A CLASS OF POSITIVE LINEAR OPERATORS

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1. Introduction. Let $F[a,b]$ be the linear space of all real valued functions defined on $[a,b]$. A linear operator $L : C[a,b] \to F[a,b]$ is called positive (and hence monotone) on $C[a,b]$ if $L(f) \geq 0$ whenever $f \geq 0$. There has been a considerable amount of research concerned with the convergence of sequences of the form $\{L_n(f)\}$ to f where $\{L_n\}$ is a sequence of positive linear operators on $C[a,b]$. Much of the recent research has made use of the following theorem of Korovkin [4]:

THEOREM. *Let* $\{L_n\}$ *be a sequence of positive linear operators on* $C[a,b]$. *Let* $e^i \in C[a,b]$ *be defined by* $e^i(x) = x^i$, *for* $i = 0, 1, 2$. *Then* $\lim_{n\to\infty} L_n(f)(x) = f(x)$ *uniformly on* $[a,b]$ *for each* $f \in C[a,b]$ *if and only if* $\lim_{n\to\infty} L_n(e^i)(x) = x^i$ *uniformly on* $[a,b]$ *for* $i = 0, 1, 2$.

A natural collection of operators which are defined on subclasses of $C[0,1]$ are those of the form

$$L_n(f)(x) = \sum_{k=0}^{\infty} a_{nk}(x) f(k/n),$$

where the matrix $(a_{nk}(x))$ is generated by a function $\phi(x,w)$ by means of the relation

$$[\phi(x,w)]^n = \sum_{k=0}^{\infty} a_{nk}(x) w^k.$$

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The matrices $(a_{nk}(x))$ which are defined in this manner were introduced by Sonnenschein [6], and have important applications in summability theory when the function ϕ satisfies certain regularity conditions.

It is the object of this note to establish a convergence theorem for a wide class of positive linear operators of this form, to point out that the complex analogs of these operators have interesting convergence properties, and to demonstrate that the summability matrix associated with the operators is a regular matrix if the operators have the proper convergence properties.

Operators of the form $L_n(f)(x) = \sum_{k=0}^{n} a_{nk}(x)f(\frac{k}{n})$ have recently been studied by Jakimovski and Leviatan [2]. In [2], however, the coefficients are of the form

$$a_{nk}(x) = \frac{1}{g(x-1)}(-1)^{n-k} x^k (1-x)^{n-k} \zeta^k_{n-k}(-k-1),$$ where the ζ^k_{n-k} are generalized Boole polynomials and g is the uniform limit of a certain sequence of analytic functions. The matrix $A = (a_{nk}(x))$ which determines the operator of [2] is, consequently, not a Sonnenschein matrix and the results of [2] are, therefore, not directly related to those of this paper.

Even more recently, Jakimovski and Leviatan [3] have considered sequences of positive linear operators of the form

$$L_n(f)(x) = \sum_{k=0}^{\infty} a_{nk}(x)f(\frac{n}{n+k})$$ where $A = (a_{nk}(x))$ is defined in terms of generalized Boole polynomials in a manner similar to that of [2]. The matrix $A = (a_{nk}(x))$ is again not a Sonnenschein matrix so that the operators of [3] are not the same as those considered in this article.

2. The operators and convergence theorem.

DEFINITION 1. Let $\phi(z, w)$ be defined on $S \times \Omega$ where S and Ω are subsets of the complex plane with $[0, 1] \subset S$ and $1 \in \Omega$. Let $\phi(z, w)$ be analytic on Ω for each $z \in S$. For each $z \in S$ let the matrix $A = (a_{nk}(z))$ be defined by

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$$[\phi(z,w)]^n = \sum_{k=0}^{\infty} a_{nk}(z)w^k, \quad n = 1,2,\ldots,$$

1)

$$a_{oo}(z) = 1, \quad a_{ok}(z) = 0, \quad k = 1,2,\ldots.$$

DEFINITION 2. Let ϕ and A satisfy definition 1. Let B denote the collection of functions which are bounded on $\{x : x \geq 0\}$. For each $f \epsilon C[0,1] \cap B$ and $x \epsilon [0,1]$ let $L_n(f)$ be defined by

2) $$L_n(f)(x) = \sum_{k=0}^{\infty} a_{nk}(x)f(k/n), \quad n = 1,2,\ldots.$$

LEMMA 1. The operator L_n defined by (2) is a positive linear operator on $C[0,1] \cap B$ for each $n = 1,2,\ldots$ if and only if $a_{nk}(x) \geq 0$ for each $k = 0,1,\ldots,$ $n = 1,2,\ldots,$ and $x \epsilon [0,1]$.

Proof. It is clear that L_n is linear for each $n = 1,2,\ldots$. If $a_{nk}(x) \geq 0$ for each $n = 1,2,\ldots,$ $k = 0,1,2,\ldots,$ and $x \epsilon [0,1]$ then $f \geq 0$ implies $L_n(f) \geq 0$ for each $n = 1,2,\ldots$ so that L_n is a positive operator for each $n = 1,2,\ldots$. It is also evident that $a_{nk}(x) \geq 0$ for each $n = 1,2,\ldots,$ $k = 0,1,\ldots,$ and $x \epsilon [0,1]$ implies that the series in (2) converges for each $f \epsilon C[0,1] \cap B$, $x \epsilon [0,1]$, and $n = 1,2,\ldots,$ since

$$\sum_{k=0}^{\infty} a_{nk}(x) = [\phi(x,1)]^n.$$

Suppose that L_n is a positive linear operator on $C[0,1] \cap B$ for each $n = 1,2,\ldots$. Let $g_o^n \epsilon C[0,1] \cap B$ be defined by

3) $$g_o^n(\zeta) = \begin{cases} 1 & \zeta = 0 \\ -n\zeta + 1 & 0 < \zeta \leq 1/n \\ 0 & 1/n < \zeta \end{cases}$$

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for each $n = 1, 2, \ldots$. Since $g_o^n \geq 0$ it follows that

$L_n(g_o^n)(x) = a_{on}(x) \geq 0$ for $n = 1, 2, \ldots$ and $x \in [0, 1]$.

For each $k = 1, 2, \ldots$ let $g_k^n \in C[0, 1] \bigcap B$ be defined by

$$4) \qquad g_k^n(\zeta) = \begin{cases} 0 & 0 \leq \zeta \leq \frac{k+1}{n} \\ n\zeta - k + 1 & \frac{k-1}{n} < \zeta \leq \frac{k}{n} \\ -n\zeta + k + 1 & \frac{k}{n} < \zeta \leq \frac{k+1}{n} \\ 0 & \frac{k+1}{n} < \zeta \end{cases}$$

for each $n = 1, 2, \ldots$. Then $g_k^n \geq 0$ so that

$L_n(g_k^n)(x) = a_{nk}(x) \geq 0$ for each $n = 1, 2, \ldots$, $k = 1, 2, \ldots$ and $x \in [0, 1]$.

This proves the lemma.

THEOREM 1. Let ϕ, A and L_n satisfy the conditions of definitions 1 and 2. Let $\phi(x, 1) = 1$ for each $x \in [0, 1]$ and suppose that there exists M such that $|\phi_{ww}(x, 1)| < M$ for all $x \in [0, 1]$. Suppose that L_n is a positive linear operator on $C[0, 1] \bigcap B$ for each $n = 1, 2, \ldots$. Then $\lim_{n \to \infty} L_n(f)(x) = f(x)$ uniformly on $[0, 1]$ for each $C[0, 1] \bigcap B$ if and only if $\phi_w(x, 1) = x$ for each $x \in [0, 1]$.

Proof. Suppose that $\phi_w(x, 1) = x$. Since $\phi(x, 1) = 1$, it follows that $L_n(e^o)(x) = 1$ for each $x \in [0, 1]$ and $n = 1, 2, \ldots$. It follows from (1) that

$$5) \qquad n[\phi(x, w)]^{n-1} \phi'(x, w) = \sum_{k=0}^{\infty} k\, a_{nk}(x) w^{k-1}$$

where the differentiation is with respect to w.

6) $$n(n-1)[\phi(x,w)]^{n-2}[\phi'(x,w)]^2 + n[\phi(x,w)]^{n-1}\phi''(x,w)$$

$$= \sum_{k=0}^{\infty} k(k-1)a_{nk}(x)w^{k-2}$$

for each $(x, w) \in [0, 1] \times \Omega$. Hence

7) $$n(n-1)[\phi'(x,1)]^2 + n\phi''(x,1) = \sum_{k=0}^{\infty} k(k-1)a_{nk}(x),$$

since $\phi(x, 1) = 1$ for each $x \in [0, 1]$. It follows from (1) and (2) that

8) $$L_n(e^1)(x) = \sum_{k=0}^{\infty} a_{nk}(x)\frac{k}{n} = \phi'(x,1).$$

Therefore (7) implies

$$\sum_{k=0}^{\infty} k^2 a_{nk}(x) = n(n-1)[L_n(e^1)(x)]^2 + n\phi''(x,1) + \sum_{k=0}^{\infty} k\, a_{nk}(x),$$

and hence that

9) $$L_n(e^2)(x) = \sum_{k=0}^{\infty} \frac{k^2}{n^2} a_{nk}(x)$$

$$= \left(\frac{n-1}{n}\right)[L_n(e^1)(x)]^2 + \frac{\phi''(x,1)}{n} + \frac{L_n(e^1)(x)}{n}.$$

Since $\phi''(x, 1)$ is uniformly bounded on $[0, 1]$ and $\phi'(x, 1) = x$ for each $x \in [0, 1]$, it follows that

$$\lim_{n\to\infty} L_n(e^2)(x) = x^2$$

uniformly on $[0, 1]$. The sufficiency part of the theorem now follows from Korovkin's theorem. The necessity follows from 8).

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Two classical examples of operators satisfying Theorem 1 are given below:

Example 1. Let $\phi(z, w) = (zw + 1 - z)$. In this case $\phi''(x, 1) = 0$, and

$$L_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)$$

is the nth Bernstein polynomial [5] for each $n = 1, 2, \ldots$.

Example 2. Let $\phi(z, w) = e^{z(w-1)}$. It follows that $\phi''(x, 1) = 1 + x^2$, and

$$L_n(f)(x) = e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{k!} f\left(\frac{k}{n}\right).$$

is the nth Szasz operator [7] for each $n = 1, 2, \ldots$.

3. Some complex operators. Let ϕ and A be as in definition 1) and let L_n be defined by

10) $$L_n(f)(z) = \sum_{k=0}^{\infty} a_{nk}(z) f\left(\frac{k}{n}\right)$$

for each $z \in S$ and each f and n for which the series converges.

It is clear that the formal computations involved in equations 5) through 9) hold for the complex operators $L_n(f)(z)$. It is also clear from equations 8) and 9) that if $\phi'(z, 1)$ is a polynomial of degree less than or equal to 1, and if $\phi''(z, 1)$ is a polynomial of degree less than or equal to 2, then $L_n(e^1)(z)$ and $L_n(e^2)(z)$ are, respectively, polynomials of degree less than or equal to 1 and of degree less than or equal to 2. These computations can be extended in a straightforward manner to establish the following lemma:

LEMMA 2. Let ϕ and A satisfy definition 1 and let L_n be defined by 10). Suppose that $\phi(z, 1) = 1$ and that $\phi^{(v)}(z, 1)$ is a polynomial of degree less than or equal to

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v for each $v = 1, 2, \ldots,$ where the differentiation is with respect to w. Let $e^m(z) = z^m$ for each $m = 1, 2, \ldots$. Then $L_n(e^m)(z)$ is a polynomial of degree less than or equal to m for each $m = 1, 2, \ldots,$ $n = 1, 2, \ldots$.

The following theorem was inspired by a similar result of Cheney and Sharma [1] for the complex Szasz operator. The proof follows from Lemma 1 and Bernstein's lemma [5, p. 90] in exactly the same manner as the result of [1].

THEOREM 2. *Let* E *be an ellipse with foci* 0 *and* 1 *and let* R *denote the bounded component of the complement of* E. *Let* ϕ *and* A *satisfy definition* 1 *with* $R \subset S$ *and* $a_{nk}(z) \geq 0$ *for each* $0 \leq z \leq 1$ *and:* $a_{nk}(z) = 0$ *for* $k > n$. *Let* L_n *be defined by* 10). *Let* $\phi(z, 1) = 1$ *and* $\phi'(z,1) = z$ *for each* $z \in S$. *Let* $\phi^{(v)}(z, 1)$ *be a polynomial of degree not greater than* v *for each* $v = 1, 2, \ldots$ *and each* $z \in S$, *where the differentiation is with respect to* w. *Then* $\lim_{n\to\infty} L_n(f)(z) = f(z)$ *uniformly on each compact subset of* R.

4. *The summability method associated with* $L_n(f)(x)$. The operators $L_n(f)(x)$ were defined in terms of the Sonnenschein summability matrix $(a_{nk}(x))$. The following theorem shows that if the operators satisfy the conclusion of Theorem 1 then the associated matrix is regular.

THEOREM 3. *Let* $L_n(f)$ *be defined by* 2) *for each* $n = 1, 2, \ldots$. *Suppose that* L_n *is a positive linear operator on* $C[0, 1] \cap B$ *for each* $n = 1, 2, \ldots$ *and that* $\lim_{n\to\infty} L_n(f)(x) = f(x)$ *uniformly on* $[0, 1]$ *for each* $f \in C[0, 1] \cap B$. *Then the matrix* $(a_{nk}(x))$ *is regular for* $x \in (0, 1]$.

Proof. Since $\lim_{n\to\infty} L_n(e^0)(x) = 1$ uniformly on $[0, 1]$, it follows that $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk}(x) = 1$ for each $x \in [0, 1]$. Lemma 1

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implies that $a_{nk}(x) \geq 0$ for each n, k and $x \in [0, 1]$. Therefore

$\sup \{ \sum_{k=0}^{\infty} |a_{nk}(x)| : n = 1, 2, \ldots \}$ is finite. Let $x \in (0, 1]$ and let k be a fixed non-negative integer. If $k = 0$ let N be such that $n \geq N$ implies that $0 < \frac{1}{n} < x$. Let $g_k^n = g_o^n \in C[0, 1] \cap B$ be defined by 3) for $n \geq N$. Then

$$L_n(g_o^n)(x) = \sum_{v=0}^{\infty} a_{nv}(x) g_o^n(\frac{v}{n}) = a_{no}(x)$$

for $n \geq N$. So $0 = g_o^N(x) = \lim_{n\to\infty} L_n(g_o^N)(x) = \lim_{n\to\infty} a_{no}(x)$.

If $k \neq 0$ let N be such that $0 \leq \frac{k-1}{n} < \frac{k}{n} < \frac{k+1}{n} \leq x$ for $n \geq N$. Let $g_k^n \in C[0, 1] \cap B$ be defined by 4). Then

$$L_n(g_k^n)(x) = \sum_{v=0}^{\infty} a_{nv}(x) g_k^n(\frac{v}{n}) = a_{nk}(x)$$

for $n \geq N$. Hence $0 = g_o^N(x) = \lim_{n\to\infty} L_n(g_o^N)(x) = \lim_{n\to\infty} a_{nk}(x)$.

The Silverman-Toeplitz theorem is, therefore, satisfied Hence $(a_{nk}(x))$ is regular for each $x \in (0, 1]$.

Example 1 shows that the associated matrix $(a_{nk}(x))$ may not be regular at $x = 0$.

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