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A Module-theoretic Characterization of Algebraic Hypersurfaces

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Abstract. In this note we prove the following surprising characterization: if $X \subset \mathbb{A}^n$ is an (embedded, non-empty, proper) algebraic variety defined over a field k of characteristic zero, then X is a hypersurface if and only if the module $T_{\mathcal{O}_{\mathbb{A}^n}/k}(X)$ of logarithmic vector fields of X is a reflexive $\mathcal{O}_{\mathbb{A}^n}$-module. As a consequence of this result, we derive that if $T_{\mathcal{O}_{\mathbb{A}^n}/k}(X)$ is a free $\mathcal{O}_{\mathbb{A}^n}$-module, which is shown to be equivalent to the freeness of the t-th exterior power of $T_{\mathcal{O}_{\mathbb{A}^n}/k}(X)$ for some (in fact, any) $t \leq n$, then necessarily X is a Saito free divisor.

1 Introduction

Freeness of modules (sheaves) of logarithmic vector fields is a celebrated property that was first investigated in the influential paper by Saito [13], originally in the complex analytic category. Several authors have enriched the theory and established beautiful connections to algebraic geometry, topology, singularity theory, combinatorics, and commutative algebra. Most of the techniques have focused mainly on detecting families of examples and their features, as well as criteria for the freeness of modules of logarithmic vector fields along *divisors* (hypersurfaces) in smooth ambient spaces, especially in the important case of hyperplane arrangements, which was first highlighted and exploited by Terao (and extended by Traves–Wakefield to the situation of subspace arrangements). Divisors possessing a free module of logarithmic vector fields have been dubbed *free divisors*. Some of the numerous references are [1–12, 15, 16, 19, 20, 22–26].

In the forthcoming paper [11], we investigate the problem in the more general, algebraic setting of logarithmic derivations of *not* necessarily principal ideals in (possibly singular) factorial domains essentially of finite type over a field of characteristic zero, and we readily derive freeness criteria for logarithmic derivation modules from the reflexiveness characterization given therein. Here, by using similar methods, our goal is to report the following surprising result, which had gone unnoticed and in fact turns out to be a new module-theoretic characterization of hypersurfaces: if $X \subset \mathbb{A}^n$ is an (embedded, non-empty, proper) algebraic variety defined over a field k of characteristic zero and $T_{\mathcal{O}_{\mathbb{A}^n}/k}(X)$ stands for its module of logarithmic vector fields, that is, polynomial vector fields tangent along the smooth part of X, then X is a hypersurface if and only if $T_{\mathcal{O}_{\mathbb{A}^n}/k}(X)$ is reflexive as a module over the polynomial ring $\mathcal{O}_{\mathbb{A}^n}$ (Theorem 3.1). In particular, we obtain that $T_{\mathcal{O}_{\mathbb{A}^n}/k}(X)$ is a free $\mathcal{O}_{\mathbb{A}^n}$-module, which, by

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means of a general argument due to Vasconcelos (Lemma 3.2), is shown to be equivalent to the freeness of the t-th exterior power $\bigwedge^t T_{\mathcal{O}_{\mathbb{A}^n}/k}(X)$ for some (in fact, any) $t \le n$, if and only if X is a free divisor (Corollary 3.3).

We point out that, for a hypersurface X, the reflexiveness of $T_{\mathcal{O}_{\mathbb{A}^n}/k}(X)$ was first noticed by Saito himself, in the local analytic setting (*cf.* [13, Corollary 1.7]). Therefore, our Theorem 3.1 states the converse of Saito's result, and, moreover, that it is true in the affine context as well.

Behind such a rigid characterization is the elementary fact that the defining ideal $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{A}^n}$ of X is *radical*. Note that $T_{\mathcal{O}_{\mathbb{A}^n}/k}(X)$ can be regarded algebraically as the module formed by the k-derivations of the polynomial ring $A = \mathcal{O}_{\mathbb{A}^n}$ preserving $\mathcal{I}_X$; analogously, under a natural scheme-theoretic viewpoint, we can consider the logarithmic derivation module $T_{A/k}(Y)$ of a given scheme $Y = \operatorname{Spec}(A/\mathcal{I})$, for a fixed ideal $\mathcal{I} \subset A$, as the A-module formed by the k-derivations of A preserving $\mathcal{I}$. However, the aforementioned main result may fail if $\mathcal{I}$ is t not radical; that is, in general, it is possible for $T_{A/k}(Y)$ to be reflexive (even free) with $\mathcal{I}$ non-principal; this is illustrated in Example 3.4.

2 Setup and Preparatory Results

In this part, we fix the setup that will be in force throughout the paper, and we establish some preparatory facts that will be used in our main results, proved in the next section.

We let k be a field of characteristic zero and $A = k[x_1, \dots, x_n]$ be a polynomial ring in indeterminates $x_1, \dots, x_n$ over k, so that A can be geometrically regarded as the ring $\mathcal{O}_{\mathbb{A}^n}$ of regular functions on the n-dimensional affine space $\mathbb{A}^n = \mathbb{A}^n_k$. We point out that the results we will prove in this note are also valid in the graded context; that is, we could replace the ambient $\mathbb{A}^n$ by a projective space, but in our proofs no grading on A will be needed and thus we consider solely the affine setting.

As it is well known, the module $\operatorname{Der}_k(A)$ of k-derivations of A is free, a basis being formed by the usual partial derivations $\partial/\partial x_1, \dots, \partial/\partial x_n$ (for the reader's convenience, we recall that if $S \subset R$ is an extension of (commutative, unital) rings, then an *S-derivation of R* is an additive map $\delta\colon R \to R$ vanishing on S and satisfying Leibniz's rule: $\delta(fg) = f\delta(g) + g\delta(f)$ for every $f, g \in R$; such maps constitute an R-module $\operatorname{Der}_S(R)$).

Given an algebraic variety $X \subset \mathbb{A}^n$, we consider its defining ideal

$$\mathcal{I}_X = \left\{ f \in A \mid f(p) = 0, \forall p \in X \right\}$$

(which is, obviously, a radical A-ideal) as well as its *logarithmic derivation module*

$$T_{A/k}(X) = \left\{ \delta \in \operatorname{Der}_k(A) \mid \delta(\mathcal{I}_X) \subseteq \mathcal{I}_X \right\}.$$

Naturally, if $\delta = \sum_{i=1}^n g_i \partial/\partial x_i$, for certain g_i's in A, then by $\delta(\mathcal{I}_X) \subseteq \mathcal{I}_X$, we mean that the polynomial function $\delta(f) = \sum_{i=1}^n g_i \partial f/\partial x_i$ vanishes on X whenever f has this property. Thus, geometrically, this module collects the polynomial vector fields tangent along X (the so-called *logarithmic vector fields* of the embedded variety $X \subset \mathbb{A}^n$), and it is quite often denoted by $\operatorname{Der}_k(-\log X)$. It is immediately seen to be A-torsionfree.

Definition 2.1 An algebraic hypersurface $X \subset \mathbb{A}^n$ is said to be a *Saito free divisor* (*free divisor*, for short) if the A-module $T_{A/k}(X)$ is free.

There are some well-known freeness criteria available in the literature. The very first of them is due to Saito [13] and essentially asserts the following: if $k = \mathbb{C}$ and $\delta_j = \sum_{i=1}^n g_{ij}\partial/\partial x_i$, $j = 1, \dots, n$, are n vector fields tangent to a divisor $X \subset \mathbb{A}^n$, say defined by the reduced equation $f = 0$, then X is free if and only if the determinant of the matrix (g_{ij}) has the form αf, for some $\alpha \in \mathbb{C} \setminus \{0\}$. Terao [22, Proposition 2.4] realized that a hyperplane arrangement in $\mathbb{A}^n$ is free if and only if its Jacobian ring is Cohen–Macaulay of Krull dimension equal to $n-2$. This characterization raised a beautiful connection to commutative ring theory, and it is also true for general algebraic hypersurfaces (a fact typically stated in the projective context; *cf., e.g.,* [10, Lemma 4.1]).

Originally, the results of Saito and Terao were established in the local setting; however, it is a classical fact that finitely generated projective modules over the polynomial ring A are free, so that we can harmlessly adopt a global concept as in Definition 2.1. Moreover, if X is a free divisor then, necessarily, $T_{A/k}(X) \simeq A^n$; this is an immediate consequence of Proposition 2.2. We recall that if a finitely generated module E over a Noetherian ring R (whose total fraction ring we denote by K) satisfies $K \otimes_R E \simeq K^r$ for some integer $r \geq 0$, then E is said to have a (generic, constant) rank, equal to r, which we denote by $\mathrm{rk}_R(E) = r$. It is clear that if $0 \to E \to F \to G \to 0$ is a short exact sequence of finitely generated R-modules with rank, then $\mathrm{rk}_R(E) + \mathrm{rk}_R(G) = \mathrm{rk}_R(F)$.

Proposition 2.2 *For any algebraic variety $X \subset \mathbb{A}^n$, we have $\mathrm{rk}_A(T_{A/k}(X)) = n$.*

Proof Notice that $\mathcal{I}_X \operatorname{Der}_k(A) \subset T_{A/k}(X)$. Hence, $\mathcal{I}_X$ annihilates the cokernel C_X of the inclusion $T_{A/k}(X) \subset \operatorname{Der}_k(A) \simeq A^n$, which yields $\mathrm{rk}_A(C_X) = 0$. Thus, from the exact sequence

$$0 \longrightarrow T_{A/k}(X) \longrightarrow \operatorname{Der}_k(A) \longrightarrow C_X \longrightarrow 0,$$

we obtain $\mathrm{rk}_A(T_{A/k}(X)) = \mathrm{rk}_A(A^n) = n$. ∎

Remark 2.3 Since the (free) A-module $\operatorname{Der}_k(A)$ has rank n, it would be natural to ask whether we might have $T_{A/k}(X) = \operatorname{Der}_k(A)$ in a non-trivial situation. However, it is a well-known classical fact that the polynomial ring A is *differentially simple*; that is, if an ideal $\mathcal{J} \subset A$ is *differential* in the sense of Seidenberg [18] (which by definition means that $\mathcal{J}$ is preserved by every k-derivation of A) then necessarily $\mathcal{J} = (0)$ or $\mathcal{J} = A$. As a consequence, if the given algebraic variety $X \subset \mathbb{A}^n$ is proper and non-empty, then necessarily

$$T_{A/k}(X) \neq \operatorname{Der}_k(A),$$

or equivalently, there exist $\ell \in \{1, \dots, n\}$, $f \in \mathcal{I}_X$ and $p \in X$ such that $(\partial f/\partial x_\ell)(p) \neq 0$. This holds, in more generality, for *regular* rings that are localizations of finitely generated algebras over a field of characteristic zero (*cf.* [17, Proposition 2.4]).

For the next result (which was first noticed by Saito in the local analytic setting; *cf.* [13, Corollary 1.7]), recall that a finitely generated module E over a Noetherian ring R is *reflexive* if the canonical map $E \to \operatorname{Hom}_R(\operatorname{Hom}_R(E, R), R)$ is an isomorphism. It is a standard fact that if R is a normal domain (*e.g.*, a regular ring) and E is a second-order syzygy module over R, then E must be reflexive.

Proposition 2.4 *If the algebraic variety $X \subset \mathbb{A}^n$ is a hypersurface, then the A-module $T_{A/k}(X)$ is reflexive.*

Proof Fix $f \in A$ such that $\mathcal{I}_X = (f)$ and consider the ideal

$$\mathcal{J}_X = (\partial f/\partial x_1, \dots, \partial f/\partial x_n, f).$$

Let $\psi_X\colon A^{n+1} \to \mathcal{J}_X$ be the natural epimorphism

$$(g_1, \dots, g_n, g) \longmapsto \sum_{i=1}^{n} g_i \frac{\partial f}{\partial x_i} - gf.$$

If $\pi\colon A^{n+1} \twoheadrightarrow A^n$ is the projection map that omits the last coordinate, then the restriction of π to $\ker(\psi_X)$ yields an isomorphism $\ker(\psi_X) \simeq T_{A/k}(X)$, which gives an exact sequence

$$0 \longrightarrow T_{A/k}(X) \longrightarrow A^{n+1} \longrightarrow \mathcal{J}_X \longrightarrow 0$$

so that $T_{A/k}(X)$ is a second-order syzygy A-module (of $A/\mathcal{J}_X$) and, hence, reflexive. ∎

We end this auxiliary section by studying logarithmic derivations of height 1 ideals of A. Given a k-scheme $Y = \operatorname{Spec}(A/\mathcal{I})$ for some fixed ideal $\mathcal{I} \subset A$, we let $T_{A/k}(Y)$ be the module formed by the k-derivations δ of A satisfying $\delta(\mathcal{I}) \subset \mathcal{I}$, which means $\delta(g) \in \mathcal{I}$, for every $g \in \mathcal{I}$. If $\mathcal{I} = A$, then $T_{A/k}(Y) = \operatorname{Der}_k(A)$. Now, suppose that $\mathcal{I}$ has the form $\mathcal{I} = f\mathcal{J}$, for some non-constant polynomial $f \in A$ and some ideal $\mathcal{J} \subset A$ containing an $A/(f)$-regular element, that is, a non-zero-divisor modulo (f). Set

$$Z = \operatorname{Spec}(A/(f)), \quad W = \operatorname{Spec}(A/\mathcal{J}).$$

Proposition 2.5 *In the situation above, we have*

$$T_{A/k}(Y) = T_{A/k}(Z) \cap T_{A/k}(W).$$

Proof First, pick $\delta \in T_{A/k}(Y)$. Then δ preserves the ideal $\mathcal{I} = f\mathcal{J}$. We claim that δ preserves (f) and $\mathcal{J}$ as well. By hypothesis, $\mathcal{J}$ contains a polynomial f_0 which is a non-zero-divisor modulo (f). Since $ff_0 \in \mathcal{I}$, we have

$$\delta(ff_0) \in \mathcal{I} \subset (f),$$

and hence, by Leibniz's rule, $f_0\delta(f) \in (f)$, which in turn implies $\delta(f) \in (f)$ as f_0 is $A/(f)$-regular. Now, given an arbitrary $g \in \mathcal{J}$, we get: $fg \in \mathcal{I} \Rightarrow f\delta(g) + g\delta(f) \in \mathcal{I} \Rightarrow f\delta(g) \in \mathcal{I}$. This yields $\delta(g) \in \mathcal{J}$ (since $f \neq 0$ and A is a domain), thus showing the claim. The converse is even easier. If $\delta \in T_{A/k}(Z) \cap T_{A/k}(W)$ then δ preserves both (f) and $\mathcal{J}$, and therefore, for any $g \in \mathcal{J}$, we have $f\delta(g), g\delta(f) \in \mathcal{I}$, whence $\delta(fg) \in \mathcal{I}$, as needed. ∎

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3 Main Result

We now prove the surprising characterization whose statement was anticipated in our introduction. Precisely, it guarantees that the converse of Proposition 2.4 is true, which will lead us, as a consequence, to freeness criteria for the module of logarithmic vector fields. In the proof, we denote the height of a proper ideal $\mathcal{J} \subset A = k[x_1, \ldots, x_n]$ by $\mathrm{ht}(\mathcal{J})$. Moreover, given A-modules $E \subset F$, we can consider the conductor ideal $E{:}F = \{g \in A \mid gF \subset E\}$.

Theorem 3.1 *Let $X \subset \mathbb{A}^n$ be an (embedded, non-empty, proper) algebraic variety. Then X is a hypersurface if and only if the A-module $T_{A/k}(X)$ is reflexive.*

Proof If X is a hypersurface, a proof of the (well-known) reflexiveness of $T_{A/k}(X)$ was given in Proposition 2.4. Now, assume that $T_{A/k}(X)$ is reflexive. We claim, first, that $\mathrm{ht}(\mathcal{I}_X) = 1$. Suppose that $\mathrm{ht}(\mathcal{I}_X) \geq 2$. Recall that, by the discussion given in Remark 2.3, we must have $T_{A/k}(X) \neq \mathrm{Der}_k(A)$, since X is non-empty and proper. Now, since $\mathcal{I}_X \,\mathrm{Der}_k(A) \subset T_{A/k}(X)$, we get

$$\mathcal{I}_X \subset T_{A/k}(X){:}\,\mathrm{Der}_k(A)$$

so that $\mathrm{ht}(T_{A/k}(X){:}\,\mathrm{Der}_k(A)) \geq 2$, which clearly means $(T_{A/k}(X))_P = (\mathrm{Der}_k(A))_P$, for every prime ideal $P \subset A$ with $\mathrm{ht}(P) \leq 1$. Thus, using a classical result due to Samuel (*cf.* [14, Proposition 1]), we obtain that the reflexive module $T_{A/k}(X)$ can be expressed, in the K-vector space $T_{A/k}(X) \otimes_A K \simeq T_{A/k}(X)_{(0)}$ (where $K = A_{(0)} = k(x_1, \ldots, x_n)$, the fraction field of A), as

$$T_{A/k}(X) = \bigcap_{\mathrm{ht}(P)=1} (T_{A/k}(X))_P = \bigcap_{\mathrm{ht}(P)=1} (\mathrm{Der}_k(A))_P = \mathrm{Der}_k(A),$$

a contradiction. Therefore, we necessarily have $\mathrm{ht}(\mathcal{I}_X) = 1$, the case $\mathcal{I}_X = (0)$ being ruled out by hypothesis. It follows that the defining ideal of X has the form $\mathcal{I}_X = f\mathcal{J}$ for some non-constant polynomial $f \in A$ and some non-zero ideal $\mathcal{J} \subset A$. We will show that $\mathcal{J} = A$.

Clearly, by absorbing the greatest common divisor of a generating set of $\mathcal{J}$ into f if necessary, we can assume that $\mathcal{J}$ contains an A-sequence of length at least 2. Hence, as every associated prime ideal of $A/(f)$ has height 1 in A (by the normality of A), there exists a polynomial in $\mathcal{J}$ that is $A/(f)$-regular. Setting $Z = \mathrm{Spec}(A/(f))$ and $W = \mathrm{Spec}(A/\mathcal{J})$, Proposition 2.5 yields $T_{A/k}(X) = T_{A/k}(Z) \cap T_{A/k}(W)$. In particular, $T_{A/k}(X) \subset T_{A/k}(Z)$. Further, it is easy to check that

$$\mathcal{J} \subset T_{A/k}(X){:}\, T_{A/k}(Z).$$

Localizing at any height 1 prime ideal $Q \subset A$ and using that $\mathcal{J}$ contains a length 2 regular sequence, we obtain $(T_{A/k}(X))_Q = (T_{A/k}(Z))_Q$. Applying Samuel's result again, as before, we get that the reflexive A-module $T_{A/k}(X)$ must equal $T_{A/k}(Z)$, and therefore $T_{A/k}(Z) = T_{A/k}(Z) \cap T_{A/k}(W)$, which means

$$T_{A/k}(Z) \subset T_{A/k}(W).$$

In particular, $f\,\mathrm{Der}_k(A) \subset T_{A/k}(W)$ and hence, for every $\delta \in \mathrm{Der}_k(A)$ and every $g \in \mathcal{J}$, we must have

$$f\delta(g) \in (f) \cap \mathcal{J}.$$

Since $\mathcal{I}_X$ is radical, it can be written as $\mathcal{I}_X = (f) \cap \mathcal{J}$ (this is elementary: $\mathcal{I}_X \subset (f) \cap \mathcal{J} \subset \sqrt{(f)} \cap \sqrt{\mathcal{J}} = \sqrt{f\mathcal{J}} = \mathcal{I}_X$). Therefore, $f\delta(g) \in f\mathcal{J}$, which yields $\delta(g) \in \mathcal{J}$ (as $f \neq 0$ and A is a domain) and then the non-zero ideal $\mathcal{J}$ is preserved by every derivation of A; by the differential simplicity of A (see Remark 2.3), this forces $\mathcal{J} = A$, as needed. ∎

For one of the equivalences established in Corollary 3.3, we will apply the following important, general observation from [21, Lemma 2.1], where it is attributed to W. Vasconcelos, and whose proof we supply for the reader's convenience.

Lemma 3.2 *Let R be a Noetherian local ring, let E be a finitely generated R-module, and let $t \geq 1$ be an integer. If the t-th exterior power $\bigwedge^t E$ is non-zero and R-free, then E is R-free.*

Proof We proceed by induction on t. We assume that $t \geq 2$, since the assertion is obvious for $t = 1$. There is a surjective R-homomorphism $E \otimes_R \bigwedge^{t-1} E \to \bigwedge^t E$. Since $\bigwedge^t E$ is non-zero and R-free, we have a surjective R-homomorphism $E \otimes_R \bigwedge^{t-1} E \to R$. Composing this map with the map $E \to E \otimes_R \bigwedge^{t-1} E$ (induced by tensoring with any element of $\bigwedge^{t-1} E$) yields an R-homomorphism $E \to R$. However, since R is local, one of these R-homomorphisms must be surjective. Therefore, we have a splitting $E = R \oplus E'$. Applying $\bigwedge^t$ to this decomposition, we get that $\bigwedge^{t-1} E'$ is a direct summand of $\bigwedge^t E$. Hence, the R-module $\bigwedge^{t-1} E'$ is free.

Now, we claim that $\bigwedge^{t-1} E' \neq 0$, or equivalently, $T \otimes_R \bigwedge^{t-1} E' \neq 0$, where T is the total quotient ring of R. Since $E = R \oplus E'$, we obtain

$$T \otimes_R \bigwedge^{t} E \simeq \Big(T \otimes_R \bigwedge^{t-1} E' \Big) \oplus \Big(T \otimes_R \bigwedge^{t} E' \Big).$$

However $T \otimes_R \bigwedge^t E \neq 0$ (since, by assumption, $\bigwedge^t E \neq 0$), so that $T \otimes_R \bigwedge^{t-1} E' \neq 0$ or $T \otimes_R \bigwedge^t E' \neq 0$, which indeed forces

$$T \otimes_R \bigwedge^{t-1} E' \neq 0$$

thus showing the claim. By the induction hypothesis, E' is R-free, and hence so is E, as needed. ∎

Corollary 3.3 *Let $X \subset \mathbb{A}^n$ be an (embedded, non-empty, proper) algebraic variety. The following assertions are equivalent:*

(i) *$T_{A/k}(X)$ is a free A-module;*
(ii) *$\bigwedge^t T_{A/k}(X)$ is a free A-module for some (in fact, any) $t \leq n$;*
(iii) *X is a free divisor.*

Proof The equivalence between statements (i) and (iii) follows readily from our Theorem 3.1. To prove that (i) and (ii) are equivalent, we first recall that $\mathrm{rk}_A(T_{A/k}(X)) = n$ (see Proposition 2.2), so that, for any given prime ideal $P \subset A$, we clearly have

$$\mathrm{rk}_{A_P}((T_{A/k}(X))_P) = n,$$

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and hence the A_P-module $(T_{A/k}(X))_P$ is minimally generated by no fewer than n elements. Consequently,

$$\bigwedge^t (T_{A/k}(X))_P \neq 0, \quad \forall t \leq n.$$

This puts us in a position to apply Lemma 3.2, which in our context yields that the A_P-module $(T_{A/k}(X))_P$ is free if (and only if) so is the A_P-module $\bigwedge^t (T_{A/k}(X))_P \simeq (\bigwedge^t T_{A/k}(X))_P$ for some (in fact, any) $t \leq n$. Since P was taken arbitrary and, as it is well known, finitely generated projective modules over polynomial rings are globally free, we are done. ∎

Example 3.4 Let $W \subset \mathbb{A}^3$ be the z-axis. Then, W is defined by the ideal $\mathcal{J} = (x, y) \subset A = k[x, y, z]$. By Theorem 3.1, the module $T_{A/k}(W)$ cannot be reflexive; explicitly,

$$T_{A/k}(W) = \mathcal{J}\frac{\partial}{\partial x} \oplus \mathcal{J}\frac{\partial}{\partial y} \oplus A\frac{\partial}{\partial z}.$$

However, setting

$$Y = \operatorname{Spec}(A/\mathcal{I}), \quad \mathcal{I} = (x^2yz, xy^2z) \subset A$$

we have that $T_{A/k}(Y)$ is free, even though $\mathcal{I}$ is non-principal. In fact, taking the free hyperplane arrangement $Z \subset \mathbb{A}^3$ defined by $f = xyz$, we have

$$\mathcal{I} = f\mathcal{J},$$

$$T_{A/k}(Y) = T_{A/k}(Z) = (x)\frac{\partial}{\partial x} \oplus (y)\frac{\partial}{\partial y} \oplus (z)\frac{\partial}{\partial z} \simeq A^3$$

as pointed out in [11, Remark 3.9]. This illustrates that our result may fail if $\mathcal{I}$ is not assumed to be radical (*i.e.,* the defining ideal of an algebraic variety).

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