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EXISTENCE THEOREM FOR THE INITIAL-BOUNDARY VALUE PROBLEM FOR A SINGULAR PARABOLIC PARTIAL DIFFERENTIAL EQUATION

BY
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1. **Introduction.** We consider the initial-boundary value problem for the parabolic partial differential equation

$$L_k U = \frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{k}{y}\frac{\partial U}{\partial y} = \frac{\partial U}{\partial t} \tag{1.1}$$

in the bounded domain D, contained in the upper half of the xy-plane, where a part of the x-axis lies on the boundary B (see Fig. 1).

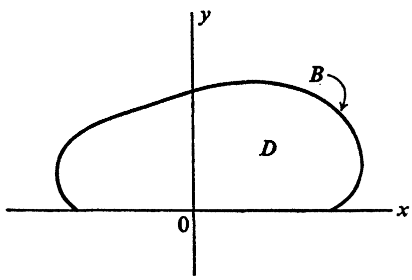

FIGURE 1.

By applying a finite difference-differential method, as opposed to the straight numerical approach used by Jamet [4], we will show that one of the conditions on the coefficients for this special case can be weakened from $|k|<1$ (the condition given by Jamet [4]) to $k<1$.

Rothe's [6], [7] finite difference method in the variable t will be used in the proof of this existence theorem. We get a difference-differential equation

$$L_k U_{n+1} = \frac{U_{n+1}-U_n}{h} \tag{1.2}$$

which is an approximation of (1.1). Here $h>0$ is a parameter which defines a sequence of mesh points along the t-axis.

The results of Schechter [9] will be applied to the Dirichlet problem obtained due to this method. This will show the existence and uniqueness of U_{n+1}.

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From our approach we will obtain a limit function as the mesh size parameter h converges towards zero. Finally we will show that this function is the solution to our initial-boundary value problem.

In order to prove convergence to this limit function as $h \to 0$ we must first derive *a priori* estimates of the solution of the difference equation (1.2). For this purpose we will make use of the following maximum principle.

2. **Maximum principle.** Let the differential equation

$$L_k v - \lambda^2 v = -g, \quad \lambda > 0 \tag{2.1}$$

be defined in the domain D with boundary B. Suppose v vanishes on B, and v, $g \in C^2(D) \cap C(\bar{D})$. Then

$$|v| \leq \frac{1}{\lambda^2} \max_{D} |g|. \tag{2.2}$$

Certainly at any point of D where v assumes its maximum value we have

$$L_k v \leq 0 \tag{2.3}$$

whence by (2.1)

$$\lambda^2 v \leq g. \tag{2.4}$$

Similarly, the inequality

$$\lambda^2 v \geq g \tag{2.5}$$

holds at any minimum of v inside D. Hence from (2.4) and (2.5)

$$\max_{D} |v| \leq \frac{1}{\lambda^2} \max_{D} |g| \tag{2.6}$$

from which (2.2) follows.

3. **Estimates.** We define the difference $U_n^{(j)}, j=0, 1, 2, \ldots$ of the sequences U_n by the following formulas:

$$U_n^{(1)} = U_{n+1} - U_n. \tag{3.1}$$

$$U_n^{(j+1)} = U_{n+1}^{(j)} - U_n^{(j)}, \quad j = 1, 2, 3, \ldots. \tag{3.2}$$

By a procedure similar to the method used by Garabedian [2], and applying the maximum principle of §2, we obtain the following important estimates:

$$U_n^{(1)} = O(h) \tag{3.3}$$

$$U_n^{(2)} = O(h^2). \tag{3.4}$$

4. **The existence theorem.** Let D be a bounded domain in the xy-plane bounded by a smooth curve B which contains part of the x-axis (see Fig. 1). Then for

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$T>0$, $k<1$, there exists in $Dx[0, T]$, a solution $U \in C^2(Dx[0, T]) \cap C(\bar{D}x[0, T])$ for the following initial-boundary value problem:

$$L_k U = \frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{k}{y}\frac{\partial U}{\partial y} = \frac{\partial U}{\partial t} \tag{4.1}$$

$$U(x, y, 0) = f(x, y) \quad \text{for } (x, y) \in D \tag{4.2}$$

$$U(x, y, t) = 0 \qquad \text{on } Bx[0, T]. \tag{4.3}$$

We assume that $f(x, y) \in C^4(\bar{D})$ and satisfies

$$L_k f = L_k(L_k f) = 0 \quad \text{on } B. \tag{4.4}$$

Proof. Let $h>0$ be a parameter which divides the interval $[0, T]$ in a sequence of mesh points

$$t = nh, \quad n = 0, 1, 2, \ldots. \tag{4.5}$$

The difference-differential equation

$$L_k U_{n+1} = \frac{U_{n+1}-U_n}{h} \tag{4.6}$$

represents an approximation of (4.1).

An iteration procedure to solve (4.6) is begun for each h by setting

$$U_0(x, y, 0) = f(x, y), \quad (x, y) \in D, \tag{4.7}$$

$$U_n(x, y, nh) = 0; \quad n = 0, 1, 2, \ldots \text{ on } Bx[0, T]. \tag{4.8}$$

The main problem is to solve (4.6) with the conditions (4.7) and (4.8) for $n=0, 1, 2, \ldots$. Then we show that the rule

$$U(x, y, t) = \lim_{h\to 0} U_n(x, y, nh), \quad nh \to t \tag{4.9}$$

defines the desired solution U of the initial-boundary value problem (4.1), (4.2), (4.3).

Let us write (4.6) in the following form

$$\frac{\partial^2}{\partial x^2} U_{n+1}+\frac{\partial^2}{\partial y^2} U_{n+1}+\frac{k}{y}\frac{\partial U_{n+1}}{\partial y}-\frac{1}{h} U_{n+1} = -\frac{1}{h} U_n. \tag{4.10}$$

By mathematical induction we suppose U_n is known throughout D, which is true for the case $n=0$. Since U_{n+1} vanishes along $Bx[0, T]$, it is determined by the Dirichlet problem (4.10), (4.8).

The existence and uniqueness of U_{n+1} is assured by Schechter [9] due to the minus sign on the coefficient of U_{n+1} and the fact that $k<1$. Hence $U_0, U_1, U_2, \ldots$ are all well defined. (It is interesting to note here that a numerical methods technique could be used to obtain the solution to (4.10) provided $|k|<1$. See Jamet and Parter [5].)

5. **Convergence.** We will restrict our attention to mesh sizes of the form

$$h_m = \frac{\epsilon}{2^m} \tag{5.1}$$

for fixed $\epsilon > 0$, $m \in \{0, 1, 2, \ldots\}$.

For each m we obtain a function U_m which represents an approximation to the value of the function U we seek. Using this notation we may express (3.3) and (3.4) as follows:

$$U_m(x, y, (n+1)h_m) - U_m(x, y, nh_m) = O(h_m) \tag{5.2}$$

$$U_m^{(1)}(x, y, (n+1)h_m) - U_m^{(1)}(x, y, nh_m) = O(h_m^2). \tag{5.3}$$

By the use of the standard Cantor diagonalization process and other classical techniques it can be shown that an increasing sequence $m_1, m_2, \ldots$, can be found with the corresponding sequence $h_{m_1}, h_{m_2}, \ldots$, such that

$$U(x, y, t) = \lim_{\nu \to \infty} U_{m_\nu}(x, y, t) \tag{5.4}$$

$$t = nh_{m_\nu} = \frac{n\epsilon}{2^{m_\nu}} \quad \text{for } 0 \le t \le T, \tag{5.5}$$

defines a solution to our initial-boundary value problem. By a similar procedure it can be shown that

$$\tilde{U}_t(x, y, t) = \lim_{\nu \to \infty} \frac{U_{m_\nu}^{(1)}(x, y, t)}{h_{m_\nu}} \tag{5.6}$$

is actually the partial derivative of U with respect to t.

Moreover, the estimates (3.3), (3.4) or (5.2), (5.3) enable us to define both functions U and $\tilde{U}_t$ by continuity for arbitrary $t \in [0, T]$.

The uniqueness of U satisfying the initial-boundary value problem (4.1), (4.2) and (4.3) is assured due to the maximum principle for parabolic partial differential equations. (See Rubinstein [8, p. 368].)

6. **Green's function.** It can be shown that the Green's function for $L_k U = 0$ in the domain D (see Fig. 1) does not exist for $k \ge 1$ (due to Huber [3]). Hence we have assumed $k < 1$.

In order to verify that (5.4) yields a solution U to the heat equation (4.1), we convert the difference-differential equation (4.6) into an integral equation using the Green's function [1].

The existence and explicit expression for the fundamental solution for the operator L_k for $y \ge 0$ was shown by Weinstein [11], [12]. Hence the Green's function $G(x, y; \xi, \eta)$ for the bounded domain D can be obtained.

Equation (4.6) and the boundary condition (4.8) may be combined, by using standard techniques, into the single equation (cf. [1], [10])

$$U_{n+1}+\iint_D \frac{U_{n+1}-U_n}{h}\, G(x, y; \xi, \eta)\, d\xi\, d\eta = 0. \tag{6.1}$$

The limit may be taken under the integral sign as $(U_{n+1}-U_n)/h$ is uniformly bounded due to the maximum principle (§2) and estimates (§3). Hence, by making use of (5.4) and (5.6), we obtain from (6.1) the integrodifferential equation

$$U+\iint_D U_t G(x, y; \xi, \eta)\, d\xi\, d\eta = 0 \tag{6.2}$$

for the function U. This result is equivalent to the heat equation (4.1) with the boundary condition (4.3.).

Finally, we observe that by construction (cf. (4.7), (4.9))

$$U(x, y, 0) = f(x, y). \tag{6.3}$$

This completes the proof of the existence theorem.

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