

SUBOPTIMALITY IN LINEAR CONTROL SYSTEMS WITH TIME DELAY

W. L. CHAN

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Abstract

For the linear quadratic control problem with delay, a lower bound for the performance index is obtained by elementary methods. Using this bound, two important *a posteriori* error estimates are derived. The first one measures the deviation of the performance index while the second is for the deviation of the state and control variables from the optimal solution.

1. Introduction

We consider a linear controlled system modelled by the delay-differential equation

$$\dot{Y}(t) = A(t)Y(t) + C(t)Y(t - r) + B(t)U(t), \quad (1.1)$$

with $Y(t)$ an n -vector, $U(t)$ an m -vector, $A(t)$, $B(t)$ and $C(t)$ continuous matrices on $[0, T]$. Further, $U(t) \in L_2[0, T]$, the space of square integrable functions. The continuous initial data is specified by $Y(t) = h(t)$ for $-r \leq t \leq 0$. The cost functional of control is

$$J_1(U) = \frac{1}{2} \int_0^T (Y'(t)Q(t)Y(t) + U'(t)R(t)U(t)) dt \quad (1.2)$$

where $Q(t) = Q'(t) > 0$ and $R(t) = R'(t) > 0$ are positive-definite matrices of appropriate dimensions; r and T are positive real numbers $T > r$. This problem has been extensively studied by Chyung and Lee [7] (see also [1]) who proved that

the minimization of $J_1(U)$, subject to (1), is equivalent to solving the following two point boundary value problem with both delayed and advanced arguments:

$$\dot{y}(t) = A(t)y(t) + C(t)y(t-r) - B(t)R^{-1}(t)B'(t)p(t), \quad (1.3)$$

$$-\dot{p}(t) = A'(t)p(t) + C'(t+r)p(t+r) + Q(t)y(t), \quad (1.4)$$

$$y(t) = h(t) \text{ for } -r \leq t \leq 0 \quad \text{and} \quad p(t) = 0 \text{ for } t > T. \quad (1.5)$$

The optimal controller is then given by

$$u(t) = -R^{-1}(t)B'(t)p(t). \quad (1.6)$$

Existence and uniqueness of absolutely continuous optimal solutions $y(t)$ and $p(t)$ were also proved in [7]. For any admissible control $U(t)$ in $L_2[0, T]$, by definition of the optimal control $u(t)$,

$$J^* \equiv J_1(u(t)) \leq J_1(U(t)).$$

Our problem at hand is to establish a lower bound $J_2(U(t))$, where $U(t)$ belongs to a certain subset of $L_2[0, T]$, with the property

$$J_2(U(t)) \leq J_2(u(t)) = J^*,$$

and more importantly, to deduce from it computable *a posteriori* estimates for the following non-negative suboptimality measures defined by

$$(i) \quad d_J = J_1(U) - J^*,$$

$$(ii) \quad d_Y = \int_0^T \|Y(t) - y(t)\|_{Q(t)}^2 dt,$$

$$(iii) \quad d_U = \int_0^T \|U(t) - u(t)\|_{R(t)}^2 dt,$$

where $\|Y(t)\|_Q^2$ denotes $Y'(t)Q(t)Y(t)$ and $Y(t)$ is the state generated by the control $U(t)$ while $y(t)$, $u(t)$ and $p(t)$ are the optimal state and costate, that is, solutions to (1.3)–(1.6).

A lower bound for the cost functional in variational control problems with time delay was first discussed by Chan [3], [4]. For the no-delay case, one can consult [9], [6], [2], [5] and the references therein. The results about d_J , d_Y and d_U , in the context of *a posteriori* suboptimal estimates are new. Notice that $d_Y^{1/2}$ is an equivalent L_2 norm between Y and y and similarly $d_U^{1/2}$ is an equivalent L_2 norm between U and u .

2. Preliminaries

From the form of the optimal controller (1.6), we are interested in considering the set of admissible controls generated by

$$U(t) = -R^{-1}(t)B'(t)W(t), \quad (2.1)$$

where $W(t)$ is an arbitrary n -vector in $L_2[0, T]$. For such a control, $U(t)$, the state associated with it, is thus

$$\dot{Y}(t) = A(t)Y(t) + C(t)Y(t - r) - M(t)W(t), \tag{2.2}$$

$$Y(t) = h(t) \quad \text{for } -r \leq t \leq 0, \tag{2.3}$$

where $M(t) = B(t)R^{-1}B'(t)$ and the corresponding costate is given by

$$-\dot{P}(t) = A'(t)P(t) + C'(t+r)P(t+r) + Q(t)Y(t), \tag{2.4}$$

$$P(t) = 0 \quad \text{for } t \geq T. \tag{2.5}$$

In what follows, suboptimal bounds on d_J , d_Y and d_U will be given in terms of $W(t)$ and $P(t)$ alone.

Alternatively, we may consider the question of assigning a meaningful ‘cost’ to any n -vector $\bar{Y}(t) \in L_2[0, T]$, which may be, say, arrived at by some approximating scheme to the state equation (1.1). To this end, for such a given $\bar{Y}(t)$, we consider the function $\bar{P}(t)$ satisfying

$$-\dot{\bar{P}}(t) = A'(t)\bar{P}(t) + C'(t+r)\bar{P}(t+r) + Q(t)\bar{Y}(t), \tag{2.6}$$

$$\bar{P}(t) = 0 \quad \text{for } t \geq T, \tag{2.7}$$

and $Z(t)$ given by

$$\dot{Z}(t) = A(t)Z(t) + C(t)Z(t - r) - M(t)\bar{P}(t), \tag{2.8}$$

$$Z(t) = h(t) \quad \text{for } -r \leq t \leq 0. \tag{2.9}$$

Suboptimal bounds on $d_{J(\bar{Y})}$, $d_{\bar{Y}}$ and $d_{\bar{P}}$, defined analogously to d_J , d_Y and d_U , will be given in terms of $\bar{Y}(t)$ and $Z(t)$. We could interpret (2.1)–(2.5) as a kind of ‘forward-backward sweep’ while (2.6)–(2.9) form a ‘backward-forward sweep’.

LEMMA 1. *The optimal cost functional $J(u(t))$ can be expressed as*

$$J = p'(0)h(0) - \frac{1}{2} \int_0^T \left[\|p(t)\|_{M(t)}^2 + \|y(t)\|_{Q(t)}^2 \right] dt + \int_{-r}^0 p'(t+r)C(t+r)h(t) dt. \tag{2.10}$$

PROOF. Multiply (1.3) by $p'(t)$ and the transpose of (1.4) by $y(t)$. Integrating the difference from 0 to T gives

$$\begin{aligned} p'(t)y(t)|_0^T &= - \int_0^T \left[\|p(t)\|_{M(t)}^2 + \|y(t)\|_{Q(t)}^2 \right] dt \\ &\quad + \int_0^T [p'(t)C(t)y(t-r)] dt - \int_0^{T-r} [p'(L+r)C(t+r)y(t)] dt \\ &= - \int_0^T \left[\|p(t)\|_{M(t)}^2 + \|y(t)\|_{Q(t)}^2 \right] dt + \int_0^r [p'(t)C(t)y(t-r)] dt. \end{aligned}$$

Hence, using (1.5), and continuity of p ,

$$p'(0)h(0) = 2J - \int_{-r}^0 [p'(t+r)C(t+r)h(t)] dt,$$

from which the result follows.

LEMMA 2. *If X, Y, Z are n -vectors and Q is an $n \times n$ symmetric matrix, then*

$$\|X - Z\|_Q^2 - \|Y - Z\|_Q^2 = \|X - Y\|_Q^2 + 2(X - Y)'Q(Y - Z). \quad (2.11)$$

3. Main result

THEOREM 1. *Assumptions and notations are as in (2.1)–(2.5). If*

$$J_1(W) = \frac{1}{2} \int_0^T (\|Y(t)\|_{Q(t)}^2 + \|W(t)\|_{M(t)}^2) dt \quad (3.1)$$

and

$$\begin{aligned} J_2(W) - P'(0)h(0) - \frac{1}{2} \int_0^T (\|Y(t)\|_{Q(t)}^2 + \|P(t)\|_{M(t)}^2) dt \\ + \int_{-r}^0 P'(t+r)C(t+r)h(t) dt, \end{aligned} \quad (3.2)$$

then

$$(a) J_2(W) \leq J \leq J_1(W) \quad \text{and} \quad (3.3)$$

$$(b) J_1(W) - J_2(W) = \frac{1}{2} \int_0^T \|W(t) - P(t)\|_M^2 dt. \quad (3.4)$$

PROOF. (a)

$$\begin{aligned} J - J_2(W) &= [p(0) - P(0)]'h(0) \\ &+ \frac{1}{2} \int_0^T [\|Y(t)\|_{Q(t)}^2 - \|y(t)\|_{Q(t)}^2 + \|P(t)\|_{M(t)}^2 - \|p(t)\|_{M(t)}^2] dt \\ &- \int_{-r}^0 [P(t+r) - p(t+r)]'C(t+r)h(t) dt. \end{aligned}$$

Using Lemma 2 in the first integral, we find

$$\begin{aligned} J - J_2(W) &= [p(0) - P(0)]'h(0) \\ &+ \int_0^T [(Y(t) - y(t))'Qy(t) + (P(t) - p(t))'M(t)p(t)] dt \\ &+ \frac{1}{2} \int_0^T [\|Y(t) - y(t)\|_{Q(t)}^2 + \|P(t) - p(t)\|_{M(t)}^2] dt \\ &- \int_{-r}^0 [P(t+r) - p(t+r)]'C(t+r)h(t) dt. \end{aligned} \quad (3.5)$$

Now, by (1.3),

$$\begin{aligned}
 & \int_0^T (P(t) - p(t))' M(t) p(t) dt \\
 &= \int_0^T (P(t) - p(t))' (-\dot{y}(t) + A(t)y(t) + C(t)y(t-r)) dt \\
 &= \int_0^T (\dot{P}(t) - \dot{p}(t))' y(t) + (P(t) - p(t))' (A(t)y(t) + C(t)y(t-r)) dt \\
 &\quad - (P(t) - p(t))' y(t) \Big|_0^T \\
 &= \int_0^T [- (Y(t) - y(t))' Q(t) y(t)] dt \\
 &\quad + \int_{-r}^0 [P(t+r) - p(t+r)]' C(t+r) y(t) dt \\
 &\quad + (P'(0) - p'(0))h(0), \tag{3.6}
 \end{aligned}$$

where the last equality follows from (1.4), (1.5), (2.4) and (2.5). Substituting (3.6) into (3.5) gives

$$\begin{aligned}
 J - J_2(W) &= \frac{1}{2} \int_0^T [\|Y(t) - y(t)\|_{Q(t)}^2 + \|P(t) - p(t)\|_{M(t)}^2] dt \\
 &= \frac{1}{2} \int_0^T [\|Y(t) - y(t)\|_{Q(t)}^2 + \|U(t) - u(t)\|_{R(t)}^2] dt, \tag{3.7}
 \end{aligned}$$

so that $J_2(W) \leq J$. The other inequality, $J \leq J_1(W)$, follows from the definition of J .

(b) Equations (2.2)–(2.5) imply

$$\frac{d}{dt} [P'(t)Y(t)] = \begin{cases} -[P'(t)M(t)W(t) + \|Y(t)\|_{Q(t)}^2] + P'(t)C(t)Y(t-r) \\ \quad -P'(t+r)C(t+r)Y(t) & \text{for } 0 \leq t \leq T-r, \\ -[P'(t)M(t)W(t) + \|Y(t)\|_{Q(t)}^2] + P'(t)C(t)Y(t-r) \\ \quad & \text{for } T-r < t \leq T. \end{cases}$$

Integrating the above from 0 to T yields,

$$\begin{aligned}
 P'(0)h(0) &= \int_0^T [P'(t)M(t)W(t) + \|Y(t)\|_{Q(t)}^2] dt \\
 &\quad - \int_{-r}^0 P'(t+r)C(t+r)h(t) dt. \tag{3.8}
 \end{aligned}$$

Putting (3.8) into (3.2) results in

$$J_2(W) = \frac{1}{2} \int_0^T [\|Y(t)\|_{Q(t)}^2 - \|P(t)\|_{M(t)}^2 + 2P'(t)M(t)W(t)] dt, \tag{3.9}$$

and therefore with (3.1)

$$\begin{aligned}
 J_1(W) - J_2(W) &= \frac{1}{2} \int_0^T \left[\|W(t)\|_{M(t)}^2 - 2P'(t)M(t)W(t) + \|P(t)\|_{M(t)}^2 \right] dt \\
 &= \frac{1}{2} \int_0^T \|P(t) - W(t)\|_{M(t)}^2 dt \geq 0.
 \end{aligned}$$

COROLLARY 1. *With the assumptions and notations in Theorem 1,*

$$(i) \quad d_J \leq \frac{1}{2} \int_0^T \|W(t) - P(t)\|_{M(t)}^2 dt \tag{3.10}$$

and

$$(ii) \quad d_Y + d_U \leq \int_0^T \|W(t) - P(t)\|_{M(t)}^2 dt. \tag{3.11}$$

PROOF. Since $J_2(W) \leq J \leq J_1(W)$ from Theorem 1(a), it follows that

$$d_J(W) = J_1(W) - J \leq J_1(W) - J_2(W) = \frac{1}{2} \int_0^T \|W(t) - P(t)\|_{M(t)}^2 dt.$$

Also, from Theorem 1(b),

$$\frac{1}{2}(d_Y + d_U) = J - J_2(W) \leq J_1(W) - J_2(W) = \frac{1}{2} \int_0^T \|W(t) - P(t)\|_{M(t)}^2 dt.$$

Now if we take the ‘backward-forward sweep’ view point, that is considering (2.6)–(2.9), we have

THEOREM 2. *Given an arbitrary n -vector $\bar{Y}(t)$ in $L_2[0, T]$, and if*

$$J_1(\bar{Y}) = \frac{1}{2} \int_0^T \left[\|Z(t)\|_{Q(t)}^2 + \|\bar{P}(t)\|_{M(t)}^2 \right] dt, \tag{3.12}$$

and

$$\begin{aligned}
 J_2(\bar{Y}) &= -\frac{1}{2} \int_0^T \left[\|\bar{Y}(t)\|_{Q(t)}^2 + \|\bar{P}(t)\|_{M(t)}^2 \right] dt \\
 &\quad + \bar{P}(0)h(0) + \int_{-r}^0 \bar{P}(t+r)C(t+r)h(t) dt,
 \end{aligned} \tag{3.13}$$

then

$$\begin{aligned}
 (a) \quad &J_2(\bar{Y}) \leq J \leq J_1(\bar{Y}) \quad \text{and} \\
 (b) \quad &J_1(\bar{Y}) - J_2(\bar{Y}) = \int_0^T \|Z(t) - \bar{Y}(t)\|_{Q(t)}^2 dt.
 \end{aligned} \tag{3.14}$$

This theorem leads to bounds for $d_J(\bar{Y})$ and $d_{\bar{Y}} + d_{\bar{P}}$ in Corollary 2.

COROLLARY 2. *With the assumptions and notations in Theorem 2,*

$$(i) d_J(\bar{Y}) \equiv J_1(\bar{Y}) - J \leq \frac{1}{2} \int_0^T \|Z(t) - \bar{Y}(t)\|_{Q(t)}^2 dt, \tag{3.15}$$

and

$$(ii) d_{\bar{y}} + d_{\bar{p}} \equiv \int_0^T \left[\|\bar{Y}(t) - y(t)\|_{Q(t)}^2 + \|\bar{P}(t) - p(t)\|_{M(t)}^2 \right] dt \\ \leq \int_0^T \|Z(t) - \bar{Y}(t)\|_{Q(t)}^2 dt.$$

The proofs of Theorem 2 and Corollary 2 are similar to those of Theorem 1 and Corollary 1, and hence are omitted. Instead, we give an example in the following section for illustration.

4. An example

Consider the example given in [8]:

$$\text{Min } J_1 = \frac{1}{2} \int_0^2 (y^2 + u^2) dt, \tag{4.1}$$

subject to

$$\dot{y} = y(t) + y(t - 1) + u(t) \quad \text{for } 0 \leq t \leq 2, \\ y(t) = 1 \quad \text{for } -1 \leq t \leq 0. \tag{4.2}$$

Our aim here is to demonstrate the implications of Theorem 2. The costate equation of (4.2) is

$$-\dot{p}(t) = \begin{cases} p(t) + y(t) & \text{for } 1 \leq t \leq 2, \\ p(t) + y(t) + p(t + 1) & \text{for } 0 \leq t \leq 1, \end{cases} \tag{4.3} \\ p(2) = 0,$$

and the optimal control is given by the solutions of (4.2), (4.3) with $u(t) = -p(t)$. Suppose we have chosen \bar{Y} in (3.12) of Theorem 2 as

$$\bar{Y}(t) = \begin{cases} 1, & -1 \leq t \leq 0 \\ 1 - 13t + (-a + 2)t^2, & 0 \leq t \leq 1 \\ (a - 8) - (a + 3)t - (a - 5)t^2, & 1 \leq t \leq 2 \end{cases}. \tag{4.4}$$

Then, the solution to (4.3) (compare with (2.6)), with $\bar{Y}(t)$ taking the place of $y(t)$, can be shown to be

$$-\bar{P}(t) = \begin{cases} (a - 2) + (-2 - a)t + t^2, & 0 \leq t \leq 1 \\ 5 + (a - 13)t + (-a + 5)t^2, & 1 \leq t \leq 2 \end{cases}, \tag{4.5}$$

where $a = -0.5$.

Now $Z(t)$ of (2.8), needed for the evaluation of $J_1(\bar{Y})$ in (3.12), can also be seen to be

$$Z(t) = \begin{cases} 1, & -1 \leq t \leq 0 \\ 1 + at - t^2, & 0 \leq t \leq 1 \\ -1 + 5t + (a - 4)t^2, & 1 \leq t \leq 2 \end{cases}. \quad (4.6)$$

With $\bar{Y}(t)$, $\bar{P}(t)$ and $Z(t)$ at hand, J_1 and J_2 are then calculated according to (3.12) and (3.13) respectively, giving

$$J_1 = 18.8958333,$$

$$J_2 = -11.7345238$$

and, from [8], the optimal value of J_1 is

$$J^* = 6.1701.$$

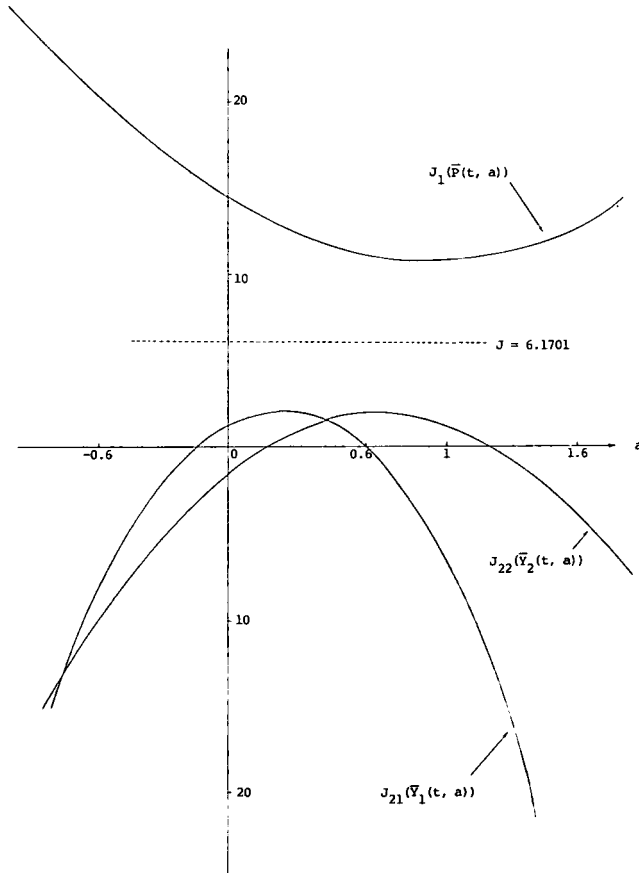


Figure 1. Suboptimality from J_1 and J_2 .

Thus, indeed, $J_2 \leq J^* \leq J_1$ but the large difference, $J_1 - J_2 = 30.6303571$, indicates there is much room for improvement. In fact if we consider a in (4.5) as a parameter and define $U(t) = -\bar{P}(t, a)$ then $Z(t, a)$ in (4.6) gives a family of states and the corresponding $J_1(a)$ has been plotted in Figure 1. To apply Theorem 2 to estimate how far away are these families of controls and states from the optimal solution, we arbitrarily choose two families of functions defined by

$$\bar{Y}_1(t, a) = \begin{cases} 1, & -1 \leq t \leq 0 \\ 1 + (2.1 - 7a)t + (-3.1 + 6a)t^2, & 0 \leq t \leq 1 \\ -0.7 + (0.7 - 2a)at^2, & 1 \leq t \leq 2 \end{cases} \quad (4.7)$$

and

$$\bar{Y}_2(t, a) = \begin{cases} 1, & -1 \leq t \leq 0 \\ 1 + (3a - 1.4)t + (0.2 - 3a)t^2, & 0 \leq t \leq 1 \\ -a + (a - 0.4)t + 0.2t^2, & 0 \leq t \leq 2 \end{cases}. \quad (4.8)$$

Substituting these into (2.6) (compare with (4.3)) and solving for $\bar{P}(t)$, gives

$$-P_1(t, a) = \begin{cases} (-5.2 + 9a) + (7.6 - 13a)t + (-3.1 + 5a)t^2, & 0 \leq t \leq 1 \\ (-1.4 + 4a) + (0.7 - 4a)t + at^2, & 1 \leq t \leq 2 \end{cases}. \quad (4.9)$$

and

$$-P_2(t, a) = \begin{cases} (0.8 - 6a) + (-0.6 + 8a)t - 3at^2, & 0 \leq t \leq 1 \\ (0.8 - 2a) + (a - 0.8)t + at^2, & 1 \leq t \leq 2 \end{cases}. \quad (4.10)$$

Using (4.7) and (4.9), $J_{21}(a)$ in (3.13) is evaluated. Similarly (4.8) and (4.10) are used to evaluate $J_{22}(a)$, (that is, $J_1(a)$ and $J_{2i}(a)$ using the functions \bar{Y}_1 and \bar{Y}_2 respectively). They are plotted against a in Figure 1. The ‘gap’ between $\min_a J_1(a)$ and $\max_{a,i=1,2} \{J_{2i}(a)\}$, gives the required estimate of suboptimality of the control $U(t) = -P(t, a)$, as in (4.5), applied to (4.2).

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Department of Mathematics
The Chinese University of Hong Kong
Shatin, N. T.
Hong Kong