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TENSOR PRODUCTS OF OPERATOR SPACES II

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ABSTRACT. Together with Vern Paulsen we were able to show that the elementary theory of tensor norms of Banach spaces carries over to operator spaces. We suggested that the Grothendieck tensor norm program, which was of course enormously important in the development of Banach space theory, be carried out for operator spaces. Some of this has been done by the authors mentioned above, and by Effros and Ruan. We give alternative developments of some of this work, and otherwise continue the tensor norm program.

Perhaps the most significant new idea in the theory of operator spaces is the following: if $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces then $B(\mathcal{H}, \mathcal{K})$ is more than a mere Banach space, there is a natural way to assign norms to the spaces $M_n\big(B(\mathcal{H}, \mathcal{K})\big)$ of $n \times n$ matrices with entries in $B(\mathcal{H}, \mathcal{K})$. This seems so evident that it is easy to overlook its remarkable consequences. The appropriate objects and morphisms in this scenario are the operator spaces and completely bounded maps respectively, and their beautiful and powerful representation theory [1,21,9,23,27,7] leads to a phenomenal theory much of which is absent if we are merely interested in norms instead of matrix norms.

It was shown in [6] that the elementary theory of tensor norms of Banach spaces carries over to operator spaces. It was suggested there that the Grothendieck tensor norm program, which was of course enormously important in the development of Banach space theory, should be carried out for operator spaces. Some of this has now been done [6,14,15], and in this paper, and in a sequel presently in preparation, we continue the program a little further.

In Section 1 we set up some of the classical machinery; the reader with a limited tolerance for tensor products should proceed directly to Section 2! We also introduce several new operator space tensor norms. In Section 2 we review the theory of the Haagerup norm and the closely related topic of factorization through a Hilbert space. We note that there is a fair amount of duplication in Section 2 between this paper and [6,14,15]. We sometimes offer alternative proofs and developments complementing these papers. Each duplication is justified for one of the following two reasons: either we have found a simpler and shorter proof, or because there is an interesting alternative route to a result. In addition, some forms of some of these duplicated results were discovered independently by the author.

It will become clear that Hilbert operator spaces are central to the entire theory; using fairly elementary properties of these spaces we are able to give elementary alternative

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proofs of some of the most important results in the theory of completely bounded maps: for example the representation theorem for completely bounded multilinear maps. In the interests of a self-contained and elementary presentation we do not use Ruan's characterization of operator spaces [27]. We remark that the results from [6] quoted here do not essentially use [27], this is because the space $\mathrm{CB}(X, Y)$ of completely bounded maps from X to Y may be seen to be an operator space by using the elementary argument of [5, Proposition 2.1] instead of appealing to [27].

In Section 3 we discuss some applications to Banach space geometry, this is related to recent work of Gilles Pisier [26] and Vern Paulsen [22].

Generally we shall use the notation of [6], except that we write min and max for the operator space injective and projective tensor products respectively. We shall not require in general that tensor product spaces be completed. A completed tensor product is denoted by a horizontal bar over the $\otimes$ symbol. Thus $X \bar{\otimes}_\alpha Y$ denotes the completion of the algebraic tensor product with respect to the norm α. We reserve the symbol $\bar{\otimes}$ (no subscript) for the W^*-algebra tensor product [30].

We now proceed to define the three major operator space tensor norms; we shall be brief since these are treated at length in [6]. The operator space *injective tensor product*, also known as the *spatial tensor product* [21], is defined as follows. If X and Y are operator spaces contained in $B(\mathcal{H})$ and $B(\mathcal{K})$ respectively then $X \otimes Y$ may be identified with a subspace of $B(\mathcal{H} \otimes \mathcal{K})$; this assigns an operator space structure to $X \otimes Y$, which is independent of the particular Hilbert spaces on which X and Y are represented. We write this operator space as $X \otimes_{\min} Y$; there is another characterization of the matrix norms on $X \otimes_{\min} Y$ given in [6].

The operator space *projective tensor product* $X \otimes_{\max} Y$, which was independently and contemporaneously discovered in [6] and [12], may be defined by specifying $\mathrm{CB}\big(X \otimes_{\max} Y, B(\mathcal{H})\big)$ for an arbitrary Hilbert space $\mathcal{H}$. A map $\phi : X \otimes_{\max} Y \longrightarrow B(\mathcal{H})$ is *completely contractive* if and only if $\| [\phi(x_{ij} \otimes y_{kl})] \|_{nm} \leq \| [x_{ij}] \|_n \| [y_{kl}] \|_m$ whenever $[x_{ij}] \in M_n(X)$, $[y_{kl}] \in M_m(Y)$. Another useful description of the norms on $X \otimes_{\max} Y$ is given in [12]. The *Haagerup tensor product* $X \otimes_h Y$ of operator spaces X and Y may also be defined by specifying $\mathrm{CB}\big(X \otimes_h Y, B(\mathcal{H})\big)$ for an aribtrary Hilbert space $\mathcal{H}$. A map $\phi : X \otimes_h Y \longrightarrow B(\mathcal{H})$ is completely contractive if and only if $\| [\Sigma_k \phi(x_{ik} \otimes y_{kj})] \|_n \leq \| [x_{ij}] \|_n \| [y_{ij}] \|_n$ whenever $[x_{ij}] \in M_n(X)$, $[y_{ij}] \in M_n(Y)$.

We refer the reader to [5] for elementary operator space duality theory. A note on our use of the word *classical*—this means that the analogous Banach space result is well known and essentially identical. We thank V. I. Paulsen for many helpful discussions.

1. **Tensor norms and the Grothendieck calculus.** Hitherto the study of operator space tensor norms has been restricted to three tensor norms: min, max and the Haagerup norm h. We shall introduce some new norms in this section. We remark that the fact that the Haagerup tensor product of two operator spaces is again an operator space is proven in [23] and later in [27]; a recent elementary proof of this is essentially contained in [29].

Because much of what follows is so similar to the classical Banach space version [17,16] we shall be terse. The following simple fact is useful in proving some of the assertions which follow:

LEMMA 1.1. *If X and Y are operator spaces then the adjoint map* $\mathrm{CB}(X, Y) \longrightarrow \mathrm{CB}(Y^*, X^*)$ *is a complete isometry.*

In what follows we shall only consider uniform operator space tensor norms (uniform norms for brevity) in the sense of [6]; for such a norm α we sometimes write $\alpha_n(U; X \otimes Y)$ for the norm of an element $U \in M_n(X \otimes_\alpha Y)$. We shall say that a uniform norm is *completely injective* if subspaces X_1 of X_2 and Y_1 of Y_2 determine a complete isometry of $X_1 \otimes_\alpha Y_1$ into $X_2 \otimes_\alpha Y_2$. A uniform norm is *completely projective* if complete quotients X_1 of X_2 and Y_1 of Y_2 determine a complete quotient map of (the completions of) $X_2 \otimes_\alpha Y_2$ onto $X_1 \otimes_\alpha Y_1$. The spatial norm min is completely injective, the projective operator space norm max is completely projective, and the Haagerup norm h is both [23,6,15]. It may be seen in several ways that min is not projective and that max is not injective.

A uniform norm α is *associative* if $(X \otimes_\alpha Y) \otimes_\alpha Z = X \otimes_\alpha (Y \otimes_\alpha Z)$ completely isometrically for all operator spaces X, Y and Z. We shall rely heavily in the sequel on the fact that min, max and h are all associative.

Now there is a natural linear map $t: X \otimes Y \longrightarrow Y \otimes X$; if α is a uniform norm then we define the *transposed* and *symmetrized* norms α^t and α^s, of α by $(\alpha^t)_n(U; X \otimes Y) = \alpha_n\big(t_n(U); Y \otimes X\big)$ and

$$(\alpha^s)_n(U; X \otimes Y) = \max\{\,\alpha_n(U; X \otimes Y), (\alpha^t)_n(U; X \otimes Y)\}\,.$$

Both these norms are uniform.

We define the dual norm α^* of α on $X \otimes Y$ via the natural inclusion of $X \otimes Y$ in $(X^* \otimes_\alpha Y^*)^*$. This is slightly different from the notation of [6]. We define the associate norm α' of α by

$$(\alpha')_n(U; X \otimes Y) = \inf\{\,(\alpha^*)_n(U; E \otimes F)\}\,,$$

where the infimum is taken over finite dimensional subspaces E and F of X and Y respectively, with $U \in M_n(E \otimes F)$. If α is uniform then α^*, and consequently α', is uniform. Of course if $\alpha \leq \beta$ then $\beta' \leq \alpha'$. If α is a completely injective uniform tensor norm then α' is completely projective, and if α is a completely projective uniform tensor norm then α^* is completely injective and $\alpha^* = \alpha'$. Thus $\max^* = \max' = \min$ [6].

We say that α is *tensorial* if $\alpha_n(U; X \otimes Y) = \inf\{\,\alpha_n(U; E \otimes F)\}$, where the infimum is taken over finite dimensional subspaces E and F of X and Y respectively, with $U \in M_n(E \otimes F)$. It is easy to see that min, max and the Haagerup norm are tensorial. Of course if α and β are both tensorial, and if $\alpha = \beta$ on finite dimensional operator spaces, then $\alpha = \beta$. From this it follows immediately that if α is tensorial then $\alpha'' = \alpha$. Thus $\min' = \max$.

If α is a uniform norm then we *define* $/\,\alpha\,\backslash$ to be the greatest completely injective uniform norm dominated by α. This is again a uniform operator space tensor norm. We note that this also has a concrete representation $(/\alpha\backslash)_n(U; X \otimes Y) = \sup\{\alpha_{npq}((S \otimes T)_n(U); B(\mathcal{H}) \otimes B(\mathcal{K}))\}$, where the supremum ranges over all Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, all positive integers p and q, and all complete contractions S and T from X and Y to $M_p(B(\mathcal{H}))$ and $M_q(B(\mathcal{K}))$ respectively. One may replace the $B(\mathcal{H})$ spaces here with any class of injective operator spaces with the property that every operator space is contained in some element of the class. In the Banach space theory one may use finite dimensional ℓ^∞ spaces instead of the $B(\mathcal{H})$ spaces; however it does not seem possible here to use finite dimensional spaces. We define $\backslash\alpha/$ to be $(/\alpha'\backslash)'$; it is not difficult to see that this is the least completely projective norm dominating α. We shall give an interesting explicit formula for $\backslash\alpha/$ in the sequel. It is easy to see that $(\backslash\alpha/)^* = (\backslash\alpha/)' = /\alpha'\backslash$. There are obvious notions of left and right injectivity and projectivity, and corresponding operations $\backslash$ and $/$ as in the classical theory. One can generate many new tensor norms if one applies the operations described above repeatedly to the norms min, max and the Haagerup norm. There is no reason to suppose that this will not give a large number of inequivalent norms. Indeed it is clear that we have $\min \leq \backslash\min/ \leq h \leq /\max\backslash \leq \max$, and since the Haagerup norm is not symmetric, and since min is not projective and max is not injective, these five norms are inequivalent. This is one major difference between the classical Grothendieck theory and our situation. By the Grothendieck inequality we know that $/\gamma\backslash$ is dominated by a constant multiple of $\backslash\lambda/$; here γ and λ are the Banach space projective and injective norms respectively.

2. **Hilbert operator spaces and factorization.** It has been well known for some time that there is a natural covariant functor and a natural contravariant functor from the category of Hilbert spaces to the category of operator spaces (see Proposition 2.2). The covariant functor takes a Hilbert space $\mathcal{H}$ to the operator space $\mathcal{H}_c = B(\mathbf{C}, \mathcal{H})$, the contravariant functor takes a Hilbert space $\mathcal{H}$ to the operator space $\mathcal{H}_r = B(\mathcal{H}, \mathbf{C})$. Henceforth we shall term them *Hilbert operator spaces:* the first we also call *Hilbert column space,* the second *Hilbert row space.* We warn the reader that the second of these notations disagrees with the notation in [15]. *Generalized* Hilbert spaces were studied in [6] (see particularly 4.3); it was observed that the space of $n \times n$ matrices over a Hilbert operator space has a natural M_n valued inner product. The finite dimensional Hilbert operator spaces played a significant role throughout [6]; we write C_n for the n-dimensional Hilbert column space, and R_n for the n-dimensional Hilbert row space.

The study of Hilbert operator spaces often reduces easily to the finite dimensional case, due to the following property [6].

PROPOSITION 2.1. *A closed subspace or quotient of Hilbert column (row) space is again a Hilbert column (row) space.*

There are many simple ways to see the result above, the reader will no doubt have his own method.

The following result is well known.

PROPOSITION 2.2. *For Hilbert spaces* $\mathcal{H}$ *and* $\mathcal{K}$ *we have* $\mathrm{CB}(\mathcal{H}_c, \mathcal{K}_c) = B(\mathcal{H}, \mathcal{K})$ *completely isometrically, and in particular* $(\mathcal{H}_c)^* = \mathcal{H}_r$. *Also* $\mathrm{CB}(\mathcal{H}_r, \mathcal{K}_r) = B(\mathcal{K}, \mathcal{H})$ *completely isometrically, and in particular* $(\mathcal{H}_r)^* = \mathcal{H}_c$.

PROOF. We prove the first identity; then $(\mathcal{H}_c)^* = \mathcal{H}_r$ is immediate, and $(\mathcal{H}_r)^* = \mathcal{H}_c$ follows by reflexivity. The remaining identity will then follow from the complete isometry

$$\mathrm{CB}(\mathcal{H}_r, \mathcal{K}_r) \longrightarrow \mathrm{CB}\big((\mathcal{K}_r)^*, (\mathcal{H}_r)^*\big) = \mathrm{CB}(\mathcal{K}_c, \mathcal{H}_c) = B(\mathcal{K}, \mathcal{H})$$

given by the Banach space adjoint map $T \longrightarrow T^*$ (see Lemma 1.1).

For simplicity we first prove the first identity in finite dimensions. It is obvious that $\mathrm{CB}(C_n, C_m) = M_{m,n}$ isometrically, and one can prove the complete isometry using the generalized Stinespring theorem as in the proof of $M_n^* = R_n \otimes_h C_n$ in [6]. However, there is a shorter direct proof of the complete isometry, which appears to use no machinery, which we shall now give. It is clear that $B_l(C_p, C_q) = M_{q,p}$ completely isometrically; where B_l is the left matrix norm structure defined in [6]. We have

$$\begin{aligned} M_p\big(\mathrm{CB}(C_n, C_m)\big) &= \mathrm{CB}\big(C_n, M_p(M_{m,1})\big) = \mathrm{CB}(C_n, M_{pm,p}) \\ &= \mathrm{CB}\big(C_n, B_l(C_p, C_{pm})\big) = \mathrm{CB}(C_n \otimes_h C_p, C_{pm}) \\ &= \mathrm{CB}(C_{pn}, C_{pm}) = M_{pm,pn} = M_p(M_{m,n}) \end{aligned}$$

isometrically. Here we used the tautological Proposition 3.7 in [6], and the trivial fact that $C_n \otimes_h C_p = C_n(C_p) = C_{np}$ (see also Proposition 2.3). It is necessary to sort through these correspondences to ensure that we do in fact obtain the canonical identification, but this is elementary (although tedious).

In the general case there is a simple reduction to finite dimensions. If $[T_{ij}] \in M_k\big(\mathrm{CB}(\mathcal{H}_c, \mathcal{K}_c)\big)$, then by definition $\|[T_{ij}]\| \leq \|[T_{ij}(\zeta_{kl})]\| + \varepsilon$, for some $\zeta_{kl} \in \mathcal{H}_c$, $\|[\zeta_{kl}]\| \leq 1$. By Proposition 2.1 we have $\operatorname{span}\{\zeta_{kl}\} = C_n$, and $\operatorname{span}\{T_{ij}(\zeta_{kl})\} = C_m$, for some integers n and m. Let $[T_{ij}\urcorner]$ be the restriction in $M_k\big(\mathrm{CB}(C_n, C_m)\big)$. Then using the finite dimensional case proved above we see that

$$\|[T_{ij}]\| \leq \|[T_{ij}(\zeta_{kl})]\| + \varepsilon \leq \|[T_{ij}\urcorner]\| + \varepsilon \leq \|[T_{ij}]\|_{Mk\left(B(\mathcal{H}, \mathcal{K})\right)} + \varepsilon.$$

The inequality in the other direction is similar.

Many of the following assertions are well known, and all appear in [15]. We offer some alternative proofs.

PROPOSITION 2.3. *If* X *is an operator space and if* $\mathcal{H}$ *and* $\mathcal{K}$ *are Hilbert spaces then we have (completely isometrically):*

(i) $\mathcal{H}_c \otimes_h X = \mathcal{H}_c \otimes_{\min} X$ *and* $X \otimes_h \mathcal{H}_r = \mathcal{H}_r \otimes_{\min} X$;

(ii) $X \otimes_h \mathcal{H}_c = \mathcal{H}_c \otimes_{\max} X$ *and* $\mathcal{H}_r \otimes_h X = \mathcal{H}_r \otimes_{\max} X$;

(iii) $\mathcal{H}_r \bar{\otimes}_{\min} \mathcal{K}_c = K(\mathcal{H}, \mathcal{K})$ *and* $\mathcal{H}_r \bar{\otimes}_{\max} \mathcal{K}_c = B(\mathcal{K}, \mathcal{H})*$,

(iv) $\mathcal{H}_c \bar{\otimes}_{\min} \mathcal{K}_c = \mathcal{H}_c \bar{\otimes}_h \mathcal{K}_c = \mathcal{H}_c \bar{\otimes}_{\max} \mathcal{K}_c = (\mathcal{H} \otimes \mathcal{K})_c$, *and* $\mathcal{H}_r \bar{\otimes}_{\min} \mathcal{K}_r = \mathcal{H}_r \bar{\otimes}_h \mathcal{K}_r = \mathcal{H}_r \bar{\otimes}_{\max} \mathcal{K}_r = (\mathcal{H} \otimes \mathcal{K})_r$

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(v) $(\mathcal{H}_r \otimes_h X \otimes_h \mathcal{K}_c)^* = (\mathcal{H}_r \otimes_{\max} X \otimes_{\max} \mathcal{K}_c)^* = \mathrm{CB}\big(X, B(\mathcal{K}, \mathcal{H})\big)$.

PROOF. By the complete injectivity of the norms we can assume the Hilbert operator spaces are finite dimensional in (i), and then this is a simpler version of Proposition 3.5 in [6].

We prove the first identity in (ii), the second is similar. We first observe that there is a simple reduction to finite dimensions: if $U \in M_n(X \otimes \mathcal{H}_c)$ then let C_p be the span in $\mathcal{H}_c$ of the elements of $\mathcal{H}_c$ occurring in the representation of U. The natural projection of $\mathcal{H}_c$ onto C_p is a complete contraction. Since max and h are uniform tensor norms it follows that the norm of U in $M_n(X \otimes_{\max} \mathcal{H}_c)$ is exactly equal to the norm of U in $M_n(X \otimes_{\max} C_p)$, and the same statement holds for $\otimes_h$. Thus we may assume without loss of generality that $\mathcal{H}_c = C_p$.

Now it follows easily from Proposition 3.7 in [6] (see proof of [6, Theorem 3.8]) that $(R_n \otimes_h X \otimes_h C_m)^* = \mathrm{CB}(X, M_{n,m})$ isometrically. Hence

$$M_n\big((X \otimes_h C_p)^*\big) = \big(R_n \otimes_h (X \otimes_h C_p) \otimes_h C_n\big)^* = \big(R_n \otimes_h X \otimes_h (C_p \otimes_h C_n)\big)^*$$
$$= (R_n \otimes_h X \otimes_h C_{pn})^* = \mathrm{CB}(X, M_{n,pn})$$

isometrically. In the last string of equalities we used the trivial $C_p \otimes_h C_n = C_{pn}$ which follows from (i). Also using the elementary properties of max [6, Section 5] the same string of equalities holds with h replaced by max. Here $C_p \otimes_{\max} C_n = C_{pn}$ follows by taking duals. Hence $(X \otimes_{\max} C_p)^* = (X \otimes_h C_p)^*$ completely isometrically, and so $X \otimes_{\max} C_p = X \otimes_h C_p$ completely isometrically.

Identity (iii) is straightforward duality theory [6, Section 5]. The first two equalities in each identity in (iv) follow from (i) and (ii), the last may be seen using a reduction to finite dimensions, or the agument in [15]. The first equality in (v) follows from (ii), the second is merely the identification

$$(\mathcal{H}_r \otimes_{\max} X \otimes_{\max} \mathcal{K}_c)^* = \mathrm{CB}(X \otimes_{\max} \mathcal{K}_c, \mathcal{H}_c)$$
$$= \mathrm{CB}\big(X, \mathrm{CB}(\mathcal{K}_c, \mathcal{H}_c)\big) = \mathrm{CB}\big(X, B(\mathcal{K}, \mathcal{H})\big).$$

The correspondences above are central to what follows. Henceforth we shall often use these results without comment.

REMARK 1. We note that the relation $B_1(\mathcal{K}_c, \mathcal{H}_c) = B(\mathcal{K}, \mathcal{H})$ could have been used in Propositions 2.2 and 2.3 to avoid a reduction to finite dimensions. To see that $B_1(\mathcal{K}_c, \mathcal{H}_c) = B(\mathcal{K}, \mathcal{H})$ completely isometrically it is only necessary to recall that $M_n(\mathcal{H}_c) = M_n\big(B(\mathbf{C}, \mathcal{H})\big)(= B(\mathbf{C}^{(n)}, \mathcal{H}^{(n)})$, so that if $[\zeta_{ij}] \in M_n(\mathcal{H}_c)$ then $\|[\zeta_{ij}]\|_n = \sup\{\Sigma_i\|\Sigma_j\lambda_j\zeta_{ij}\|^2 : \lambda_i \in \mathbf{C}, \Sigma_j|\lambda_j|^2 \leq 1\}$. From this it is easy to see that if $[T_{ij}] \in M_n\big(B(\mathcal{K}, \mathcal{H})\big)$ then $\|[\Sigma_k T_{ik}(\zeta_{kj})]\|_n \leq \|[T_{ij}]\|\,\|[\zeta_{ij}]\|_n$, from which the desired relation is evident.

REMARK 2. In fact even if X is just a matrix normed space it is true that $(\mathcal{H}_r \otimes_h X \otimes_h \mathcal{K}_c)^* = \mathrm{CB}\big(X, B(\mathcal{K}, \mathcal{H})\big)$ completely isometrically. The isometric case of this may be seen by inspection, using the correspondence of [10]; and the complete isometry follows using two applications of the isometric case:

$$\begin{aligned} M_n\big((\mathcal{H}_r \otimes_h X \otimes_h \mathcal{K}_c)^*\big) &= \mathrm{CB}(\mathcal{H}_r \otimes_h X \otimes_h \mathcal{K}_c, M_n) \\ &= (R_n \otimes_h \mathcal{H}_r \otimes_h X \otimes_h \mathcal{K}_c \otimes_h C_n)^* = (\mathcal{H}_r^{(n)} \otimes_h X \otimes_h \mathcal{K}_c^{(n)})^* \\ &= \mathrm{CB}\big(X, B(\mathcal{K}^{(n)}, \mathcal{H}^{(n)})\big) = \mathrm{CB}\Big(X, M_n\big(B(\mathcal{K}, \mathcal{H})\big)\Big) \\ &= M_n\Big(\mathrm{CB}\big(X, B(\mathcal{K}, \mathcal{H})\big)\Big). \end{aligned}$$

We now sketch how the representation theorem for completely bounded multilinear maps may be proven using Hilbert operator space theory. We need only do the bilinear case; the general case follows by induction.

THEOREM 2.4 [9,23]. *Let X and Y be operator spaces, and let $\mathcal{H}$ be a Hilbert space. A bilinear map φ from $X \times Y$ into $B(\mathcal{H})$ is completely contractive if and only if there is a Hilbert space $\mathcal{K}$ and completely contractive maps Φ and Ψ from X and Y into $B(\mathcal{K}, \mathcal{H})$ and $B(\mathcal{H}, \mathcal{K})$ respectively, such that $\varphi(x, y) = \Phi(x)\Psi(y)$ for $x \in X$ and $y \in Y$.*

PROOF. We need only prove the necessity. Now

$$\mathrm{CB}\big(X \otimes_h Y, B(\mathcal{H})\big) = \big(\mathcal{H}_r \otimes_h (X \otimes_h Y) \otimes_h \mathcal{H}_c\big)^* = \Big((\mathcal{H}_r \otimes_{\max} X) \otimes_h (Y \otimes_{\max} \mathcal{H}_c)\Big)^*,$$

isometrically, using the asociativity of h and Proposition 2.3 (ii). The first equality here follows from the isometric case of Remark 2 after Proposition 2.3. Now the projective tensor product of two operator spaces is again an operator space [6] (we note that we are not using Ruan's characterization of operator spaces [27] here—see the comments in the introduction). Using Haagerup's representation theorem [11,32], and the injectivity of the Haagerup norm [6] (later we give another simple proof of injectivity of h which does not depend on the present result), it follows that there is a Hilbert space $\mathcal{K}$ such that φ may be expressed as $\varphi = S(\cdot)T(\cdot)$, where T is a completely bounded map from $Y \otimes_{\max} \mathcal{H}_c$ into $B(\mathbf{C}, \mathcal{K}) = \mathcal{K}_c$, and where S is a completely bounded map from $\mathcal{H}_r \otimes_{\max} X$ into $B(\mathcal{K}, \mathbf{C}) = \mathcal{K}_r$. Now

$$\mathrm{CB}(Y \otimes_{\max} \mathcal{H}_c, \mathcal{K}_c) = \mathrm{CB}\big(Y, \mathrm{CB}(\mathcal{H}_c, \mathcal{K}_c)\big) = \mathrm{CB}\big(Y, B(\mathcal{H}, \mathcal{K})\big);$$

similarly

$$\mathrm{CB}(\mathcal{H}_r \otimes_{\max} X, \mathcal{K}_r) = \mathrm{CB}\big(X, \mathrm{CB}(\mathcal{H}_r, \mathcal{K}_r)\big) = \mathrm{CB}\big(X, B(\mathcal{K}, \mathcal{H})\big).$$

An untangling of these indentifications yields the desired result.

If X and Y are dual operator spaces, with preduals $X*$ and $Y*$ respectively, then we may realize X and Y as weak*-closed subspaces of some $B(\mathcal{H})$ and $B(\mathcal{K})$ respectively [5,14]. Given normal functionals φ and ψ, on $B(\mathcal{H})$ and $B(\mathcal{K})$ respectively, there are right and left slice maps R_φ and L_ψ from $B(\mathcal{H}) \otimes B(\mathcal{K})$ onto $B(\mathcal{K})$ and $B(\mathcal{H})$ respectively, given on elementary tensors by $R_\varphi(S\otimes T)=\varphi(S)T$ and $L_\psi(S\otimes T)=S\psi(T)$ respectively (see [31]).We may define the Fubini product $F\big(X,Y,B(\mathcal{H})\bar{\otimes}B(\mathcal{K})\big)$ to be the set

$$\{u\in B(\mathcal{H})\bar{\otimes}B(\mathcal{K}):R_\varphi(u)\in Y \text{ and } L_\psi(u)\in X \text{ for } \varphi \text{ and } \psi \text{ as above}\}.$$

THEOREM 2.5 [14]. *If X and Y are as above, then $(X*\otimes_{\max}Y*)^*=F\big(X,Y,B(\mathcal{H})\bar{\otimes}B(\mathcal{K})\big)$ completely isometrically. In particular if $\mathcal{M}$ and $\mathcal{N}$ are W^*-algebras then we have $\mathcal{M}\bar{\otimes}\mathcal{N}=(\mathcal{M}*\otimes_{\max}\mathcal{N}*)^*$ completely isometrically.*

PROOF. We first show that $B(\mathcal{H})\bar{\otimes}B(\mathcal{K})=\big(T(\mathcal{H})\otimes_{\max}T(\mathcal{K})\big)^*$. Now

$$\begin{aligned}B(\mathcal{H})\bar{\otimes}B(\mathcal{K})&=B(\mathcal{H}\otimes\mathcal{K})=\mathrm{CB}\Big((\mathcal{H}\otimes\mathcal{K})_c,\big((\mathcal{H}\otimes\mathcal{K})_r\big)^*\Big)\\&=\mathrm{CB}\big(\mathcal{H}_c\bar{\otimes}_{\max}\mathcal{K}_c,(\mathcal{H}_r\otimes_{\max}\mathcal{K}_r)^*\big)=(\mathcal{H}_c\bar{\otimes}_{\max}\mathcal{K}_c\bar{\otimes}_{\max}\mathcal{H}_r\bar{\otimes}_{\max}\mathcal{K}_r)^*\\&=(\mathcal{H}_r\bar{\otimes}_{\max}\mathcal{H}_c\bar{\otimes}_{\max}\mathcal{K}_r\bar{\otimes}_{\max}\mathcal{K}_c)^*=\big(T(\mathcal{H})\otimes_{\max}T(\mathcal{K})\big)^*\end{aligned}$$

completely isometrically. Unravelling these correspondences does indeed give the required identification.

Now if X and Y are dual operator spaces then the quotient maps $T(\mathcal{H})\longrightarrow X*$ and $T(\mathcal{K})\longrightarrow Y*$ induce a complete quotient map $T(\mathcal{H})\otimes_{\max}T(\mathcal{K})\longrightarrow X*\otimes_{\max}Y*$, which determines a complete isometry $(X*\otimes_{\max}Y*)^*\longrightarrow\big(T(\mathcal{H})\otimes_{\max}T(\mathcal{K})\big)^*=B(\mathcal{H}\otimes\mathcal{K})$. Now it follows from the first part and duality considerations as in [14] that $(X*\otimes_{\max}Y*)^*=F\big(X,Y,B(\mathcal{H})\bar{\otimes}B(\mathcal{K})\big)$. The last statement then follows from Tomiyama's Slice Map Theorem [31].

Of course in general $X\bar{\otimes}Y\neq F\big(X,Y,B(\mathcal{H})\bar{\otimes}B(\mathcal{K})\big)$ for arbitrary X, Y as above; and the class of spaces for which this is an equality has been extensively studied (see [20] for example). Roger Smith has recently shown that $F\big(X,Y,B(\mathcal{H})\bar{\otimes}_hB(\mathcal{K})\big)=X\bar{\otimes}_hY$ for arbitrary closed spaces X and Y [29].

The next proposition may also be proven easily using properties of the Haagerup norm, but we feel it is more natural to give a direct proof.

PROPOSITION 2.6. *The following natural identifications are complete isometries:*
(i) $(C_n\otimes_{\max}X)^*=R_n\otimes_{\min}X^*$, *and* $(R_n\otimes_{\max}X)^*=C_n\otimes_{\min}X^*$,
(ii) $(C_n\otimes_{\min}X)^*=R_n\otimes_{\max}X^*$, *and* $(R_n\otimes_{\min}X)^*=C_n\otimes_{\max}X^*$.

PROOF. The first assertion follows immediately from the duality of min and max [6]. Now $(C_n\otimes_{\min}X)^{**}=\big(C_n(X)\big)^{**}\subset M_n(X)^{**}$, and $C_n\otimes_{\min}X^{**}=C_n(X^{**})\subset M_n(X^{**})$. However Theorem 2.5 of [5] implies that $M_n(X^{**})=M_n(X)^{**}$ completely isometrically. Thus $(C_n\otimes_{\min}X)^{**}=C_n\otimes_{\min}X^{**}$; also from (i) we have that $(R_n\otimes_{\max}X^*)^*=C_n\otimes_{\min}X^{**}$, from which it follows that $R_n\otimes_{\max}X^*=(C_n\otimes_{\min}X)^*$. The last assertion is similar.

We remark here that there is another way to describe the operator space structure on $\mathrm{CB}(X,Y)$ for operator spaces X and Y. An element of $M_n\big(\mathrm{CB}(X,Y)\big)$ may be regarded as acting as "left matrix multiplication" on a column of n elements of X, or acting as " right matrix multiplication" on a row of n elements of X.

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PROPOSITION 2.7. *Via the identifications described above we have* $M_n\big(\mathrm{CB}(X,Y)\big) = \mathrm{CB}(C_n \otimes_{\max} X, C_n \otimes_{\min} Y) = \mathrm{CB}(R_n \otimes_{\max} X, R_n \otimes_{\min} Y)$ *completely isometrically.*

PROOF. Using Corollary 5.2 and Proposition 5.4 of [6] we have

$$\begin{aligned}\mathrm{CB}(C_n \otimes_{\max} X, C_n \otimes_{\min} Y) &= \mathrm{CB}\big(X, \mathrm{CB}(C_n, C_n \otimes_{\min} Y)\big)\\ &= \mathrm{CB}(X, R_n \otimes_{\min} C_n \otimes_{\min} Y) = \mathrm{CB}\big(X, M_n(Y)\big) = M_n\big(\mathrm{CB}(X,Y)\big).\end{aligned}$$

Unravelling these canonical correspondences an element of $M_n\big(\mathrm{CB}(X,Y)\big)$ corresponds to a "left matrix multiplication" in $\mathrm{CB}(C_n \otimes_{\max} X, C_n \otimes_{\min} Y)$. The row case is similar.

We now turn to factorizations through Hilbert operator space. The idea for this was communicated to the author by Professor G. Pisier, and is mentioned briefly in [6] (see particularly Theorem 3.11). Let X and Y be operator spaces, and define $\Gamma_c(X,Y)$ to be the space of linear maps $X \longrightarrow Y$ which factorize through a Hilbert column space. That is $T \in \Gamma_c(X,Y)$ if and only if $T = RS$ for completely bounded maps $S\colon X \longrightarrow \mathcal{H}_c$ and $R\colon \mathcal{H}_c \longrightarrow Y$; for such T define $\gamma(T) = \inf\{\,\|R\|_{cb}\|S\|_{cb}\}$, where the infimum is taken over all factorizations of the type described above. This is identical to the Banach space case [25], and as with all such classical factorizations [17,16] it is easy to see that $\Gamma_c(X,Y)$ is a normed space (see also [15]). In any case we do not need this fact, but we remark that it also follows easily from a simple reduction to finite dimensional Hilbert operator spaces, and the fact that operator spaces are 2-summing in the sense of [6].

Identify $M_n\big(\Gamma_c(X,Y)\big)$ with $\Gamma_c(C_n \otimes_{\max} X, C_n \otimes_{\min} Y)$, where we identify a matrix $[T_{ij}] \in M_n\big(\Gamma_c(X,Y)\big)$ with "left matrix multiplication" on columns of elements of X. We leave it as an easy exercise for the reader to check that each such "left matrix multiplication" $[T_{ij}]$ does indeed factor through some $\mathcal{H}_c$. We shall write $\Gamma_c^f(X,Y)$ for the subspace of $\Gamma_c(X,Y)$ consisting of finite rank operators, and in this case we may restrict to factorizations through finite dimensional Hilbert column spaces (see proof of Lemma 2.8). The difference between our factorization and that of [15] is explained by Proposition 2.7.

Similarly we define $\Gamma_r(X,Y)$ and $\Gamma_r^f(X,Y)$ as above, but corresponding to factorization through Hilbert row space. We have immediately:

LEMMA 2.8. *If Y is completely isometric to a subspace of W and if X is completely isometric to a quotient of V, then $\Gamma_c(X,\bar{Y})$ is completely isometric to a subspace of $\Gamma_c(V,\bar{W})$. Also $\Gamma_c^f(X,Y)$ is completely isometric to a subspace of $\Gamma_c^f(V,W)$. The same statements hold for Γ_r.*

PROOF. This is classical, and probably due to Grothendieck. We establish the result for Γ_c, the result for Γ_r is similar.

Let $i\colon \bar{Y} \longrightarrow \bar{W}$ and $q\colon V \longrightarrow X$ be respectively the inclusion and quotient maps. We show that $\Phi\colon \Gamma_c(X,\bar{Y}) \longrightarrow \Gamma_c(X,\bar{Y})\colon T \longrightarrow iTq$ is a complete isometry. Since the natural maps $C_p \otimes_{\max} V \longrightarrow C_p \otimes_{\max} X$ and $C_p \otimes_{\min} \bar{Y} \longrightarrow C_p \otimes_{\min} \bar{W}$ are a complete quotient map and a complete isometry respectively, the proof for the complete isometry will be identical to that for the isometry. Certainly Φ is a contraction. Suppose $\Phi(T) = iTq = RS$ for complete contractions $S\colon V \longrightarrow \mathcal{H}_c$ and $R\colon \mathcal{H}_c \longrightarrow \bar{W}$. By factoring R through $\mathcal{H}_c/\ker R$

we may assume that R is one-to-one. Of course in the finite rank case this implies that the Hilbert column space is finite dimensional. Now by restricting to the closure of the range of S we may assume that S has dense range. Now $R(\mathcal{H}_c) = R\big(\overline{S(V)}\big) \subset \overline{iTq(V)} \subset \bar{Y}$, so R maps into $\bar{Y}$. Also $q(u) = 0$ implies that $R\big(S(u)\big) = iTq(u) = 0$, so that $S(u) = 0$ since R is one-to-one. Define $S^{\sim}\colon X \longrightarrow \mathcal{H}_c$ by $S^{\sim}\big(q(u)\big) = S(u)$, then $S^{\sim}$ is a well defined complete contraction, and $T = RS^{\sim}$. Thus Φ is an isometry.

The following is a slight generalization of Theorem 3.11 in [6], we give a proof for a reason which will be apparent shortly:

THEOREM 2.9. *If X and Y are operator spaces then $X \otimes_h Y$ is completely isometrically contained in $\Gamma^f_c(Y^*, X)$, and also in $\Gamma^f_r(X^*, Y)$, via the usual identifications of an elementary tensor with a linear map.*

PROOF. We prove the first assertion; the second is similar. We shall only prove the isometry; the complete isometry is almost identical. We note that the complete isometry also follows immediately from the isometry and Proposition 3.5 in [6], since $M_n(X \otimes_h Y) = C_n(X) \otimes_h R_n(Y)$ is contained in $\Gamma_c\big(R_n(Y)^*, C_n(X)\big) = \Gamma_c(C_n \otimes_{\max} Y^*, C_n \otimes_{\min} X)$ (using Proposition 2.6).

Since the operators concerned are finite rank it suffices to factorize (as in the lemma) through C_m, m varying. By the elementary theory of tensor products [6] we have $\mathrm{CB}(Y^*, C_m) = C_m \otimes_{\min} Y^{**} = C_m(Y^{**})$, and $\mathrm{CB}(C_m, X) = C_m^* \otimes_{\min} X = R_m(X)$. It is easy to see from this that the norm of an element of $X \otimes Y$ considered as an element of $\Gamma_c(Y^*, X)$ is precisely its norm as an element of $X \otimes_h Y^{**}$. The result now follows immediately from the injectivity of the Haagerup norm; however we finish the argument another way so as to be able to *deduce* the injectivity of the Haagerup norm next.

Suppose $u \in X \otimes Y$ corresponds to $u^{\sim} : Y^* \longrightarrow X$. The identities above show that $\gamma(u^{\sim}) \leq \|u\|_h$. Now suppose $u^{\sim} = RS$ is a factorization through some C_m. As above we may regard R as an element $[x_1, \ldots, x_m]^t$ of $C_m(X)$ and S as an element $[G_1, \ldots, G_m]$ of $R_m(Y^{**})$. We may suppose without loss of generality, as in the lemma, that R is one-to-one and S is surjective. It follows that $x_1, \ldots, x_m$ are linearly independent, and $G_1, \ldots, G_m$ are linearly independent. Now Lemma 3.3 in [6] shows that S actually lies in $R_m(Y)$. This completes the proof.

We are now able to give the shortest, and perhaps conceptually simplest, proof of the complete injectivity of the Haagerup norm (see also [23,6]). This is essentially the proof of the injectivity of Grothendieck's H norm [17]; we are indebted to Professor Gilles Pisier for suggesting this method. We remark that this approach was hinted at in [6, 3.13].

COROLLARY 2.10. *If $E_1 \subset F_1$ and $E_2 \subset F_2$ completely isometrically, then $E_1 \otimes_h E_2 \subset F_1 \otimes_h F_2$ completely isometrically.*

PROOF. By Theorem 2.9 we see that $E_1 \otimes_h E_2 \subset \Gamma^f_c(E_2^*, E_1)$ and $F_1 \otimes_h F_2 \subset \Gamma^f_c(F_2^*, F_1)$. However $\Gamma^f_c(E_2^*, E_1) \subset \Gamma^f_c(F_2^*, F_1)$ by Lemma 2.8, which completes the proof.

We recall that it was shown in [6], using an idea of E. G. Effros, that the extension theorem for completely bounded linear maps follows immediately from the injectivity of the Haagerup norm, the Hahn-Banach theorem, and the relation $\mathrm{CB}(X, M_n) = B(R_n \otimes_h X \otimes_h C_n, \mathbf{C})$ (see Proposition 2.3).

The following important observation of Effros and Ruan [15] may be viewed as a restatement of the representation theorem for completely bounded bilinear maps; we sketch a proof.

THEOREM 2.11. *If X and Y are operator spaces the $(X \otimes_h Y)^* = \Gamma_c(Y, X^*) = \Gamma_r(X, Y^*)$ completely isometrically.*

PROOF. We prove only the first identity; the second is similar. If $\psi \in M_n\big((X \otimes_h Y)^*\big) = \mathrm{CB}(X \otimes_h Y, M_n)$ then ψ has a Christensen-Sinclair representation (see Theorem 2.4) $\psi(x, y) = \Phi(x)\Psi(y)$, where Φ and Ψ are complete contractions from X and Y into $B(\mathcal{H}, l_n{}^2)$ and $B(l_n{}^2, \mathcal{H})$ respectively. Now

$$\mathrm{CB}\big(Y, B(l_n{}^2, \mathcal{H})\big) = \mathrm{CB}\big(Y, \mathrm{CB}(C_n, \mathcal{H}_c)\big) = \mathrm{CB}(C_n \otimes_{\max} Y, \mathcal{H}_c),$$

and

$$\begin{aligned} \mathrm{CB}\big(X, B(\mathcal{H}, l_n{}^2)\big) &= \mathrm{CB}\big(X, \mathrm{CB}(\mathcal{H}_c, C_n)\big) \\ &= \mathrm{CB}\big(\mathcal{H}_c, \mathrm{CB}(X, C_n)\big) \\ &= \mathrm{CB}(\mathcal{H}_c, C_n \otimes_{\min} X^*), \end{aligned}$$

which establishes the inequality in one direction. The other direction is simpler: just follow these correspondences in the reverse order.

As noted in [15] one can also deduce from 2.11 and 2.8 that $\Gamma_c(X, Y)$ and $\Gamma_r(X, Y)$ are operator spaces.

At this point one may without further calculation obtain the following corollary, which also follows from a result in [15] (cf. Proposition 2.13 (ii) below).

COROLLARY 2.12. *We have $h = h^* = h'$.*

PROOF. Consider the following diagram of linear maps:

$$\begin{array}{ccc} X \otimes_h Y & \longrightarrow & (X^* \otimes_h Y^*)^* \\ \downarrow & & \downarrow \\ \Gamma_c(Y^*, X) & \longrightarrow & \Gamma_c(Y^*, X^{**}) \end{array}$$

where all the maps are the canonical ones. Using 2.8, 2.9 and 2.11 it follows that all the maps are complete isometries except perhaps the upper one. However this diagram is obviously commutative. Thus $h = h^*$, and since h is tensorial we obtain the last relation.

We also remark that the result can be seen using a simple reduction to the finite dimensional case, and in this case the result is not difficult to see. This result is not true in the Banach space case: although $H = H'$ on certain particular nontrivial tensor products of Banach spaces (see Corollary 3.3), they are not equivalent in general [17].

PROPOSITION 2.13. *If X and Y are operator spaces then*

(i) $\Gamma_c(X, \bar{Y})$ *is (completely isometrically) contained in* $\Gamma_c(X^{**}, Y^{**})$ *via the map* $T \longrightarrow T^{**}$*;*

(ii) $X^* \otimes_h Y^*$ *is (completely isometrically) contained in* $(X \otimes_h Y)^*$*;*

(iii) $\Gamma_c(X, Y^*)$ *is (completely isometrically) a direct summand of* $\Gamma_c(X^{**}, Y^{***})$*;*

(iv) $(X \otimes_h Y)^*$ *is (completely isometrically) a direct summand of* $(X^{**} \otimes_h Y^{**})^*$.

PROOF. The isometric containment in (i) is obvious and classical. Now using Proposition 2.6 we have

$$\begin{aligned} M_n\big(\Gamma_c(X, \bar{Y})\big) &= \Gamma_c(C_n \otimes_{\max} X, C_n \otimes_{\min} \bar{Y}) \\ &\subset \Gamma_c\big((C_n \otimes_{\max} X)^{**}, (C_n \otimes_{\min} \bar{Y})^{**}\big) \\ &= \Gamma_c(C_n \otimes_{\max} X^{**}, C_n \otimes_{\min} Y^{**}) = M_n\big(\Gamma_c(X^{**}, Y^{**})\big). \end{aligned}$$

The second statement follows easily using (i) and essentially the same method as 2.12 (see [6], Theorem 5.6). Statement (iii) follows immediately from (i), using the canonical projection $Y^{***} \longrightarrow Y^*$. Statement (iv) is immediate from (iii) and Theorem 2.11.

Part (ii) was proved in [15] by different methods. We remark that the statement of Proposition 2.13 is true if one replaces Γ_c by Γ_r throughout; or indeed if one replaces (throughout) Γ_c with CB and $\otimes_h$ with $\otimes_{\max}$ (in (ii) with $\otimes_{\min}$ and $\otimes_{\max}$ respectively).

COROLLARY 2.14 [5]. *If X is an operator space then* $M_n(X^{**}) = M_n(X)^{**}$ *completely isometrically.*

PROOF. The complete isometry follows here from the isometry. Since $M_n(X) = C_n \otimes_h X \otimes_h R_n$ it follows that $M_n(X)^* = R_n \otimes_h X^* \otimes_h C_n$, and so

$$M_n(X)^{**} = C_n \otimes_h X^{**} \otimes_h R_n = M_n(X^{**}).$$

We remark that we used the self-duality of h in the form of 2.13 (ii) here, our proof of which used Proposition 2.6, which in turn uses the result we are attempting to prove. However, this is essentially the only use we made of 2.6, and in fact 2.13 (ii) may be proven without 2.6 [15]. Indeed the complete isometry in 2.13 (ii) follows fairly easily from the isometry and Proposition 3.5 of [6]; and the isometry uses the isometric form of 2.13 (i).

Finally we note that using the self-duality of the Haagerup norm, together with the elementary identifications, it follows that $B(\mathcal{F}, \mathcal{G}) \otimes_h B(\mathcal{H}, \mathcal{K})$ is contained in $\mathrm{CB}\big(K(\mathcal{K}, \mathcal{F}), B(\mathcal{H}, \mathcal{G})\big)$ completely isometrically, for Hilbert spaces $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{K}$. For we have

$$\begin{aligned} B(\mathcal{F}, \mathcal{G}) \otimes_h B(\mathcal{H}, \mathcal{K}) &= (\mathcal{G}_r \otimes_h \mathcal{F}_c)^* \otimes_h (\mathcal{K}_r \otimes_h \mathcal{H}_c)^* \\ &\subset \big((\mathcal{G}_r \otimes_h \mathcal{F}_c) \otimes_h (\mathcal{K}_r \otimes_h \mathcal{H}_c)\big)^* \\ &= \big(\mathcal{G}_r \otimes_h K(\mathcal{K}, \mathcal{F}) \otimes_h \mathcal{H}_c\big)^* \\ &= \mathrm{CB}\big(K(\mathcal{K}, \mathcal{F}), B(\mathcal{H}, \mathcal{G})\big). \end{aligned}$$

This fact is known in some form to many people. Roger Smith has recently found a proof of this which requires less machinery [29].

3. **Applications to Banach space geometry.** We recall [13,6] that if X is a normed space then there is a least operator space structure $\mathrm{MIN}(X)$ on X, obtained by identifying $\mathrm{MIN}(X)$ completely isometrically with the usual subspace of the space of continuous functions on the unit ball of the dual space of X. We defined $\mathrm{MAX}(X)$ purely abstractly [6] to be the largest operator space structure on X. In [5] it was shown that $\mathrm{MIN}(X)^* = \mathrm{MAX}(X^*)$ and $\mathrm{MAX}(X)^* = \mathrm{MIN}(X^*)$, which gives an alternative description of $\mathrm{MAX}(X)$ as a subspace of $\mathrm{MIN}(X^*)^*$.

We write λ and γ for the Banach space injective and projective norms respectively. If X and Y are normed spaces then we shall write π^* for the norm on $X \otimes Y$ induced by identifying $(X \otimes_{\pi^*} Y)^* = \Pi_2(Y, X^*)$, where $\Pi_2(\cdot, \cdot)$ is the space of 2-summing operators [25]. Then π^{*t} is the norm induced by identifying $(X \otimes_{\pi^{*t}} Y)^* = \Pi_2(X, Y^*)$. We write H' for the associate norm of Grothendieck's H norm [17]. This norm H, more recently denoted by γ_2 [25] may be defined by $X \otimes_H Y \subset \Gamma_2(Y^*, X)$, or by $X \otimes_H Y \subset \Gamma_2(X^*, Y)$, where Γ_2 is the Banach space version of Γ_c and Γ_r. It is known that $H^* = H'$ [8], and that $(X \otimes_{H'} Y)^* = \Gamma_2(Y, X^*) = \Gamma_2(Y, X^*)$ [17,16], it is interesting to note that either of these will imply the other from our considerations.

THEOREM 3.1. *Let X and Y be normed spaces. We have*

(i) $\mathrm{MIN}(X) \otimes_{\min} \mathrm{MIN}(Y) = \mathrm{MIN}(X \otimes_\lambda Y)$

(ii) $\mathrm{MAX}(X) \otimes_{\max} \mathrm{MAX}(Y) = \mathrm{MAX}(X \otimes_\gamma Y)$

completely isometrically, and also

(iii) $\mathrm{MIN}(X) \otimes_h \mathrm{MIN}(Y) = X \otimes_H Y$,

(iv) $\mathrm{MAX}(X) \otimes_h \mathrm{MIN}(Y) = X \otimes_{\pi*} Y$,

(v) $\mathrm{MIN}(X) \otimes_h \mathrm{MAX}(Y) = X \otimes_{\pi*^t} Y$,

(vi) $\mathrm{MAX}(X) \otimes_h \mathrm{MAX}(Y) = X \otimes_{H'} Y$

isometrically.

PROOF. The first two identities are clear from the elementary theory of operator space tensor norms [6]. The third identity appears in [6] and [4]. Identity (vi) follows from Corollary 2.12, part (iii) above, and the duality of MIN and MAX. Identity (iv) is an immediate corollary of Theorem 2.11 and [15, Theorem 5.7]. The fifth identity follows in the same way from the row space version of [15, Theorem 5.7], which in turn follows immediately with our definition of Hilbert row space.

REMARK 1. We showed in [6] that Pisier's gamma norms are special cases of the Haagerup norm. In fact results (iii)–(vi) are closely related to results in Section 3 of [26]. Let B denote the set of all positive sesquilinear forms on X of norm less than or equal to one. In [6] it was shown how to associate an operator space structure on X with a given set of sesquilinear forms on X. Doing this for the set B then we obtain an operator space $\mathcal{L}_B(X)$. We also defined a related space $\mathcal{R}_B(X)$ in [6]. If $\mathrm{MAX}(X) = \mathcal{L}_B(X) = \mathcal{R}_B(X)$ completely isometrically, then (iii)–(vi) would also follow immediately from results in Section 3 of [26] and Section 4 of [6]. Unfortunately in general the matrix norms of $\mathrm{MAX}(X)$ strictly dominate those of $\mathcal{L}_B(X)$ and $\mathcal{R}_B(X)$. Consequently it is a little surprising that we get the same Banach space tensor norms in (iv)–(vi) that Pisier obtained (and not larger ones). It is also of interest to note that Pisier realized these four norms in terms of factorizations through Hilbert space, via judicious selections of certain sets $I(X)$ and $I(Y)$ of finite subsets of X and Y respectively. One may obtain realizations of all the gamma norms in terms of Hilbert space factorization from Theorem 2.9 and the fact that each γ-norm is a particular case of the Haagerup norm [6, Section 4].

REMARK 2. In the light of (i) and (ii) it is natural to ask if $\mathrm{MIN}(X) \otimes_h \mathrm{MIN}(Y) = \mathrm{MIN}(X \otimes_H Y)$ and $\mathrm{MAX}(X) \otimes_h \mathrm{MAX}(Y) = \mathrm{MAX}(X \otimes_{H'} Y)$ (completely isometrically). However, this probably happens vary rarely if at all in nontrivial cases. Nonetheless it may be difficult to think of examples where this fails. The nonassociativity of H (we do not have a convenient reference for this; it was communicated to the author by Professor G. Pisier) shows that counterexamples do exist. We now display another class of low dimensional examples where this fails. Let $\mathcal{A}$ be an n-dimensional unital noncommutative operator algebra (so $n \geq 3$), for instance the upper triangular 2×2 matrices. We claim that $\mathrm{MIN}(\mathcal{A}^*) \otimes_h \mathrm{MIN}(\mathcal{A}^*) \neq \mathrm{MIN}(\mathcal{A}^* \otimes_H \mathcal{A}^*)$ and (equivalently) $\mathrm{MAX}(\mathcal{A}) \otimes_h \mathrm{MAX}(\mathcal{A}) \neq \mathrm{MAX}(\mathcal{A} \otimes_{H'} \mathcal{A})$. For suppose the latter was an equality. Since $\mathcal{A}$ is an operator algebra, the canonical multiplication map $\mathcal{A} \otimes_h \mathcal{A} \longrightarrow \mathcal{A}$ is a contraction, and since $H' \geq h$, the map $\mathcal{A} \otimes_{H'} \mathcal{A} \longrightarrow \mathcal{A}$ is a contraction. This last sentence may be rephrased as the fact the every operator algebra is an H'-algebra, which we believe is due originally to Ph. Charpentier. Thus the multiplication map is a complete contraction $\mathrm{MAX}(\mathcal{A}) \otimes_h \mathrm{MAX}(\mathcal{A}) = \mathrm{MAX}(\mathcal{A} \otimes_{H'} \mathcal{A}) \longrightarrow \mathrm{MAX}(\mathcal{A})$. From the main theorem in [7] we see that $\mathrm{MAX}(\mathcal{A})$ is an operator algebra, and consequently commutative by [4, Theorem 3]. This is the desired contradiction.

REMARK 3. In the Résumé [17] Grothendieck stated that $H \leq \rho H'$, for some (least) universal constant ρ whose value he was unable to ascertain. In the list of problems at the end he conjectured that $\rho = 1$. This is in fact true and was probably first observed by Kwapien. We are indebted to Professors W. B. Johnson and N. Tomczak-Jaegermann for pointing this out. That $H \leq H'$ is evident here: since the operator space norms on $\mathrm{MAX}(X)$ dominate the norms on $\mathrm{MIN}(X)$ we have a complete contraction $\mathrm{MAX}(X) \otimes_h \mathrm{MAX}(Y) \longrightarrow \mathrm{MIN}(X) \otimes_h \mathrm{MIN}(Y)$. This gives a contraction $X \otimes_{H'} Y \longrightarrow X \otimes_H Y$, and so $H \leq H'$. A more direct approach is to use a Hahn-Banach type separation argument as in [24] (see also [11,32]).

COROLLARY 3.2. *If X is a normed space such that $\mathrm{MIN}(X) = \mathrm{MAX}(X)$ then $\Gamma_2(X, Y) = \Pi_2(X, Y)$ and $\Gamma_2(X^*, Y) = \Pi_2(X^*, Y)$ isometrically for any Banach space Y. Also if Y is either X or X^* then $H = H'$ on $X \otimes Y$. If X is finite dimensional then X is at maximal Banach-Mazur distance from the l^2 space of the same dimension.*

PROOF. It follows immediately from the above the $\Gamma_2(X, Y^*) = \Pi_2(X, Y^*)$ for any Banach space Y. However, $\Gamma_2(X, Y) \subset \Gamma_2(X, Y^{**})$ and $\Pi_2(X, Y) \subset \Pi_2(X, Y^{**})$, from which the first result follows. The second result follows by duality of MIN and MAX. The last result follows by putting $Y = X$ in the first identity. If $\dim(X) = n$ then since $\pi_2(I_X) = n^{1/2}$ [25] it follows that $\gamma_2(I_X) = n^{1/2}$, or equivalently that $d(X, {l_n}^2) = n^{1/2}$ (which is maximal by a result of F. John [19]).

V. I. Paulsen has obtained the last of these results in [22]; moreover he has shown that if MIN(X) = MAX(X) then X has dimension at most 4!. This question is related to questions about representations of function algebras, we refer the reader to [22] for details. Perhaps the formulation above in terms of classical Banach space theory will further reduce the class of candidates.

COROLLARY 3.3. *For* $X = l_2^\infty$ *or* l_2^1 *we have* $\Gamma_2(X, Y) = \Pi_2(X, Y)$ *isometrically for any Banach space* Y. *Indeed if in addition* $Y = l_2^\infty$ *or* l_2^1 *then on* $X \otimes Y$ *we have* $H = \pi^* = \pi^{*t} = H'$.

PROOF. Haagerup proved that MIN(l_2^∞) = MAX(l_2^∞) [18,22], and by duality of MIN and MAX the same statement holds for l_2^1. Now use Theorem 3.1.

Many of these results also follow from [26, Section 3] in conjunction with [6, Section 4], as in Remark 1 after Theorem 3.1.

After this paper was submitted to the journal, and a limited number of preprints circulated, we received a preprint from Z-J. Ruan entitled *On the predual of dual algebras.* The proof of Proposition 2.2 and the results in Propositions 4.1 and 4.3 of Ruan's preprint coincide with Theorem 2.5 of this paper, which in turn is a simplification and extension of an earlier result of Effros and Ruan.

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