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Almost Periodicity and Lyapunov's Functions for Impulsive Functional Differential Equations with Infinite Delays

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Abstract. This paper studies the existence and uniqueness of almost periodic solutions of nonlinear impulsive functional differential equations with infinite delay. The results obtained are based on the Lyapunov–Razumikhin method and on differential inequalities for piecewise continuous functions.

1 Introduction

Impulsive differential equations arise naturally from a wide variety of applications such as aircraft control, inspection process in operations research, drug administration, and threshold theory in biology. However, due to numerous theoretical and technical difficulties, not much has been developed in the direction of impulsive functional differential equations. In the few publications dedicated to this subject, early works were by Anokhin [1] and by Gopalsamy and Zhang [3]. Some qualitative properties (oscillation, asymptotic behavior and stability, almost periodicity) were investigated by several authors (see [2, 13, 14]).

One of the most important parts of the qualitative theory of differential equations is the theory of the existence and stability of the almost periodic solutions. In the present paper we consider the problem of existence and stability of almost periodic solutions of nonlinear impulsive functional differential equations with infinite delay. Moreover, the technique of investigation essentially depends on the choice of minimal subsets of a suitable space of piecewise continuous functions, which are used to estimate the derivatives of Lyapunov's functions [6, 12]. It is well known that Lyapunov–Razumikhin methods have been widely used in the study of qualitative properties for functional differential equations without impulses [10].

The paper is organized as follows. In Section 2 we give some preliminaries and main definitions. In Section 3 we investigate the existence and stability of almost periodic solutions of nonlinear impulsive functional differential equations with infinite delay. Sufficient conditions are obtained by means of piecewise continuous auxiliary functions that are analogues of the classical Lyapunov functions. The investigations are carried out by also using a comparison principle that permits us to reduce the study of nonlinear impulsive functional differential equations to the study of a scalar differential equation.

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2 Preliminary Notes and Definitions

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with norm $\|\cdot\|$, $\mathbb{R}^+ = [0,\infty)$, $B_\nu = \{x \in \mathbb{R}^n : \|x\| \leq \nu\}, \nu > 0, \Omega \subset \mathbb{R}^n, B_\nu \subset \Omega, \Omega \neq \varnothing$. Consider the following sets:

- $\mathbb{B} = \{\{\tau_k\} : \tau_k \in \mathbb{R}, \tau_k < \tau_{k+1}, k \in \mathbb{Z}\}$, *i.e.*, $\{\tau_k\}$ is unbounded and strictly increasing, with distance $\rho(\{\tau_k^{(1)}\}, \{\tau_k^{(2)}\})$.
- $PC = PC[\mathbb{R},\mathbb{R}^n] = \{\varphi \colon \mathbb{R} \to \mathbb{R}^n, \varphi$ is a piecewise continuous function with points of discontinuity of the first kind at τ_k, with $\{\tau_k\} \in \mathbb{B}$, at which $\varphi(\tau_k - 0)$ and $\varphi(\tau_k + 0)$ exist and $\varphi(\tau_k - 0) = \varphi(\tau_k)\}$.
- $PC^1[\mathbb{R},\mathbb{R}^n] = \{\varphi \colon \mathbb{R} \to \mathbb{R}^n, \varphi$ is continuously differentiable everywhere except for points τ_k, with $\{\tau_k\} \in \mathbb{B}$, at which $\dot\varphi(\tau_k - 0)$ and $\dot\varphi(\tau_k + 0)$ exist and $\dot\varphi(\tau_k - 0) = \dot\varphi(\tau_k)\}$.

Let $\varphi_0 \in PC[\mathbb{R},\Omega]$ and $|\varphi_0| = \sup_{t\in\mathbb{R}} \|\varphi_0(t)\|$.

We will consider the system of impulsive functional differential equations

$$(2.1) \qquad \begin{cases} \dot x(t) = f(t, x_t), t > t_0, t \neq \tau_k, \\ \Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k - 0) = I_k(x(\tau_k - 0)),\ k \in \mathbb{Z}, \end{cases}$$

where $t_0 \in \mathbb{R}$, $f \colon \mathbb{R} \times PC[\mathbb{R},\mathbb{R}^n] \to \mathbb{R}^n$; $I_k \in C[\Omega,\mathbb{R}^n], k \in \mathbb{Z}$; $\{\tau_k\} \in \mathbb{B}$, and for $t > t_0, x_t \in PC[\mathbb{R},\mathbb{R}^n]$ is defined by $x_t = x(t + s), -\infty < s \leq 0$.

Denote by $x(t) = x(t; t_0, \varphi_0)$, $\varphi_0 \in PC[\mathbb{R},\Omega]$ the solution to system (2.1) satisfying the initial conditions

$$(2.2) \qquad \begin{cases} x(t; t_0, \varphi_0) = \varphi_0(t - t_0),\ t \leq t_0, \\ x(t_0 + 0; t_0, \varphi_0) = \varphi_0(0), \end{cases}$$

and let $J^+(t_0, \varphi_0)$ be the maximal interval of type $[t_0, \beta)$ in which the solution $x(t; t_0, \varphi_0)$ is defined.

Recall from [14] that the solution $x(t) = x(t; t_0, \varphi_0)$ of the initial value problem (2.1) (2.2) is characterized by the following:

(i) For $t \leq t_0, t_0 \in [\tau_{k_0}, \tau_{k_1}), \tau_{k_0} < \tau_{k_1}, \tau_{k_i} \in \{\tau_k\}, i = 0, 1$ the solution $x(t)$ satisfies the initial conditions (2.2);

(ii) For $t_0 < t \leq \tau_{k_1}$, $x(t)$ coincides with the solution to the problem

$$\dot x(t) = f(t, x_t),\ t > t_0,$$
$$x_{t_0} = \varphi_0(s),\ -\infty < s \leq 0.$$

At the moment $t = \tau_{k_1}$ the mapping point $(t, x(t; t_0, \varphi_0))$ of the extended phase space jumps instantaneously from position $(\tau_{k_1}, x(\tau_{k_1}; t_0, \varphi_0))$ to position $(\tau_{k_1}, x(\tau_{k_1}; t_0, \varphi_0) + I_1(x(\tau_{k_1}; t_0, \varphi_0))$;

(iii) For $\tau_{k_1} < t \leq \tau_{k_2}, \tau_{k_2} \in \{\tau_k\}$ the solution $x(t)$ coincides with the solution to

$$\begin{cases} \dot y(t) = f(t, y_t),\ t > \tau_{k_1}, \\ y_{\tau_{k_1}} = \varphi_1,\ \varphi_1 \in PC[\mathbb{R},\Omega], \end{cases}$$

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where

$$\varphi_1(t-\tau_{k_1})=\begin{cases}\varphi_0(t-\tau_{k_1}) & t\in(-\infty,\tau_{k_1}],\\ x(t;t_0,\varphi_0)+I_{k_1}(t;t_0,\varphi_0) & t=\tau_{k_1}.\end{cases}$$

At the moment $t=\tau_{k_2}$ the mapping point $(t,x(t))$ jumps instantaneously, etc.

Thus in interval $J^+(t_0,\varphi_0)$ the solution $x(t;t_0,\varphi_0)$ to the problem (2.1), (2.2) is a piecewise continuous function with points of discontinuity of the first kind at the moments $t=\tau_k$, $k\in\mathbb{Z}$, where it is continuous from the left.

For convenience, let us state the following hypotheses.

(H1) $f\in C[\mathbb{R}\times PC[\mathbb{R},\Omega],\mathbb{R}^n]$, $f(t,0)=0$, $t\in[t_0,\infty)$.

(H2) The function $f(t,\varphi)$ is Lipchitzian with respect to $\varphi\in PC[\mathbb{R},\Omega]$ uniformly on $t\in[t_0,\infty)$.

(H3) $I_k\in C[\Omega,\mathbb{R}^n]$, $I_k(0)=0$ and $(I+I_k)\colon\Omega\to\Omega$, $k\in\mathbb{Z}$ where I is the identity in Ω.

Lemma 2.1 *Let the conditions* (H1)–(H3) *hold. Then* $J^+(t_0,\varphi_0)=[t_0,\infty)$.

Proof Since conditions (H1)–(H3) hold, then from the existence theorem for the equation without impulses $\dot{x}=f(t,x_t)$ [4, Theorem 2.2.1], it follows that the solution $x(t)=x(t;t_0,\varphi_0)$ to problem (2.1), (2.2) is defined on each of the intervals $(\tau_{k-1},\tau_k]$, $k\in\mathbb{Z}$. From the property of the sequence $\{\tau_k\}$ we conclude that it is continuous for $t\geq t_0$. ■

We note that the problems of existence, uniqueness, and continuity of the solutions of functional differential equations without impulses have been investigated [5,7].

Since the solutions to (2.1), (2.2) are piecewise continuous functions, we adopt the following definitions for almost periodicity.

For $T,P\in\mathbb{B}$, let $s(T\cup P)\colon\mathbb{B}\to\mathbb{B}$ be a map such that the set $s(T\cup P)$ forms a strictly increasing sequence and if $D\subset\mathbb{R}$, let $\theta_\varepsilon(D)=\{t+\varepsilon,\ t\in D\}$, and $F_\varepsilon(D)=\bigcap\{\theta_\varepsilon(D):\varepsilon>0\}$.

By $\phi=(\varphi(t),T)$ we denote an element from the space $PC\times\mathbb{B}$, and for every sequence of real numbers $\{\alpha_n\}$, $n=1,2,\dots$, let $\theta_{\alpha_n}\phi$ denote the sets $\{\varphi(t+\alpha_n),T-\alpha_n\}\subset PC\times\mathbb{B}$, where $T-\alpha_n=\{\tau_k-\alpha_n,k\in\mathbb{Z},n=1,2,\dots\}$.

Definition 2.2 ([11]) The set of sequences $\{\tau_k^j\}$, $\tau_k^j=\tau_{k+j}-\tau_k$, $k\in\mathbb{Z}$, $j\in\mathbb{Z}$, $\{\tau_k\}\in\mathbb{B}$, is said to be *uniformly almost periodic* if for any $\varepsilon>0$ there exists a relatively dense set in $\mathbb{R}$ of ε-almost periods, common for all the sequences $\{\tau_k^j\}$.

Lemma 2.3 ([11]) *The set of sequences* $\{\tau_k^j\}$ *is uniformly almost periodic if and only if from each infinite sequence of shifts* $\{\tau_k-\alpha_n\}$, $k\in\mathbb{Z}$, $n=1,2,\dots,\alpha_n\in\mathbb{R}$ *we can choose a subsequence that is convergent in* $\mathbb{B}$.

Definition 2.4 The sequence $\{\phi_n\}$, $\phi_n=(\varphi_n(t),T_n)\in(PC\times\mathbb{B})$ is convergent to ϕ, $\phi=(\varphi(t),T)$, $(\varphi(t),T)\in(PC\times\mathbb{B})$ if and only if for any $\varepsilon>0$ there exists

$n_0 > 0$ such that for $n \geq n_0$,

$$\rho(T, T_n) < \varepsilon, \quad |\varphi_n(t) - \varphi(t)| < \varepsilon$$

holds uniformly for $t \in \mathbb{R} \setminus F_\varepsilon(s(T_n \cup T))$.

Definition 2.5 The function $\varphi \in PC[\mathbb{R}, \Omega]$ is said to be an *almost periodic piecewise continuous function* with points of discontinuity of the first kind from the set T if for every sequence of real numbers $\{\alpha'_m\}$ there exists a subsequence $\{\alpha_n\},\ \alpha_n = \alpha'_{m_n}$, such that $\theta_{\alpha_n}\phi$ is compact in $PC \times \mathbb{B}$.

We introduce the following assumptions.

(H4) The function $f(t, \varphi)$ is almost periodic in $t \in \mathbb{R}$, uniformly with respect to $\varphi \in PC[\mathbb{R}, \Omega]$.
(H5) The sequence $\{I_k(x)\}$ is almost periodic, uniformly with respect to $x \in \Omega$, $\Omega \in \mathbb{R}^n$.
(H6) The function $\varphi_0 \in PC[\mathbb{R}, \mathbb{R}^n]$ is almost periodic.
(H7) The set of sequences $\{\tau_k^j\}$, $\tau_k^j = \tau_{k+j} - \tau_k$, $k \in \mathbb{Z}$, $j \in \mathbb{Z}$, $\{\tau_k\} \in \mathbb{B}$, is uniformly almost periodic.

Let conditions (H4)–(H7) hold and let $\{\alpha_m{}'\}$ be an arbitrary sequence of real numbers. Then there exist a subsequence $\{\alpha_n\},\ \alpha_n = \alpha_{m_n}{}'$, such that the sequence $\{f(t+\alpha_n, x, y)\}$ is convergent uniformly on $x \in \Omega$, $y \in \Omega$ to the function $f^\alpha(t, x, y)$, the sequence $\varphi_0(t + \alpha_n)$ is convergent uniformly to the function $\varphi_0^\alpha(t)$, and the set of sequences $\{\tau_k - \alpha_n\},\ k \in \mathbb{Z}$ is convergent to the sequence τ_k^α, uniformly with respect to $k \in \mathbb{Z}$ as $n \to \infty$.

By $\{k_{n_i}\}$ we denote the sequence of integers such that the subsequence $\{\tau_{k+n_i}\}$ is convergent to τ_k^α, uniformly with respect to k as $i \to \infty$. From (H2) it follows that there exists a subsequence of the sequence $\{k_{n_i}\}$ such that the sequence $\{I_{k+k_{n_i}}(x)\}$ is convergent uniformly to the limit denoted by $I_k^\alpha(x)$.

Then for every sequence $\{\alpha'_m\}$ the system (2.1), (2.2) satisfies the system E^α in the form

$$\begin{cases} \dot{x}(t) = f^\alpha(t, x_t), t \neq \tau_k^\alpha, \\ x(t) = \varphi_0^\alpha(t), t \in (-\infty, t_0], \\ \Delta x(t) = I_k^\alpha(x(\tau_k^\alpha)), t = \tau_k^\alpha,\ k \in \mathbb{Z}. \end{cases} \tag{2.3}$$

Definition 2.6 The set of all systems E^α is said to be the *module* of the system (2.1), (2.2), and we denote this set by $\operatorname{mod}(f, \varphi_0, I_k, \tau_k)$.

Definition 2.7 ([14]) The zero solution to the system E^α is said to be

- *uniformly stable* if $(\forall\varepsilon> 0)(\exists\delta = \delta(\varepsilon))(\forall t_0 \in \mathbb{R})(\forall\varphi_0 \in PC[\mathbb{R}, \mathbb{R}^n] \cap B_\delta)$ $(\forall t > t_0)\colon \|x(t; t_0, \varphi_0)\| < \varepsilon$;
- *uniformly attractive* if $(\exists\lambda> 0)(\forall\varepsilon> 0)(\exists T = T(\varepsilon))(\forall\varphi_0 \in PC[\mathbb{R}, \mathbb{R}^n] \cap B_\lambda)$ $(\forall t_0 \in \mathbb{R})(\forall t > t_0 + T)\colon\ \|x(t; t_0, \varphi_0)\| < \varepsilon$;
- *uniformly asymptotically stable* if it is uniformly stable and uniformly attractive.

Consider the sets:

$$G_k = \{(t, x, y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n : \tau_{k-1} < t < \tau_k, k \in \mathbb{Z}, \}, G = \bigcup_{k\in\mathbb{Z}} G_k;$$

$$Q = \{a \in C[\mathbb{R}^+, \mathbb{R}^+] : a \text{ is strictly increasing in } \mathbb{R}^+ \text{ and } a(0) = 0\}.$$

Definition 2.8 We shall say that the function $V : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$ belongs to the class V_0 if

- V is continuous in G and $V(t, 0, 0) = 0, t \in \mathbb{R}$.
- For each $k \in \mathbb{Z}$ and each point $(x_0, y_0) \in B_\nu \times B_\nu$ the limits

$$V(\tau_k - 0, x_0, y_0) = \lim_{\substack{(t,x,y)\to(\tau_k,x_0,y_0)\\(t,x,y)\in G_k}} V(t, x, y),$$

$$V(\tau_k + 0, x_0, y_0) = \lim_{\substack{(t,x,y)\to(\tau_k,x_0,y_0)\\(t,x,y)\in G_{k+1}}} V(t, x, y)$$

  exist and are finite and the equality $V(\tau_k - 0, x_0, y_0) = V(\tau_k, x_0, y_0)$ holds.
- V is locally Lipshitz in x, y, *i.e.*, there exists a positive constant L such that

$$\|V(t, x_1, y_1) - V(t, x_2, y_2)\| \le L(\|x_1 - x_2\| + \|y_1 - y_2\|), \tag{2.4}$$

  for $t \in \mathbb{R}, (x_i, y_i) \in B_\nu \times B_\nu,\ i = 1, 2$.

Let $V \in V_0,\ t > t_0,\ t \neq \tau_k,\ x \in PC[\mathbb{R}, \mathbb{R}^n],\ y \in PC[\mathbb{R}, \mathbb{R}^n])$. Introduce

$$D_-V(t, x(t), y(t)) = \liminf_{\delta\to 0} \delta^{-1} \left\{V(t + \delta, x(t) + \delta f(t, x_t), y(t) + \delta f(t, y_t)) - V(t, x(t), y(t))\right\}.$$

Introduce the following classes of functions:

$$\Omega_1 = \left\{(x, y) : x, y \in PC[\mathbb{R}, \Omega], V(s, x(s), y(s)) \le V(t, x(t), y(t)), -\infty < s \le t,\ t \ge t_0,\ V \in V_0\right\}.$$

Definition 2.9 We shall say that the function $W : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+$ belongs to the class W_0 if the following hold:

- W is continuous for $(t, x) \in \mathbb{R} \times \mathbb{R}^n, t \neq \tau_k^\alpha,\ k \in \mathbb{Z}$ and $W(t, 0) = 0,\ t \in \mathbb{R}$.
- For each $k \in \mathbb{Z}$ and each point $x_0 \in B_\nu$ the limits

$$W(\tau_k^\alpha - 0, x_0) = \lim_{\substack{(t,x)\to(\tau_k^\alpha,x_0)\\t<\tau_k^\alpha}} W(t, x),$$

$$W(\tau_k^\alpha + 0, x_0) = \lim_{\substack{(t,x)\to(\tau_k,x_0)\\t>\tau_k^\alpha}} W(t, x)$$

  exist and are finite and the equality $W(\tau_k^\alpha - 0, x_0) = W(\tau_k^\alpha, x_0)$ holds.

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- W is locally Lipshitz along x.

Let $W \in W_0,\ t > t_0,\ t \neq \tau_k^\alpha,\ x \in PC[\mathbb{R}, \mathbb{R}^n]$. Introduce the function

$$D_-W(t, x(t)) = \liminf_{\delta \to 0} \delta^{-1}\{W(t+\delta, x(t) + \delta f^\alpha(t, x_t)) - W(t, x(t))\}.$$

In the proof of the main results we shall use the following lemmas.

Lemma 2.10 *Let the following conditions hold.*

(i) *Conditions* (H1)–(H3).

(ii) *The function* $g\colon (t_0, \infty) \times \mathbb{R}^+ \to \mathbb{R}$ *is continuous in each of the sets* $(\tau_{k-1}, \tau_k] \times \mathbb{R}^+$, $k \in \mathbb{Z}$, *and* $g(t, 0) = 0$ *for* $t \in (t_0, \infty)$.

(iii) $\gamma_k \in C[\mathbb{R}^+, \mathbb{R}^+]$, $\gamma_k(0) = 0$, *and* $\psi_k(u) = u + \gamma_k(u)$, $k \in \mathbb{Z}$ *are nondecreasing with respect to u.*

(iv) *The maximal solution* $r(t; t_0, u_0)$ *to the problem*

$$\begin{cases} \dot{u} = g(t, u),\ t > t_0,\ t \neq \tau_k,\ u(t_0 + 0) = u_0 \geq 0, \\ \Delta u(\tau_k) = \gamma_k(u(\tau_k)),\ \tau_k > t_0,\ k \in \mathbb{Z}. \end{cases} \tag{2.5}$$

is defined in the interval $[t_0, \infty)$.

(v) *The solution* $x(t) = x(t; t_0, \varphi_0)$, $y(t) = y(t; t_0, \varphi_0)$ *to the problem* (2.1), (2.2) *is such that* $x(t), y(t) \in PC[\mathbb{R}, \Omega] \cap PC^1[\mathbb{R}, \Omega]$.

(vi) *The function* $V \in V_0$ *is such that* $V(t_0 + 0, \varphi_0, \varphi_0) \leq u_0$ *and the inequalities*

$$D_-V(t, x(t), y(t)) \leq g(t, V(t, x(t), y(t))),\ \textit{for } t \neq \tau_k,$$
$$V(t+0, x(t) + I_k(x(t)), y(t) + I_k(y(t))) \leq \psi_k(V(t, x(t), y(t))),\ \textit{for } t = \tau_k,\ k \in \mathbb{Z},$$

are valid for each $t > t_0$ *and* $x, y \in \Omega_1$.

Then

$$V(t, x(t; t_0, \varphi_0), y(t; t_0, \varphi_0))) \leq r(t; t_0, u_0) \textit{ as } t \geq t_0. \tag{2.6}$$

Proof From Lemma 2.1 it follows that $J^+(t_0, \varphi_0) = [t_0, \infty)$. The maximal solution $r(t; t_0, u_0)$ to problem (2.5) is defined by

$$r(t; t_0, u_0) = \begin{cases} r(t; t_0, u_0^+), & t_0 < t \leq \tau_{k_1}, \\ r_1(t; \tau_{k_1}, u_1^+), & \tau_{k_1} < t \leq \tau_{k_2}, \\ \vdots \\ r_i(t; \tau_{k_i}, u_i^+), & t_{k_i} < t \leq t_{k_{i+1}}, \\ \vdots \end{cases}$$

where $r_i(t; \tau_{k_i}, u_i^+)$ is the maximal solution to the equation without impulses $\dot{u} = g(t, u)$ in the interval $(\tau_{k_i}, \tau_{k_{i+1}}]$, $\{\tau_{k_i}\} \subset \{\tau_k\}$ for which

$$u_i^+ = \psi_i(r_{k_{i-1}}(\tau_{k_i}; \tau_{k_{i-1}}, u_{i-1}^+)), \quad k \in \mathbb{Z}, \quad u_0^+ = u_0.$$

Let $t \in (t_0, \tau_{k_1}]$. Then from corresponding comparison lemma for the continuous case [8, Theorem 1.4.1] it follows that

$$V(t, x(t; t_0, \varphi_0), y(t; t_0, \varphi_0)) \leq r(t; t_0, u_0),$$

i.e., inequality (2.6) is valid for $t \in (t_0, \tau_{k_1}]$.

Suppose that (2.6) is satisfied for $t \in (\tau_{k_{i-1}}, \tau_{k_i}]$, $k_i > k_1$. Then from condition (vi) of Lemma 2.10 and the fact that the functions ψ_{k_i} are nondecreasing, we obtain

$$\begin{aligned} V\big(\tau_{k_i} + 0, &x(\tau_{k_i} + 0; t_0, \varphi_0), y(\tau_{k_i} + 0; t_0, \varphi_0)\big) \\ &\leq \psi_{k_i}\big(V(\tau_{k_i}, x(\tau_{k_i}; t_0, \varphi_0), y(\tau_{k_i}; t_0, \varphi_0))\big) \\ &\leq \psi_{k_i}(r(\tau_{k_i}; t_0, \varphi_0)) = \psi_{k_i}(r_{k_{i-1}}(\tau_{k_i}; \tau_{k_{i-1}}, u^+_{k_{i-1}})) = u^+_{k_i}. \end{aligned}$$

We apply again the comparison lemma for the continuous case in the interval $(\tau_{k_i}, \tau_{k_{i+1}}]$ and obtain

$$V\big(t, x(t; t_0, \varphi_0), y(t; t_0, \varphi_0)\big) \leq r_{k_i}(t; \tau_{k_i}, u^+_{k_i}) = r(t; t_0, u_0),$$

i. e., inequality (2.6) is valid for $t \in (\tau_{k_i}, \tau_{k_{i+1}}]$. The proof is completed by induction. ■

Lemma 2.11 *Let the following conditions hold.*

(i) *Conditions* (H1)–(H7).

(ii) *For any* $E^\alpha \in \operatorname{mod}(f, \varphi_0, I_k, \tau_k)$ *there exist functions* $W \in W_0$, $a, b \in Q$ *such that*

(a) $a(\|x(t)\|) \leq W(t, x(t)) \leq b(\|x(t)\|),\ t \in \mathbb{R},\ x(t) \in PC[\mathbb{R}, \mathbb{R}^n]$;

(b) *for any* $t > t_0, x \in PC[(t_0, \infty), \mathbb{R}^n]$ *for which* $W(s, x(s)) \leq W(t, x(t))$ $s \in [t_0, t]$ *the following inequalities hold*

$$D_- W(t, x(t)) \leq -cW(t, x(t)), t \neq \tau_k^\alpha, c = \text{const} > 0,$$
$$W(t + 0, x(t + I_k^\alpha(x(t))) \leq W(t, x(t)), t = \tau_k^\alpha, k \in \mathbb{Z}.$$

Then the zero solution to the system E^α *is uniformly asymptotically stable.*

Proof The proof of Lemma 2.11 is analogous to the proof of Lemma 2.10 . ■

3 Main Results

Theorem 3.1 *Let the following conditions hold.*

(C1) *Conditions* (H1)–(H7).

(C2) *The functions* $V \in V_0$ *and* $a, b \in Q$ *are such that*

$$a(\|x(t) - y(t)\|) \leq V(t, x(t), y(t)) \leq b(\|x(t) - y(t)\|), \tag{3.1}$$

for $x(t) \in PC[\mathbb{R}, \Omega]$, $y(t) \in PC[\mathbb{R}, \Omega]$.

(C3) *The inequalities*

$$D_-V(t, x(t), y(t)) \le -cV(t, x(t), y(t)), \quad t \ne \tau_k, \ c = \text{const} > 0, \tag{3.2}$$

$$V(t + 0, x(t) + I_k(x(t)), y(t) + I_k(x(t))) \le V(t, x(t), y(t)), \tag{3.3}$$

$t = \tau_k, \ x, y \in \Omega_1, \ k \in \mathbb{Z}$.

(C4) *There exists a solution* $x(t; t_0, \varphi_0)$ *of* (2.1),(2.2) *such that* $\|x(t; t_0, \varphi_0)\| < \nu_1$, *where* $t \ge t_0$, $\nu_1 < \nu$.

Then for the system (2.1),(2.2) *there exists a unique almost periodic solution* $\omega(t)$ *such that:*

(i) $\|\omega(t)\| \le \nu_1$*;*
(ii) $\operatorname{mod}(\omega(t), \tau_k) \subset \operatorname{mod}(f, \varphi_0, I_k, \tau_k)$*;*
(iii) $\omega(t)$ *is uniformly asymptotically stable.*

Proof Let $\{\alpha_i\}$ be any sequence of real numbers such that $\alpha_i \to \infty$ as $i \to \infty$ and $\{\alpha_i\}$ moving the system (2.1),(2.2) in the system E^α, $E^\alpha \in \operatorname{mod}(f, \varphi_0, I_k, \tau_k)$.

For any real number β, let $i_0 = i_0(\beta)$ be the smallest value of i such that $\alpha_{i_0} + \beta \ge t_0$. Since $\|x(t; t_0, \varphi_0)\| < \nu_1$ for all $t \ge t_0$, $x(t + \alpha_i; t_0, \varphi_0) \in B_{\nu_1}$ for $t \ge \beta, \ i \ge i_0$.

Let $I \subset (\beta, \infty)$ be compact. Then for any $\varepsilon > 0$, choose an integer $n_0(\varepsilon, \beta) \ge i_0(\beta)$ so large that for $l \ge i \ge n_0(\varepsilon, \beta)$ and $t \in (\beta, \infty)$ it follows that

$$b(2\nu_1)e^{-c(\beta+\alpha_i-t_0)} < \frac{a(\varepsilon)}{2}, \quad \|f(t + \alpha_l, x_t) - f(t + \alpha_i, x_t)\| < \frac{a(\varepsilon)c}{2L},$$

where $x \in PC[(t_0, \infty), B_{\nu_1}]$, $c = \text{const} > 0$.

Consider the function $V(\sigma, x(\sigma), x(\sigma + \alpha_l - \alpha_i))$.
For $\sigma > t_0$, $x(\sigma) \in \Omega_1$, $x(\sigma + \alpha_l - \alpha_i) \in \Omega_1$, from (2.4). (2.6), and (3.2) we obtain

$$\begin{aligned} D_-V(\sigma, x(\sigma), x(\sigma + \alpha_l - \alpha_i)) &\\ \le -cV(\sigma, x(\sigma), x(\sigma + \alpha_l - \alpha_i)) &\\ + L\|f(\sigma + \alpha_l - \alpha_i, x_{\sigma+\alpha_l-\alpha_i}) - f(\sigma, x_{\sigma+\alpha_l-\alpha_i})\| &\\ \le -cV(\sigma, x(\sigma), x(\sigma + \alpha_l - \alpha_i)) + \frac{a(\varepsilon)c}{2}. & \end{aligned} \tag{3.4}$$

On the other hand, from (3.3), (3.4), and Lemma 2.10

$$V(t+\alpha_i, x(t+\alpha_i), x(t+\alpha_l)) \le e^{-c(t+\alpha_i-t_0)}V(t_0, x(t_0), x(t_0+\alpha_l-\alpha_i)) + \frac{a(\varepsilon)c}{2} < a(\varepsilon).$$

Then from (3.1) we have $\|x(t + \alpha_i) - x(t + \alpha_l)\| < \varepsilon$, for $l \ge i \ge n_0(\varepsilon, \beta), \ t \in I$. Consequently there exists a function $\omega(t)$ such that $x(t+\alpha_i) - \omega(t) \to \infty$ for $i \to \infty$. Since β is arbitrary it follows that $\omega(t)$ is defined uniformly on $t \in I$.

Next we shall show that $\omega(t)$ is the solution to (2.3). Since $x(t; t_0, \varphi_0)$ is solution to (2.1), (2.2) we have

$$\begin{aligned} \|\dot{x}(t + \alpha_i) - \dot{x}(t + \alpha_l)\| \le{} & \|f(t + \alpha_i, x_{t+\alpha_i}) - f(t + \alpha_l, x_{t+\alpha_i})\| \\ & + \|f(t + \alpha_l, x_{t+\alpha_i}) - f(t + \alpha_l, x_{t+\alpha_l})\|, \end{aligned}$$

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for $t+\alpha_j \neq \tau_k,\ j=i,k$, and $k \in \mathbb{Z}$.

As $x(t+\alpha_i) \in B_{\nu_1}$ for large α_i, for each compact subset of $\mathbb{R}$ there exists an $n_1(\varepsilon) > 0$ such that if $l \geq i \geq n_1(\varepsilon)$, then

$$\|f(t+\alpha_i, x_{t+\alpha_i}) - f(t+\alpha_l, x_{t+\alpha_i})\| < \frac{\varepsilon}{2}.$$

Since $x(t+\alpha_j) \in B_{\nu_1},\ j = i,l$, it follows from Lemma 2.3 that there exists $n_2(\varepsilon) > 0$ such that if $l \geq i \geq n_2(\varepsilon)$, then

$$\|f(t+\alpha_l, x_{t+\alpha_i}) - f(t+\alpha_l, x_{t+\alpha_l})\| < \frac{\varepsilon}{2}.$$

For $l \geq i \geq n(\varepsilon),\ n(\varepsilon) = \max\{n_1(\varepsilon), n_2(\varepsilon)\}$ we obtain

$$\|\dot{x}(t+\alpha_i) - \dot{x}(t+\alpha_l)\| \leq \varepsilon,\ t+\alpha_i \neq \tau_k^\alpha,$$

which shows that $\lim_{i\to\infty} \dot{x}(t+\alpha_i)$ exists uniformly on all compact subsets of $\mathbb{R}$. Then $\lim_{i\to\infty} \dot{x}(t+\alpha_i) = \dot{\omega}(t)$, and

$$\text{(3.5)} \qquad \begin{cases} \dot{\omega}(t) = \lim\limits_{i\to\infty} [f(t+\alpha_i, x_{t+\alpha_i}) - f(t+\alpha_i, \omega(t)) + f(t+\alpha_i, \omega(t))] \\ \qquad\ = f^\alpha(t, \omega(t)),\ t \neq \tau_k^\alpha. \end{cases}$$

On the other hand, for $t+\alpha_i = \tau_k^\alpha$,

$$\text{(3.6)} \qquad \begin{cases} \omega(\tau_k^\alpha + 0) - \omega(\tau_k^\alpha - 0) = \lim\limits_{i\to\infty} (x(\tau_k^\alpha + \alpha_i + 0) - x(\tau_k^\alpha + \alpha_i - 0)) \\ \qquad = \lim\limits_{i\to\infty} I_k^\alpha(x(\tau_k^\alpha + \alpha_i)) = I_k^\alpha(\omega(\tau_k^\alpha)). \end{cases}$$

From (H6) we get that for sequence $\{\alpha_i\}$ there exists a subsequence $\{\alpha_n\},\ \alpha_n = \alpha_{i_n}$ such that the sequence $\{\varphi_0(t+\alpha_n)\}$ converges uniformly to the function φ_0^α. From (3.5) and (3.6) it follows that $\omega(t)$ is a solution to (2.3).

To show that $\omega(t)$ is an almost periodic function, let the sequence $\{\alpha_i\}$ move the system (2.1) to $\operatorname{mod}(f, \varphi_0, I_k, \tau_k)$. For any $\varepsilon > 0$ there exists $m_0(\varepsilon) > 0$ such that if $l \geq i \geq m_0(\varepsilon)$, then $e^{-c\alpha_i b(2\nu_1)} < a(\varepsilon)/4$ and $\|f(\kappa+\alpha_i, x_{\kappa+\alpha_i}) - f(\kappa+\alpha_l, x_{\kappa+\alpha_l})\| < a(\varepsilon)/4L$, where $x \in PC[((t_0,\infty), \mathbb{R}^n]$, $c = \text{const} > 0$.

For each fixed $t \in \mathbb{R}$ let τ_ε be an $a(\varepsilon)/4L$-translation number of f such that $t+\tau_\varepsilon \geq 0$. Consider the function $V(\tau_\varepsilon+\sigma, \omega(\sigma), \omega(\sigma+\alpha_l-\alpha_i))$, where $t \leq \sigma \leq t+\alpha_i$. Then

$$\text{(3.7)} \qquad \begin{aligned} D_-V(\tau_\varepsilon+\sigma,&\omega(\sigma), \omega(\sigma+\alpha_l-\alpha_i)) - cV(\tau_\varepsilon+\sigma, \omega(\sigma), \omega(\sigma+\alpha_l-\alpha_i)) \\ &+ L\|f^\alpha(\sigma, \omega(\sigma)) - f^\alpha(\tau_\varepsilon+\sigma, \omega(\sigma))\| \\ &+ L\|f^\alpha(\sigma+\alpha_l-\alpha_i, \omega(\sigma+\alpha_l-\alpha_i)) \\ &- f^\alpha(\tau_\varepsilon+\sigma, \omega(\sigma+\alpha_l-\alpha_i))\| \\ &\leq -cV(\tau_\varepsilon+\sigma, \omega(\sigma)) + \frac{3a(\varepsilon)}{4}. \end{aligned}$$

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On the other hand,

$$V(\tau_\varepsilon+\tau_k^\alpha,\omega(\tau_k^\alpha)+I_k^\alpha(\omega(\tau_k^\alpha)),\omega(\tau_k^\alpha+\alpha_l-\alpha_i)+I_k^\alpha(\omega(\tau_k^\alpha+\alpha_l-\alpha_i))) \le V(\tau_\varepsilon+\tau_k^\alpha,\omega(\tau_k^\alpha),\omega(\tau_k^\alpha+\alpha_l-\alpha_i)). \tag{3.8}$$

From (3.7), (3.8), and Lemma 2.10 it follows that

$$V(\tau_\varepsilon+t+\alpha_i,\omega(t+\alpha_i),\omega(t+\alpha_l)) \le e^{-c\alpha_i}V(\tau_\varepsilon+t,\omega(t),\omega(t+\alpha_i-\alpha_l))+\frac{3a(\varepsilon)}{4}<a(\varepsilon). \tag{3.9}$$

From (3.9) we have

$$\|\omega(t+\alpha_i)-\omega(t+\alpha_l)\|<\varepsilon,\ l\ge i\ge m_0(\varepsilon). \tag{3.10}$$

From the definition of the sequence $\{\alpha_i\}$ for $l\ge i\ge m_0(\varepsilon)$ it follows that

$$\rho(\tau_k+\alpha_i,\tau_k+\alpha_l)<\varepsilon.$$

Then from (3.10) and the last inequality we obtain that the sequence $\omega(t+\alpha_i)$ converges uniformly to the function $\omega(t)$.

Assertions (i) and (ii) of Theorem 3.1 follow immediately. We will prove assertion (iii). Let $\overline{\omega}(t)$ be an arbitrary solution to (iii), and set

$$u(t)=\overline{\omega}(t)-\omega(t),\quad g^\alpha(t,u(t))=f^\alpha(t,u(t)+\omega(t))-f^\alpha(t,\omega(t)),$$
$$\gamma_k^\alpha(u)=I_k^\alpha(u+\omega)-I_k^\alpha(u).$$

Now we consider the system

$$\begin{cases}\dot u=g^\alpha(t,u(t)),\ t\ne\tau_k^\alpha,\\ \Delta u(\tau_k^\alpha)=\gamma_k^\alpha(u(\tau_k^\alpha)),\ k\in\mathbb{Z},\\ u(t_0+0)=u_0,t_0\in\mathbb{R},\end{cases}$$

and let $W(t,u(t))=V(t,\omega(t),\omega(t)+u(t))$. Then from Lemma 2.10 it follows that the zero solution $u(t)=0$ of the last system is uniformly asymptotically stable for $t_0\ge 0$ and $\omega(t)$ is uniformly asymptotically stable. ■

Example 3.2 Consider the equation

$$\begin{cases}\dot x(t)=-a(t)x(t)+\displaystyle\int_{-\infty}^t c(t-s)x(s)ds+f(t),\ t>t_0,\ t\ne\tau_k,\\ \Delta x(\tau_k)=b_kx(\tau_k),\ k\in\mathbb{Z},\end{cases} \tag{3.11}$$

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where $a, c, f \in C[\mathbb{R}, \mathbb{R}]$ are almost periodic in the sense of Bohr. The function $f(t)$ is Lipchitzian in $\mathbb{R}$; $\{b_k\}$, $b_k \geq 0$, $k \in \mathbb{Z}$, is an almost periodic sequence of real numbers, and condition (H7) for sequence $\{\tau_k\} \in \mathbb{B}$ is met. Let

$$-a(t) + M \int_0^\infty |c(u)| du \leq -\lambda,$$

where $\lambda > 0$ and $M = \prod_{-\infty}^{\infty}(1 + b_k)$. For the function $V(t, x, y) = |x| + |y|$, (C2) and (C3) of Theorem 3.1 hold for the equations (3.11), and from (3.6) it follows that there exists a uniformly bounded solution for equation (3.11). Then the conditions of Theorem 3.1 hold and consequently there exists a unique almost periodic solution for equation (3.11).

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