}|p(y_j)|$$

of a Müntz polynomial $p(x)=a_0+\sum_{j=1}^{n}a_jx^{\lambda_j}$ which are equivalent to each other. If $p\in M_n$, $|p(y_j)|\leqq 1$ $(j=1,2,\dots,n)$ and $p(1)=0$, then $\|p\|_2\leqq n$, hence $\|p\|_1\leqq K$ with some constant K depending only on n. Therefore there is a constant $K'=K'(n)\geqq 1$ such that $\max_{1/2\leqq x\leqq 1}|p'(x)|\leqq K'$ for every $p\in H_n$, which, together with the Mean Value Theorem and $p(1)=0$, implies

$$|p(x)|=|p(1)-p(x)|=(1-x)|p'(\xi)|\leqq(1-x)K'\leqq(n+1)^{-1},\quad p\in M_n$$

if $1-x\leqq((n+1)K')^{-1}$. Hence

$$y_{n+1}:=\max\left\{y_n+\frac{1-y_n}{2},\ 1-((n+1)K')^{-1}\right\}$$

is suitable. Obviously $\lim_{n\to\infty}y_n=1$. Let $S:=\{y_n\}_{n=1}^{\infty}\cup\{1\}$. We show that *span*$\{1,x^{\lambda_1},x^{\lambda_2},\dots\}$ is dense in $C[S]$. Let f be continuous on S, without loss of generality we may assume that $f(1)=0$. Let $L:=\max_{x\in S}|f(x)|$. Choose $p_n\in M_n$ $(n=1,2,\dots)$ such that

$p_n(y_j)=f(y_j)$ $(j=1,2,\dots,n)$ and $p_n(1)=0$ (this is the interpolation property of a Haar space). Then, from the choice of $\{y_j\}_{j=1}^{\infty}$ we easily deduce that

$$\max_{x\in S}|f(x)-p_n(x)|\leqq\frac{L}{n+1}$$

which proves the theorem. □

Our next theorem, together with Müntz's and Tietze's theorems, will show that if $\{\lambda_i\}$ is a sequence of real numbers satisfying $\inf_i(\lambda_{i+1}-\lambda_i)>0$, then *span*$\{1, x^{\lambda_1}, x^{\lambda_2},\dots\}$ is dense in $C[S]$ in the uniform norm for every countable closed $S\subset[0,1]$ if and only if $\sum_{i=1}^{\infty}1/\lambda_i=\infty$.

Theorem 5. *Let $\{\lambda_i\}_{i=1}^{\infty}$ be a sequence of positive real numbers such that $\sum_{i=1}^{\infty}1/\lambda_i<\infty$ and $\inf_i(\lambda_{i+1}-\lambda_i)>0$. Then there is a countable closed subset S of $[0,1]$ so that span$\{1,x^{\lambda_1},x^{\lambda_2},\dots\}$ fails to be dense in $C[S]$ in the uniform norm.*

Proof. For every $n\in\mathbb{N}$, let E_n be the collection of the $n+1$ extreme points of the nth Chebyshev polynomial $T_n=T_n\{[0,\lambda_1,\dots,\lambda_n]:[0,1]\}$ of M_n and let $E=\cup_{n=1}^{\infty}E_n$. Since $\sum_{i=1}^{\infty}1/\lambda_i<\infty$, Lemma 4 gives $0<c:=\inf\{E\backslash\{0\}\}$, hence we can choose three points $0<y_1<y_2<y_3<c$. Now let $S=\bar{E}\cup\{y_1,y_2,y_3\}$. By an observation of [1] we have $E''=\{1\}$, where E' denotes the collection of the limit points of E. Therefore $S''=\{1\}$ as well, hence S is a closed countable set. Now let f be continuous on S, $f(x)=0$ on $[c,1]$, $f(y_1)=f(y_3)=2$ and $f(y_2)=-2$. Assume that there is a $p\in M_n$ such that $\max_{x\in S}|p(x)-f(x)|\leqq 1/2$. Then it is easy to check that $p-T_n\in M_n$ has at least $n+1$ zeros in $(0,1)$, which is a contradiction. This finishes the proof. □

We remark that if $\{\lambda_i\}$ is an arbitrary sequence of distinct positive real numbers then there is a non-empty perfect set $S\subset[0,1]$ such that *span*$\{x^{\lambda_1},x^{\lambda_2},\dots\}$ is dense in $C[S]$ in the uniform norm. This can be obtained by straightforward modifications of the proof of Theorem 3.

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