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ON GLOBAL CLUSTER SETS FOR FUNCTIONS MEROMORPHIC ON SOME RIEMANN SURFACES

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0. Consider a single-valued meromorphic function $w = f(p)$ defined on an open Riemann surface R with an ideal boundary β. In [1], Collingwood and Cartwright introduced the global cluster set for a function meromorphic on the unit disk. Generalizing the definition of global cluster sets to our present setting, we define the *global cluster set* for $w = f(p)$ as follows;

A value w in the extended complex plane is called a cluster value at β if there exists a sequence $\{p_n\}_{n=1}^{\infty}$ in R converging to β such that

$$\lim_{n\to\infty} f(p_n) = w \ .$$

The set consisting of cluster values is called the global cluster set for $w = f(p)$ and denoted by $C_R(f)$.

In the same way, the *range of values* $R_R(f)$ and the *asymptotic set* $A_R(f)$ can be defined as usual in our present setting. Collingwood and Cartwright obtained the following so-called their main theorem in their setting;

THEOREM. $C_R(f) = \{\mathrm{Int}\, R_R(f)\} \cup \overline{A_R(f)}$, *where* Int *and the bar indicate the interior and the closure, respectively.*

Using the wholly analogous discussion to their proof, we can prove, although we omit the proof, that the above theorem is valid for our present setting.

By the reason that the realization of ideal boundaries in our definition of global cluster sets is extremely rough, one might doubt that any refined function-theoretic information can be derived from global cluster sets. However, not only global cluster sets are convenient to

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Riemann surfaces in the sense that they do not refer to the specific individual point of ideal boundaries which is originally undetermined uniquely but also there are many important instances of Riemann surfaces where global cluster sets supply knowledges that determine the conformal shapes of them when the function-theoretic sizes of their ideal boundaries are, in a sense, small.

From this point of view, we classify Riemann surfaces according to certain properties determined by global cluster sets and study the relations between these classes and certain known classes, which is the purpose of this paper.

1. DEFINITION. C_{AB} *(resp.* C_{HB}*): the class of open Riemann surfaces on which the global cluster sets for any non-constant meromorphic function is either total or AB-removable (resp. HB-removable).*

Let O_G, O_{HB} and O_{AB} denote classes of Riemann surfaces which admit no Green's functions, no non-constant bounded harmonic functions and no non-constant bounded analytic functions, respectively. It can be seen that $C_{HB} \subset C_{AB}$ and that $O_G = O_{HB} = C_{HB}$ and $O_{AB} = C_{AB}$ for surfaces of finite genus.

Kuroda [4] introduced a class O°_{AB} of Riemann surfaces, on any subregion of which there exists no non-constant bounded analytic function whose real part vanishes on the relative boundary continuously.

THEOREM 1. $O^\circ_{AB} \subset C_{AB}$.

Proof. Let R belong to O°_{AB} and $w = f(p)$ be an arbitrary non-constant meromorphic function on R such that $C_R(f)$ is sub-total. We must only show that $C_R(f)$ is AB-removable. To the contrary, suppose that $C_R(f)$ is not AB-removable. Since $w = f(p)$ has Iversen's property ([4]), Stoïlow's principle ([7], [8]) holds. Hence every point of $\mathscr{C} C_R(f)$ is covered by $w = f(p)$ just n (>0, integer) times and

$$C_R(f) = \{w\,;\, n_f(w) < n\}\ ,$$

where $\mathscr{C}$ indicates the complement with respect to the whole complex plane and $n_f(w)$ indicates the number of points of $f^{-1}(w) \cap R$ (here and hereafter multiple points are counted repeatedly). Then there exists an integer m $(0 \le m < n)$ such that $e_m = \{w\,;\, n_f(w) \le m\}$ is not AB-removable and $e_{m-1} = \{w\,;\, n_f(w) \le m-1\}$ is AB-removable, since $C_R(f) = \bigcup_{0 \le i < n} e_i$ and each e_i is compact. Choose a point w_0 in $e_m - e_{m-1}$ such that $e_m \cap \{|w - w_0| \le \rho\}$ is not AB-removable for any $\rho > 0$. For sufficiently small

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ρ, the inverse image $f^{-1}(\{|w - w_0| < \rho\})$ consists of m compact components (multiplicity is considered) and of at least one non-compact component Δ, since $w_0 \in C_R(f)$. Then $w = f(p)$ does not cover any point of e_m on Δ. Since $e_m \cap \{|w - w_0| \leq \rho/2\}$ is not AB-removable, we can find a non-constant bounded analytic function $\varphi(w)$ on $\{|w - w_0| < \rho\} - e_m \cap \{|w - w_0| \leq \rho/2\}$ whose real part vanishes on $\{|w - w_0| = \rho\}$ continuously. Then $\Phi(p) = \varphi(f(p))$ is a non-constant bounded analytic function on Δ whose real part vanishes on the relative boundary of Δ continuously. This contradicts that R belongs to O°_{AB}. ■

The method of the above proof owes to [6, p. 98].

Next, we deal with the class C_{HB}.

LEMMA. $O_G \subset C_{HB}$.

Proof. Let R belong to O_G and $w = f(p)$ be an arbitrary non-constant meromorphic function on R such that $C_R(f)$ is sub-total. We have only to show that $C_R(f)$ is HB-removable. Suppose that $C_R(f)$ is not HB-removable. Since $O_G \subset O^\circ_{AB}$, $w = f(p)$ has Iversen's property. Hence, using the analogous argument in the proof of Theorem 1, we can find a subregion Δ on R and a non-constant bounded harmonic function on Δ which vanishes on the relative boundary of Δ continuously. Hence R has positive boundary, i.e., $R \notin O_G$, which is a contradiction. ■

Heins [3] introduced a class O_L of Riemann surfaces, on which there exists no non-constant Lindelöfian meromorphic function. Here a conformal mapping of a Riemann surface R_1 into another R_2, $q = f(p)$, is said to be Lindelöfian if

$$\sum_{f(r)=q} n(r) G(p, r) < +\infty$$

for any $f(p) \neq q$, where $G(p, \cdot)$ is a Green's function on R_1 and $n(r)$ denotes the multiplicity at r. He proved that $O_{HB} \subsetneqq O_L \subsetneqq O_{AB}$ and that $O_G = O_{HB} = O_L$ for surfaces of finite genus.

THEOREM 2. $O_L \subset C_{HB}$.

Proof. Since $O_G \subset C_{HB}$ by the above lemma and $O_G \subset O_L$, we must only show that $O_L - O_G \subset C_{HB}$. Let R belong to $O_L - O_G$. Then any non-constant meromorphic function on R assumes every value infinitely often with the exception of a set of capacity zero ([3, p. 428]). This shows that $R \in C_{HB}$. ■

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Next theorem is originally obtained by Tsuji ([9]).

THEOREM 3. *Let R be an open Riemann surface. Suppose that there exists a non-constant meromorphic function $w = f(p)$ on R such that $C_R(f)$ is HB-removable. Then $R \in O_G$.*

Proof. Evidently, $w = f(p)$ has Iversen's property. By Stoïlow's principle, $w = f(p)$ covers every point of $\mathscr{C}C_R(f)$ just n times. For each $w \in \mathscr{C}C_R(f)$, we denoted by $\{p_1^w, p_2^w, \cdots, p_n^w\}$ the set $f^{-1}(w) \cap R$. Suppose that there exists a Green's function $G(p_0, p)$ on R. On $\mathscr{C}C_R(f) - \{f(p_0)\}$, we consider a function

$$H(w) = \sum_{i=1}^{n} G(p_0, p_i^w) .$$

Then $H(w)$ is a non-constant single-valued positive harmonic function on $\mathscr{C}C_R(f) - \{f(p_0)\}$. This contradicts that $C_R(f)$ is HB-removable.

From the above theorem, we see that if R belongs to C_{HB}, one of the following mutually exclusive alternatives holds:

(i) there exists a non-constant meromorphic function on R whose global cluster set is HB-removable,

(ii) the global cluster set for any non-constant meromorphic function on R is total.

Moreover, if the case (i) occurs, R belongs to O_G.

2. Now, we shall show the strictness of the inclusions obtained above.

For surfaces of finite genus, it is seen that $O_{HB} = O_L = C_{HB}$, $C_{AB} = O_{AB}$ and $O_{HB} \subsetneqq O_{AB}$. Hence $C_{HB} \subsetneqq C_{AB}$.

Let R_M be the Myrberg's example ([5]). It is well-known that $R_M \in O_{AB}$. While, we see that $R_M \notin C_{AB}$, since the global cluster set for the projection of R_M onto the complex plane is sub-total and not AB-removable. Hence $C_{AB} \subsetneqq O_{AB}$.

Heins ([2, *p*. 298]) introduced a subregion R_H, with compact complement, of a Riemann surface of null boundary. It is seen that $R_H \in O_L$ and $R_H \notin O_{AB}^{\circ}$. Hence $O_{AB}^{\circ} \subsetneqq C_{AB}$ and $O_{AB}^{\circ} \not\supset C_{HB}$.

Kuroda ([4]) proved that $O_{HB} \subsetneqq O_{AB}^{\circ}$ for the surfaces of finite genus. Hence $O_{AB}^{\circ} \not\subset C_{HB}$.

To prove $O_L \subsetneqq C_{HB}$, we construct an example which is a modification of Heins' example ([2, p. 298]). Let $\{a_n\}$ and $\{b_n\}$ denote increasing sequences of positive numbers converging to ∞. Here $\{a_n\}$ is chosen such that $\sum 1/a_n = \infty$. Moreover, let e be a compact set in the w-plane such that e is AB-removable and not HB-removable and that e does not intersect with the real axis. Let E_1 be the w-plane less the slits $[a_{2n}, a_{2n+1}]$ on the real axis (all n), E_2 be the region E_1 less the slits $[-b_{2n+1}, -b_{2n}]$ (all n) and σ_n $(n = 0, 1, \cdots)$ be the w-plane less the slit $[-b_{2n+1}, -b_{2n}]$ and a compact set e. Joining these copies along all common slits identifying in the usual manner the upper edge of each slit with lower edge of the corresponding slit and vice versa, we obtain the desired covering surface, which we denote by F. In the same way of Heins' example, we see that $F \in C_{HB}$. Also, $F \notin O_G$. Consider the projection $w = P(p)$ of F onto the w-plane. Then $w = P(p)$ covers every point of e just twice. Since e is not HB-removable, $w = f(p)$ is a non-constant Lindelöfian meromorphic function on F ([3, p. 428]). Therefore, $F \notin O_L$.

3. Combining known inclusion relations, we obtain the following table:

$$\begin{tikzcd}
 & & O_L \arrow[r] & C_{HB} \arrow[d] & \\
O_G \arrow[r] & O_{HB} \arrow[ur] \arrow[dr] & & & \\
 & & O^\circ_{AB} \arrow[r] & C_{AB} \arrow[r] & O_{AB}
\end{tikzcd}$$

where $\rightarrow$ indicates strict inclusion and there doesn't exist any inclusion relation between O_L and O°_{AB}, C_{HB} and O°_{AB}. For surfaces of finite genus,

$$O_G = C_{HB} \subsetneqq O^\circ_{AB} \subset C_{AB} = O_{AB} .$$

References

[1] E. F. Collingwood and M. L. Cartwright: Boundary theorems for a function meromorphic in the unit circle. Acta Math. **87** (1952), 83–146.

[2] M. Heins: Riemann surfaces of infinite genus. Ann. of Math. **55** (2) (1952), 296–317.

[3] ——: Lindelöfian maps. Ann. of Math. **62** (1955), 418–446.

[4] T. Kuroda: On analytic functions on some Riemann surfaces. Nagoya Math. J. **10** (1956), 27–50.

[5] P. J. Myrberg: Über die analytische Fortsetzung von beschränkten Funktionen. Ann. Acad. Sci. Fenn. A.I. 58 (1949).

[6] K. Noshiro: Cluster Sets. Springer (1960).

[7] S. Stoïlow: Lecon sur les principes topologiques de la théorie des fonctions analytiques. Gauthier-Villars (1956).

[8] ——: Sur la théorie topologique des recouvrements Riemanniens. Ann. Acad. Sci. Fenn. A.I. 250/35 (1958).

[9] M. Tsuji: On non-prolongable Riemann surfaces. Proc. Imp. Acad. Japan 19 (1943).

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