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Nagoya Math. J.
Vol. 82 (1981), 27–56

SEMINORMAL RINGS AND WEAKLY NORMAL VARIETIES

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Introduction

In the late sixties and early seventies the operation of weak normalization was formally introduced first in the case of analytic spaces and later in the abstract scheme setting (cf. [6] & [4]). The notion arose from a classification problem. An unfortunate phenomenon in this area occurs when one tries to parametrize algebraic objects associated with a space by an algebraic variety; the resulting variety is, in general, not uniquely determined and may, for example, depend on the choice of coordinates. Under certain conditions one does know that the normalization of the parameter variety is unique. The price one pays for passing to the normalization is that often this variety no longer parametrizes what it was intended to; one point on the original parameter variety may split into several in the normalization. This problem is avoided if one passes instead to the weak normalization of the original variety. One then obtains a variety homeomorphic to the original variety with universal mapping properties that guarantee uniqueness.

In recent years weakly normal complex spaces have been systematically studied by several people and many interesting results have been obtained. On a complex space X define the sheaf of c-holomorphic functions $\mathcal{O}_X^c$ on X as follows. For an open subset U of X let $\Gamma(U, \mathcal{O}_X^c)$ consist of all continuous complex valued functions on U which are holomorphic at the regular points of U. X is weakly normal if $\mathcal{O}_X = \mathcal{O}_X^c$ (where $\mathcal{O}_X$ denotes the sheaf of holomorphic functions on X).

In [2] a generic type singularity called a multicross was defined and was shown to be what most frequently occurs in weakly normal spaces. More precisely, the complement of the multicrosses is an analytic subset of codimension at least two. One has a Hartogs theorem for weak normality (cf. [5] & [2] for a refinement) and an Oka theorem which com-

Received April 26, 1979.

https://doi.org/10.1017/S0027763000019279 Published online by Cambridge University Press

pletely classifies weakly normal spaces which are pure dimensional, local complete intersections (cf. [1]). Within the class of weakly normal spaces there is the subclass of locally optimal spaces; these spaces can be described by requiring that a certain local cohomology sheaf be identically zero.

A purely algebraic notion, called seminormality, appeared in a paper by Traverso [20] in 1970. It was then observed that if one studies the affine rings of schemes over a field of characteristic 0, the affine ring is seminormal if and only if the associated affine scheme is weakly normal in the sense of Andreotti and Bombieri in [4]. Let A be a commutative ring and let B denote its normalization. For a (commutative) ring C let $R(C)$ denote the Jacobson radical of C. The seminormalization ${}^{+}A$ of A is defined by ${}^{+}A = \{b \in B \mid b_x \in A_x + R(B_x)\ \forall x \in \operatorname{Spec}(A)\}$. A is said to be seminormal if $A = {}^{+}A$.

In recent years the algebraic notion has been studied from several different points of view. In [20] Traverso established the connection between the seminormality of a ring R and a certain property of the Picard group $\operatorname{Pic}(R)$. In [9] Gilmer and Heitmann extended this result using a generalization of a criterion for seminormality which we believe first appeared in a paper by Hamann [10]. The K-theorists have also been studying the notion particularly as it applies to curves. The geometric nature of weakly normal curves has been understood for some time. In [7] Bombieri classified the singularities of an irreducible weakly normal curve over an algebraically closed field of characteristic 0 and proved that ordinary singularities arising from generic projections of surfaces in $\boldsymbol{P}^3$ are weakly normal.

Perhaps it is a consequence of the diversity of the investigators' particular fields of interest that the algebraic theory is still disjoint. In the first half of section one of this paper we have gathered the scattered known results of the theory.

The material found in the latter half of section one (as well as results 1.6 and 1.7) is all new. We begin by showing that for a finite integral extension $A \subset B$ the seminormalization of A in B is equal to

$$\bigcap_{x \in \operatorname{Ass}(B/A)} {}^{+}_{x}A \qquad \text{where } {}^{+}_{x}A$$

is the ring obtained from B by gluing over x. This enables us to prove that for a seminormal extension $A \subset B$ as above, the associated primes of B/A give an exact description of the gluing sequence whose existence was established by Traverso's Structure Theorem ([20], Theorem 2.1). In section three we give an example of a seminormal extension $A \subset B$ such that B/A has an embedded prime; hence one must also glue over the embedded primes of B/A in any type of structure theorem. We completely classify the seminormal rings amongst the reduced Cohen-Macaulay rings having finite normalization. We conclude section one by showing that for algebro-geometric local rings seminormality is preserved when one passes to completions.

In the second section we focus our attention on algebraic varieties over an algebraically closed field of characteristic 0. As the notions of seminormality and weak normality are identical in this case we use the latter terminology. The starting point of our treatment is the following theorem (2.2): Let $f\colon Y \to X$ be a dominating finite morphism of affine varieties whose affine coordinate rings are B and A respectively. Then ${}^{+}_{B}A$ consists of all regular functions on Y that agree on the fibres of f. Using this result we develop the close connection that exists between the algebraic and complex space theories.

We define the sheaf of c-regular functions on a variety and then show that a variety is weakly normal if and only if every c-regular function is regular. We pursue the function theoretic approach and in order to provide a firm foundation for future study we provide the proofs of several results that are probably well known by many. This point of view enables us to quickly obtain many results. For example, we are easily able to define the weak normalization of a variety and clearly state its universal mapping properties. We also show that the product of weakly normal varieties is weakly normal. We establish the equivalence of the weak normality of a complex algebraic variety and the weak normality of the associated complex analytic space. Finally, we give criteria for determining when the union of weakly normal varieties is again weakly normal and use these results to establish the weak normality of several general classes of varieties in section three.

In the future we plan to develop the algebro-geometric analogue of the (complex space) multicross singularity and determine the role that this class of singularities and local cohomology play in the singularity theory of weakly normal varieties.

§ 1. Preliminaries

All rings in this paper are commutative with identity. If A is a ring we will let $R(A)$ denote the Jacobson radical of A, i.e. $R(A)$ is the intersection of all maximal ideals of A. If $x \in \operatorname{Spec}(A)$ corresponds to the prime ideal $\mathfrak{p}_x$ and M is an A-module we let M_x denote $S^{-1}M$ where the multiplicative subset $S = A - \mathfrak{p}_x$. If $m \in M$ we let m_x and $m(x)$ denote the images of m under the canonical homomorphisms: $M \to M_x$ and $M \to M_x/\mathfrak{p}_x M_x$ respectively. Finally we let $\kappa(x) = A_x/\mathfrak{p}_x A_x$ and $\Omega(A) = \{x \in \operatorname{Spec}(A) \mid x \text{ is a closed point}\}$. We now give a simplified version of a result by Andreotti and Bombieri ([4], § 1, Proposition 3)

PROPOSITION 1.1. *Let $A \subset B$ be an integral extension of rings. Then*
(1) $R(B) \cap A = R(A)$
(2) *if $(A, \mathfrak{m})$ is local then $C = A + R(B)$ is a local rings with maximal ideal $R(B)$ and the canonical map: $A/\mathfrak{m} \to C/R(B)$ is an isomorphism.*

Proof. (1) follows from knowing that the induced map $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is surjective and that if $\mathfrak{p} = P \cap A$ for some prime ideal P of B then $\mathfrak{p}$ is maximal if and only if P is maximal.

(2) Suppose (A, $\mathfrak{m}$) is local. Then $A + R(B)$ is clearly a subring of B and the canonical map $A/\mathfrak{m} \to A + R(B)/R(B)$ is an isomorphism. Hence $R(B)$ is a maximal ideal of C. But by (1) we have $R(C) = R(B) \cap C = R(B)$ so that $R(B)$ is the unique maximal ideal of C.

DEFINITIONS 1.2. Let $A \subset B$ be an integral extension of rings. We define

$$ {}^{+}_{B}A = \{b \in B \mid b_x \in A_x + R(B_x),\ \forall x \in \operatorname{Spec}(A)\} $$

${}^{+}_{B}A$ is called the seminormalization of A in B and if $A = {}^{+}_{B}A$ then we say that A is seminormal in B. If B is the normalization of A (i.e. the integral closure of A in its total ring of quotients) we set ${}^{+}A = {}^{+}_{B}A$ and we say that A is seminormal if $A = {}^{+}A$. ${}^{+}A$ is called the seminormalization of A.

We now recall Traverso's characterization of the operation of seminormalization.

PROPOSITION 1.3 (Traverso [20]). *${}^{+}_{B}A$ is the largest subring A' of B containing A such that:*

(1) *For each* $x \in \operatorname{Spec}(A)$ *there is exactly one* $x' \in \operatorname{Spec}(A')$ *over* x, *and*

(2) *The canonical homomorphism* $\kappa(x) \to \kappa(x')$ *is an isomorphism.*

We note that this characterization of ${}^{+}_{B}A$ entails that if A is seminormal in B then A is seminormal in every intermediary ring C lying between A and B.

We now present several equivalent criteria for determining when A is seminormal in an overring B. In the case where A is a pseudo-geometric ring and B is the normalization of A the equivalence of (1)–(3) below was first proven by Hamann in [10]. In [9] Gilmer and Heitmann prove this equivalence for an arbitrary ring A and its normalization B. We include criterion (4) here as it is the most manageable in actual computations. We include the proof of Gilmer and Heitmann for (2) implies (1).

PROPOSITION 1.4. *For an integral extension* $A \subset B$ *the following statements are equivalent.*

(1) *A is seminormal in B.*

(2) *For each b in B, the conductor of A in $A[b]$ is a radical ideal of $A[b]$.*

(3) *A contains each element b of B such that $b^n, b^{n+1} \in A$ for some positive integer n.*

(4) *For a fixed pair of relatively prime integers $e > f > 1$, A contains each element b of B such that $b^e, b^f \in A$.*

Proof. Statements (3) and (4) are each equivalent to demanding that A contain each element b of B such that $b^m, b^{m+1}, \cdots$ are in A, for some positive integer m.

$(3) \Rightarrow (2)$. Suppose (3) is valid, $f \in A[b]$ and $f^n \in (A\colon A[b])$. Then for any $g \in A[b]$ we have $(fg)^n, (fg)^{n+1} \in A$ so that $fg \in A$ by (3). Thus $f \in (A\colon A[b])$ as desired.

$(1) \Rightarrow (3)$. Suppose $A = {}^{+}_{B}A$ and that $b \in B$, $b^n, b^{n+1} \in A$ for some positive integer n. Let $x \in \operatorname{Spec}(A)$ be arbitrary. Then $b_x^n, b_x^{n+1} \in A_x$. If b_x^n is a unit in A_x, then $b_x = b_x^{n+1}/b_x^n \in A_x$. Otherwise, $b_x^n \in \mathfrak{p}_x A_x \subset R(B_x)$ and since the latter is a radical ideal in B_x, $b_x \in R(B_x)$. In any case, $b_x \in A_x + R(B_x)$ and since x was arbitrary, $b \in {}^{+}_{B}A = A$.

$(2) \Rightarrow (1)$. Suppose that (2) holds but A is not seminormal in B. Choose $b \in {}^{+}_{B}A \backslash A$ and a minimal prime $\mathfrak{p}_x$ of $(A\colon A[b])$ in A. Let $\mathfrak{c} =$

https://doi.org/10.1017/S0027763000019279 Published online by Cambridge University Press

$(A: A[b])$, $C = A[b]$. Since $A[b]$ is finitely generated as an A-module we have $\mathfrak{p}_x A_x = \mathfrak{c}_x = (A_x: C_x)$. By assumption, $\mathfrak{c}$ is a radical ideal of C and hence $\mathfrak{c}_x$ is a radical ideal of C_x. Viewing $\mathfrak{c}_x$ as the intersection of its minimal primes in C_x we obtain $\mathfrak{p}_x A_x = \mathfrak{c}_x = R(C_x)$. Now $({}^+_B A)_x$ is local with residue field canonically isomorphic to $\kappa(x)$ and $A_x \subset C_x \subset ({}^+_B A)_x$. Hence C_x is local with maximal ideal $R(C_x)$ and residue field canonically isomorphic to $\kappa(x)$. Furthermore, $\mathfrak{p}_x A_x = R(C_x)$ and $\bar{1}$ generates $C_x/\mathfrak{p}_x C_x$ as a $\kappa(x)$-module so by Nakayama's lemma 1 generates C_x as an A_x-module. That is, $A_x = C_x$, contradicting the fact that $(A_x: C_x) = \mathfrak{p}_x A_x$.

COROLLARY 1.5. *Let $A \subset B$ be an integral extension of rings.*

(1) *If A is seminormal in B then $(A: B)$ is a radical ideal of B.*

(2) *Suppose that $A \subset B \subset C$. If A is seminormal in B and B is seminormal in C, then A is seminormal in C.*

Proof. (1) Assume that A is seminormal in B. Suppose that $b \in B \backslash (A: B)$ but $b^n \in (A: B)$ for some $n > 1$. Choose $c \in B$ such that $bc \notin A$. Then $(bc)^n = b^n c^n \in A$ since $b^n \in (A: B)$ and similarly $(bc)^{n+1} \in A$, contradicting criterion (3) of (1.4) above.

(2) is an immediate consequence of criterion (3) of (1.4).

COROLLARY 1.6. *If A is seminormal in B and S is any multiplicative subset of A, then $S^{-1}A$ is seminormal in $S^{-1}B$. Moreover, the operations of seminormalization and localization commute.*

Proof. We show that condition (3) of (1.4) is valid for $S^{-1}A \subset S^{-1}B$ if it is valid for $A \subset B$.

Suppose that $b/s \in S^{-1}B$ and that $(b/s)^n, (b/s)^{n+1} \in S^{-1}A$ for some positive integer n. Then there exist elements t and t' of S such that $tb^n, t'b^{n+1} \in A$. Thus $(tt'b)^n, (tt'b)^{n+1} \in A$ and hence $tt'b \in A$. Therefore $b/s \in S^{-1}A$.

In general if we let $C = {}^+_B A$ then $S^{-1}C$ is seminormal in $S^{-1}B$ and clearly $S^{-1}C \subseteq {}_{S^{-1}}{}^+_B S^{-1}A$. Then $S^{-1}C$ is seminormal in ${}_{S^{-1}}{}^+_B S^{-1}A$ and since conditions (1)–(2) of (1.3) hold for $S^{-1}C \subset {}_{S^{-1}}{}^+_B S^{-1}A$ we must have $S^{-1}C = {}_{S^{-1}}{}^+_B S^{-1}A$ by (1.3).

PROPOSITION 1.7. *A is seminormal in B if and only if A_x is seminormal in B_x for each $x \in \Omega(A)$.*

Proof. The assertion in one direction follows from (1.6). Conversely, suppose that A_x is seminormal in B_x for each $x \in \Omega(A)$ but A is not seminormal in B. Using the fourth criterion for seminormality of (1.4) there exists an element $b \in B \backslash A$ such that $b^2, b^3 \in A$. As A_x is seminormal in B_x for each $x \in \Omega(A)$ we have $b_x \in A_x$ for each $x \in \Omega(A)$.

However, since $b \notin A$ there exists some maximal ideal $\mathfrak{m}_x$ of A containing $(A: b)$. Then $(A_x: b_x) = (A: b)_x \subseteq \mathfrak{m}_x A_x$, contradicting the fact that $b_x \in A_x$. Hence A is seminormal in B.

Henceforth we shall always assume that $A \subset B$ is a finite integral extension of noetherian rings. We now recall Traverso's notion of gluing.

Let $x \in \operatorname{Spec}(A)$ and let $x_1, \cdots, x_n$ denote the points in $\operatorname{Spec}(B)$ lying over x. For each i let $\omega_i: \kappa(x) \to \kappa(x_i)$ denote the canonical homomorphism.

Define ${}^+_xA = \{b \in B \mid b_x \in A_x + R(B_x)\}$. One can easily see that an element b of B lies in ${}^+_xA$ if and only if:

(1) $b(x_i) \in \omega_i(\kappa(x))$ for each i, and

(2) $\omega_i^{-1}(b(x_i)) = \omega_j^{-1}(b(x_j))$ for all i, j.

DEFINITION 1.8. We say the ring ${}^+_xA$ constructed above is the ring obtained from gluing B over x or that ${}^+_xA$ is the gluing of A in B over x.

PROPOSITION 1.9 (Traverso [20]). *${}^+_xA$ is the largest subring A' of B containing A such that*

(1) *There is exactly one $x' \in \operatorname{Spec}(A')$ over x, and*

(2) *The canonical homomorphism $\kappa(x) \to \kappa(x')$ is an isomorphism.*

In particular, ${}^+_xA$ is seminormal in B.

Remarks 1.10. (1) Let $C = {}^+_BA$ where $A \subset B$ is a finite integral extension of noetherian rings. Then $(A: C)$ is not an intersection of primes in C. For if so, let $\mathfrak{p}_x$ be a minimal prime of $(A: C)$ in A. Then $\mathfrak{p}_x A_x = (A: C)_x = (A_x: C_x) = R(C_x)$ where the latter equality follows from viewing $(A_x: C_x)$ as the intersection of its minimal primes in C_x. Arguing as we did in the proof of (1.4) this implies $A_x = C_x$, a contradiction.

(2) Suppose A is seminormal in B where $A \subset B$ is a finite integral extension and A is noetherian. Then we know $\mathfrak{c} = (A: B)$ is a radical ideal of B and hence is a radical ideal of A. Suppose that $\mathfrak{p}_x \subseteq A$ is a minimal prime of $\mathfrak{c}$.

Then A_x is seminormal in B_x; in fact A_x is its own gluing in B_x over $\mathfrak{p}_x A_x$. We have $(A_x: B_x) = \mathfrak{c}_x = \mathfrak{p}_x A_x$ (since B is an A-module of finite type). Also $\mathfrak{c}$ is a radical ideal in B so that viewing $\mathfrak{c}$ as the intersection of its minimal primes in B we have $\mathfrak{c}_x = R(B_x)$. Thus $A_x = A_x + R(B_x)$ and A_x is its own gluing in B_x over $\mathfrak{p}_x A_x$.

In particular if A is a reduced noetherian ring of dimension one with finite normalization B then A is seminormal if and only if A_x is its own gluing in B_x for all closed points $x \in \operatorname{Spec}(A)$. (cf. [7], p. 208).

(3) Let $A \subset B$ be as above and let $C = {}^+_x A$. Let x' denote the unique prime of $\operatorname{Spec}(C)$ over x. Then C is equal to the gluing of C in B over x' by the characterization of ${}^+_{x'}C$ given in (1.9).

We now prove a sequence of results that enable us to obtain refinements of Traverso's structure theorem and its consequence (cf. [20], Theorems 2.1 and 2.3).

LEMMA 1.11. *Let $A \subset B$ be as before. Suppose that $(A: B)$ is a reduced ideal of B and is a prime ideal of A. Then $A = {}^+_x A$ where $\mathfrak{p}_x = (A: B)$ if and only if* $\operatorname{Ass}_A(B/A) = \{\mathfrak{p}_x\}$.

Proof. As we assumed that $\mathfrak{p}_x = (A: B)$ is a reduced ideal of B we must have $\mathfrak{p}_x = \mathfrak{p}_x B = \sqrt{\mathfrak{p}_x B} = P_1 \cap \cdots \cap P_n$ where $P_1, \cdots, P_n$ are the minimal primes in B of $\mathfrak{p}_x B$. Then $\mathfrak{p}_x A_x = (A: B)_x = (A_x: B_x) = R(B_x)$ so that $A_x = A_x + R(B_x)$ and hence ${}^+_x A = \{b \in B \mid b_x \in A_x\} = \{b \in B \mid (A: b) \not\subset \mathfrak{p}_x\}$. Also note that $\mathfrak{p}_x = (A: B) = (0: B/A)$ entails that $\mathfrak{p}_x$ is the minimal associated prime of B/A.

Thus $A = {}^+_x A \Leftrightarrow b \in A$ whenever $(A: b) \not\subset \mathfrak{p}_x \Leftrightarrow$ every associated prime of B/A is contained in $\mathfrak{p}_x \Leftrightarrow \operatorname{Ass}_A(B/A) = \{\mathfrak{p}_x\}$.

THEOREM 1.12. *A is seminormal in B if and only if $A = \{b \in B \mid b_x \in A_x + R(B_x)\ \forall \mathfrak{p}_x \in \operatorname{Ass}_A(B/A)\}$.*

Proof. Let $A' = \{b \in B \mid b_x \in A_x + R(B_x)\ \forall \mathfrak{p}_x \in \operatorname{Ass}_A(B/A)\}$. Then ${}^+_B A \subseteq A'$ so that if $A = A'$ then A is seminormal in B.

Conversely, assume that A is seminormal in B. To see that $A = A'$ we proceed by induction on $r = \# \operatorname{Ass}_A(B/A)$ the case $r = 0$ being trivial.

Let $r \geq 1$ and assume the assertion is true whenever $\# \operatorname{Ass}_A(B/A) < r$. Let $A \subset B$ be such that $\# \operatorname{Ass}_A(B/A) = r$ and choose $x_1 \in \operatorname{Spec}(A)$ so that $\mathfrak{p}_{x_1}$ is a minimal prime of B/A. Then $\mathfrak{p}_{x_1}$ is a minimal prime of $(A: B)$ and $A_{x_1} = A_{x_1} + R(B_{x_1})$ as in (1.9).

Set $B' = {}^+_{x_1} A$ and let x' denote the unique prime of B' lying over x_1. Then $B' = {}^+_{x'} B'$ so that $\operatorname{Ass}_{B'}(B/B') = \{\mathfrak{p}_{x'}\}$ by (1.11) and hence $\operatorname{Ass}_A(B/B') = \{\mathfrak{p}_{x_1}\}$.

Consider the exact sequence of A-modules: $0 \to B'/A \to B/A \to B/B' \to 0$. As $A_{x_1} = A_{x_1} + R(B_{x_1})$ we have $B' = \{b \in B \mid (A: b) \not\subset \mathfrak{p}_{x_1}\}$ and consequently $\mathfrak{p}_{x_1}$ is not an associated prime of B'/A. Then

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$$\operatorname{Ass}_A(B/A) = \operatorname{Ass}_A(B'/A) \cup \operatorname{Ass}_A(B/B')$$

so that $\# \operatorname{Ass}_A(B'/A) < r$.

Since A is seminormal in B, A is seminormal in B' and hence by the induction hypothesis

$$A = \{b \in B' \mid b_x \in A_x + R(B'_x)\ \forall \mathfrak{p}_x \in \operatorname{Ass}_A(B'/A)\}$$

Now

$$R(B'_x) = R(B_x) \cap B'_x$$

so that

$$\begin{aligned} A &= \{b \in B' \mid b_x \in A_x + R(B_x)\ \forall \mathfrak{p}_x \in \operatorname{Ass}_A(B'/A)\} \\ &= \{b \in B \mid b_x \in A_x + R(B_x)\ \forall \mathfrak{p}_x \in \operatorname{Ass}_A(B/A)\}\ . \end{aligned}$$

Suppose that A is seminormal in B. Index the associated primes of B/A so that $\operatorname{ht} \mathfrak{p}_{x_1} \leq \operatorname{ht} \mathfrak{p}_{x_2} \leq \cdots \leq \operatorname{ht} \mathfrak{p}_{x_r}$. Define subrings of B inductively as follows: Set $B^0 = B$ and define

$$B^i = \{b \in B^{i-1} \mid b_{x_i} \in A_{x_i} + R(B^{i-1}_{x_i})\}\ , \qquad (i = 1, \cdots, r)\ .$$

As $R(B^{i-1}_{x_i}) = R(B_{x_i}) \cap B^{i-1}_{x_i}$ we also have $B^i = \{b \in B^{i-1} \mid b_{x_i} \in A_{x_i} + R(B_{x_i})\}$.

THEOREM 1.13. *Suppose that A is seminormal in B with the notation as above. Then:*

(1) *B^i is obtained from B^{i-1} by gluing over x_i $(i = 1, \cdots, r)$.*

(2) *$\operatorname{Ass}_A(B^i/A) = \{\mathfrak{p}_{x_{i+1}}, \cdots, \mathfrak{p}_{x_r}\}$ $(i = 1, \cdots, r)$.*

(3) *$A = B^r \subsetneqq \cdots \subsetneqq B^1 \subsetneqq B^0 = B$.*

Proof. (1) follows from the definition of the subrings B^i and (3) follows from (2).

To see (2) we mimic the proof of (1.11). Suppose that $\operatorname{Ass}_A(B^{i-1}/A) = \{\mathfrak{p}_{x_i}, \cdots, \mathfrak{p}_{x_r}\}$ and $1 \leq i < r$. Then $\mathfrak{p}_{x_i}$ is a minimal prime of B^{i-1}/A and hence a minimal prime of $(A : B^{i-1})$. As B^i is the gluing in B^{i-1} over x_i we have $\operatorname{Ass}_A(B^{i-1}/B^i) = \{\mathfrak{p}_{x_i}\}$ and $\operatorname{Ass}_A(B^i/A) = \{\mathfrak{p}_{x_{i+1}}, \cdots, \mathfrak{p}_{x_r}\}$ as in the proof of (1.12).

THEOREM 1.14. *Let A be seminormal and suppose in addition that A is reduced with finite normalization B. Let*

$$m = \max\{\{1\} \cup \{\operatorname{ht} \mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass}_A(B/A)\}\}\ .$$

If $a \in A$ is not a zero divisor then every associated prime of A/aA has height no greater than m.

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Proof. Index the associated primes of B/A so that $\operatorname{ht}\mathfrak{p}_{x_1} \le \cdots \le \operatorname{ht}\mathfrak{p}_{x_r}$ and let the subrings B^i $(i = 0, \cdots, r)$ be defined as above. Let $a \in A$ be a non-zero divisor. Then a is not a zero divisor in B since A is reduced and the minimal primes of A and B are in bijective correspondence. Hence a is not a zero divisor in B^i $(i = 0, \cdots, r)$.

We prove by induction on i that if $P \in \operatorname{Ass}_{B^i}(B^i/aB^i)$ then $\operatorname{ht}(P \cap A) \le m$.

Suppose $P \in \operatorname{Ass}_B(B/aB)$ so that $\operatorname{ht} P = 1$. Set $\mathfrak{p} = P \cap A$. If $\operatorname{ht}\mathfrak{p} > 1$ then $(A\colon B) \subset P \cap A = \mathfrak{p}$. For if not $A_\mathfrak{p} = B_\mathfrak{p}$ so that $\operatorname{ht}\mathfrak{p} = \dim A_\mathfrak{p} = \dim B_\mathfrak{p} = \operatorname{ht} PB_\mathfrak{p} = 1$, a contradiction. As $(A\colon B)$ is a reduced ideal of B and is not contained in any minimal prime of B we must have P is a minimal prime of $(A\colon B)$ in B. Thus $\mathfrak{p} \in \operatorname{Ass}_A(B/(A\colon B))$ and in light of the exact sequence of A-modules:

$$0 \longrightarrow A/(A\colon B) \longrightarrow B/(A\colon B) \longrightarrow B/A \longrightarrow 0$$

we have $\mathfrak{p} \in \operatorname{Ass}_A(B/A)$ so that $\operatorname{ht}\mathfrak{p} \le m$.

So let $0 \le i < r$ and assume that if $P \in \operatorname{Ass}_{B^i}(B^i/aB^i)$ then $\operatorname{ht}(P \cap A) \le m$. Let $P \in \operatorname{Ass}_{B^{i+1}}(B^{i+1}/aB^{i+1})$, let $\mathfrak{p} = P \cap A$ and assume that $\operatorname{ht}\mathfrak{p} > m$. Let $\xi \in B^{i+1}\backslash aB^{i+1}$ be such that $P = (aB^{i+1}\colon \xi)$. Then $PB^i\xi \subseteq aB^i$ and hence $\xi \in aB^i$. For if not, $PB^i \subseteq Q$ for some $Q \in \operatorname{Ass}_{B^i}(B^i/aB^i)$ and $\operatorname{ht}(\mathfrak{p}) \le \operatorname{ht}(Q \cap A) \le m$, a contradiction. Thus $\xi \in aB^i$ and $P \subseteq P'$ for some $P' \in \operatorname{Ass}_{B^{i+1}}(aB^i/aB^{i+1})$. As a is not a zero divisor in B^i we have aB^i/aB^{i+1} and B^i/B^{i+1} are isomorphic as B^{i+1}-modules and hence $P \subseteq P'$ for some $P' \in \operatorname{Ass}_{B^{i+1}}(B^i/B^{i+1})$. Then $\mathfrak{p} = P \cap A \subseteq P' \cap A = \mathfrak{p}_{x_{i+1}}$ (by 1.13) so that $\operatorname{ht}\mathfrak{p} \le \operatorname{ht}\mathfrak{p}_{x_{i+1}} \le m$, a contradiction. Hence if $P \in \operatorname{Ass}_{B^{i+1}}(B^{i+1}/aB^{i+1})$ then $\operatorname{ht}(P \cap A) \le m$ and by induction if $\mathfrak{p} \in \operatorname{Ass}_A(A/aA)$ then $\operatorname{ht}\mathfrak{p} \le m$.

Remark 1.15. The m of (1.14) is the best we can do. For in the sequence of (1.13) $\{\mathfrak{p}_r\} = \operatorname{Ass}_A(B^{r-1}/A)$. Hence there exists an element $\beta \in B^{r-1}\backslash A$ such that $\mathfrak{p}_r = (A\colon \beta)$. Suppose $\beta = b/a$ where $b, a \in A$ and a is not a zero divisor. Then $\mathfrak{p}_r = (aA\colon b)$ so that $\mathfrak{p}_r \in \operatorname{Ass}_A(A/aA)$. In particular if A is seminormal then A satisfies the condition S_2 of Serre if and only if every associated prime of B/A is minimal of height one.

We also point out that there exist seminormal extensions $A \subset B$ such that B/A has embedded primes (cf. Example 3.7). Suppose $A \subset B$ is such an extension and $\mathfrak{p} = (A\colon \beta)$ is an embedded prime of B/A. Then $\beta_x \in A_x = A_x + R(B_x)$ for every minimal prime $\mathfrak{p}_x$ of $(A\colon B)$ but $\beta \notin A$. So one must glue over the embedded primes in any type of structure theorem for seminormal extensions (cf. Theorems 1.12 and 1.13).

LEMMA 1.16. *Let A be a reduced noetherian ring with finite normalization B. If $\mathfrak{p}$ is an associated prime of B/A then* depth $A_{\mathfrak{p}} = 1$.

Proof. Since $\mathrm{Ass}_A(B/A) \subseteq \mathrm{Supp}(B/A)$ we know that every associated prime of B/A has height at least one (since A is reduced) and this entails that if $\mathfrak{p} \in \mathrm{Ass}_A(B/A)$ then depth $A_{\mathfrak{p}} \geq 1$.

Let $\mathfrak{p}_x \in \mathrm{Ass}_A(B/A)$. Then we have a short exact sequence: $0 \to A_x \to B_x \to B_x/A_x \to 0$ and in turn a long exact sequence on cohomology:

$$0 \longrightarrow \mathrm{Hom}_{A_x}(\kappa(x), A_x) \longrightarrow \mathrm{Hom}_{A_x}(\kappa(x), B_x) \longrightarrow \mathrm{Hom}_{A_x}(\kappa(x), B_x/A_x) \longrightarrow \mathrm{Ext}^1_{A_x}(\kappa(x), A_x) \longrightarrow \cdots .$$

As A_x is reduced of dimension at least one and B_x is the normalization of A_x we have $\mathrm{Hom}_{A_x}(\kappa(x), B_x) = 0$ and hence $\mathrm{Hom}_{A_x}(\kappa(x), B_x/A_x) \subseteq \mathrm{Ext}^1_{A_x}(\kappa(x), A_x)$. As $\mathfrak{p}_x A_x$ is an associated prime of B_x/A_x we have $\mathrm{Hom}_{A_x}(\kappa(x), B_x/A_x) \neq 0$. Hence $\mathrm{Ext}^1_{A_x}(\kappa(x), A_x) \neq 0$ and depth $A_x = 1$.

At this point we would like to recall another notion of seminormality as defined by Endo in [8]. Suppose that A is a reduced noetherian ring with finite normalization B. We say that A satisfies the condition R_1' of Endo if for each height one prime $\mathfrak{p}_x$ of A, A_x is equal to its own gluing in B_x over $\mathfrak{p}_x A_x$. Endo calls A seminormal if A satisfies R_1' and the condition S_2 of Serre. We now show that this entails that A is seminormal in the sense of Traverso and give an example that illustrates A can be seminormal in the sense of Traverso without satisfying S_2.

LEMMA 1.17. *Let $A \subset B$ be as above and assume that A satisfies conditions R_1' and S_2. Then $A = {}^+A$.*

Proof. If A satisfies S_2 then by (1.16) we may conclude that every associated prime of B/A has height 1. Since A satisfies R_1', $A_x = A_x + R(B_x)$ for each $\mathfrak{p}_x \in \mathrm{Ass}_A(B/A)$. Then

$$\begin{aligned} A' &= \{b \in B \mid b_x \in A_x + R(B_x)\ \forall \mathfrak{p}_x \in \mathrm{Ass}_A(B/A)\} \\ &= \{b \in B \mid b_x \in A_x\ \forall \mathfrak{p}_x \in \mathrm{Ass}_A(B/A)\} \\ &= \{b \in B \mid (A : b) \text{ is not contained in any associated prime of } B/A\} \\ &= A . \end{aligned}$$

so that $A = {}^+A$ by (1.12).

EXAMPLE 1.18 (Two planes in 4-space meeting at the origin). Let x_1,

x_2, x_3, x_4 be transcendentals over the complex numbers $\boldsymbol{C}$, let $\mathfrak{p}_1 = (x_1, x_2)$, $\mathfrak{p}_2 = (x_3, x_4)$ and set $A = \boldsymbol{C}[x_1, x_2, x_3, x_4]/\mathfrak{p}_1 \cap \mathfrak{p}_2$. Then if B denotes the normalization of A we have $B \cong \boldsymbol{C}[x_3, x_4] \times \boldsymbol{C}[x_1, x_2]$ and an exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} \boldsymbol{C} \longrightarrow 0$$

where $\alpha(f + P_1 \cap P_2) = (f(0, 0, x_3, x_4), f(x_1, x_2, 0, 0))$ and $\beta(f, g) = f(0, 0) - g(0, 0)$. Hence $(A : B) = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$ where $\bar{x}_i$ denote the image of x_i in A $(i = 1, \cdots, 4)$. Say $(f, g) \in B$ and $(f^2, g^2) \in A$, $(f^3, g^3) \in A$. Then $f(0, 0)^2 = g(0, 0)^2$ and $f(0, 0)^3 = g(0, 0)^3$ so that $f(0, 0) = g(0, 0)$ and $(f, g) \in A$. Hence A is seminormal by (1.6, (3)). As $(A : B)$ is a height 2 prime in A we see that A does not satisfy S_2 by (1.16).

COROLLARY 1.19. *Let A be a reduced Cohen-Macaulay ring with finite normalization B. Then A is seminormal if and only if $(A : B)$ is a radical ideal in B.*

Proof. As A is Cohen-Macaulay we see that every associated prime of B/A is of height 1 by (1.16).

If A is seminormal then $(A : B)$ is a radical ideal in B.

Conversely assume that $(A : B)$ is a radical ideal in B. If $A \neq B$, then $(A : B)$ is the intersection of height one primes in A. If $\mathfrak{p}_x$ is any minimal prime of $(A : B)$ then $\mathfrak{p}_x A_x = (A_x : B_x)$ is a reduced ideal of B_x and hence $\mathfrak{p}_x A_x = R(B_x)$ so that $A_x = A_x + R(B_x)$. Then A satisfies R_1' and is seminormal by (1.17).

Another important consequence of the structure theorem (1.13) is that in the geometric situation the completion of a seminormal local ring is again seminormal. Toward this end we recall a definition.

DEFINITION 1.20. We say that a local ring A is an algebro-geometric local ring if $A \cong B_{\mathfrak{p}}$ where B is a reduced and finitely generated k-algebra for some algebraically closed field k and $\mathfrak{p} \in \operatorname{Spec}(B)$.

For a local ring $(A, \mathfrak{m})$ and an A-module M, we let $\hat{M}$ denote the $\mathfrak{m}$-adic completion of M.

THEOREM 1.21. *An algebro-geometric local ring is seminormal if and only if its completion is seminormal.*

Proof. Let A be an algebro-geometric local ring and let B denote its normalization. Then B is an A-module of finite type and $\hat{B}$ is the normalization of $\hat{A}$. (See [13], Theorem 72, p. 240 and Theorem 79, p. 259).

First let us assume that A is seminormal. Suppose that $\mathrm{Ass}_A(B/A) = \{\mathfrak{p}_{x_1}, \cdots, \mathfrak{p}_{x_r}\}$ where $\mathrm{ht}\,\mathfrak{p}_{x_1} \leq \cdots \leq \mathrm{ht}\,\mathfrak{p}_{x_r}$. Let $A = B^r \subseteq \cdots \subseteq B^0 = B$ be the sequence of subrings of B as in (1.13). Letting $C^i = (B^i)^\wedge$ we proceed by induction to show that C^i is seminormal in $C = C^0$ for $i = 0, \cdots, r$, the case $i = 0$ being trivial.

So assume that $0 \leq i < r$ and that C^i is seminormal in C. Recall that B^{i+1} was defined to be the gluing of A in B^i over x_{i+1}. If $y \in \mathrm{Spec}\,(B^{i+1})$ is the unique point lying over x_{i+1}, then $B^{i+1} = {}^+_y B^{i+1}$ and $\mathrm{Ass}_{B^{i+1}}(B^i/B^{i+1}) = \{\mathfrak{p}_y\}$. Since C^{i+1} is also the $R(B^{i+1})$-adic completion of B^{i+1} we have $\mathrm{Ass}_{C^{i+1}}(C^i/C^{i+1}) = \mathrm{Ass}_{C^{i+1}}(C^{i+1}/\mathfrak{p}_y C^{i+1})$ by ([13], Theorem 12, p. 58).

As $\mathfrak{p}_y C^{i+1}$ is a reduced ideal ([13], Theorem 79, p. 259) we know that every associated prime of C^i/C^{i+1} is a minimal prime of $(C^{i+1} \colon C^i) = (B^{i+1} \colon B^i)^\wedge$ ([15], Theorem 18.1, p. 58) and the latter is a reduced ideal of C^i. Consequently, if P_z is a minimal prime (in C^{i+1}) of $(C^{i+1} \colon C^i)$ then $C_z^{i+1} = C_z^{i+1} + R(C_z^i)$.

Finally,

$$\begin{aligned} C^{i+1} &= \{\beta \in C^i \mid \beta_z \in C_z^{i+1}\ \forall\, P_z \in \mathrm{Ass}_{C^{i+1}}(C^i/C^{i+1})\} \\ &= \{\beta \in C^i \mid \beta_z \in C_z^{i+1} + R(C_z^i)\ \forall\, P_z \in \mathrm{Ass}_{C^{i+1}}(C^i/C^{i+1})\} \end{aligned}$$

and C^{i+1} is seminormal in C^i by (1.12). Since C^{i+1} is seminormal in C^i and C^i is seminormal in C we have C^{i+1} is seminormal in C by (1.5). By induction, $\hat{A} = C^r$ is seminormal in $\hat{B} = C$.

Conversely, assume that $\hat{A}$ is seminormal in $\hat{B}$. Suppose $b \in B$ and $b^2, b^3 \in A$. Then $b \in B \cap \hat{A} = A$ by (1.4). Hence A is seminormal in B by (1.4).

§ 2. The case of algebraic varieties

Let k be a fixed algebraically closed field of characteristic 0. When we use the term variety we assume that the underlying topological space is the set of closed points of a reduced, separated scheme of finite type over k. Unless otherwise stated all rings in this section are assumed to be k-algebras. We will say A is an affine ring if A is the coordinate ring of an affine variety (over k). In particular, if A is an affine ring with total ring of quotients K and L is any finite extension of K, then the integral closure A_L of A in L is a finite A-module. ([13], Theorem 72, p. 240).

By assuming char $k = 0$ we avoid all inseparability problems and as was pointed out in the introduction the operations of seminormalization and weak normalization coincide. Henceforth we will use the latter terminology; i.e., if $A \subset B$ is an integral extension of k-algebras we will call ${}^{+}_{B}A$ the weak normalization of A in B. A main result here is that if $A \subset B$ is a finite integral extension of affine rings and we let $\pi\colon Y = \mathrm{Var}(B) \to X = \mathrm{Var}(A)$ denote the induced morphism of varieties then ${}^{+}_{B}A$ consists of all regular functions f on Y such that $f(y_1) = f(y_2)$ whenever $\pi(y_1) = \pi(y_2)$. To see this we recall the following well known result.

LEMMA 2.1. *Let $\pi\colon Y \to X$ be a dominating finite morphism of irreducible varieties and let $n = [K(Y)\colon K(X)]$. Then there is a non-empty open subset U of X such that for each $x \in U$ the fibre $\pi^{-1}(x)$ consists of n distinct points.*

Proof. Replacing X by a non-empty open V such that V is normal and Y by $\pi^{-1}(V)$ we can assume that X is normal. Since char $k = 0$, π is a separated morphism and the result can be found in [19] (Theorem 7, p. 116).

THEOREM 2.2. *Let $A \subset B$ be a finite integral extension of affine rings and define A' by $A' = \{b \in B \mid b_x \in A_x + R(B_x)\ \forall x \in X = \mathrm{Var}(A)\}$. Then $A' = {}^{+}_{B}A$. Thus if $\pi\colon Y = \mathrm{Var}(B) \to X$ is the induced morphism, then ${}^{+}_{B}A$ consists of all regular functions f on Y such that $f(y_1) = f(y_2)$ whenever $\pi(y_1) = \pi(y_2)$.*

Proof. By definition we have ${}^{+}_{B}A \subset A'$. To prove equality it suffices to see that:

(1) For each point $x \in \mathrm{Spec}(A)$ there is exactly one point $x' \in \mathrm{Spec}(A')$ over x, and

(2) The canonical map $\kappa(x) \to \kappa(x')$ is an isomorphism.

Let $x \in X = \mathrm{Var}(A)$. Since $A_x \subseteq A'_x \subseteq A_x + R(B_x)$ and the latter is a local ring with residue field k (by (1.1)) we see that A'_x is local. Hence there exists a unique point x' in $\mathrm{Spec}(A')$ over x. Thus (1) and (2) are valid for all $x \in X$.

Then (1) is valid for all points in $\mathrm{Spec}(A)$ by a standard application of the Nullstellensatz. For let $\mathfrak{p}$ be an arbitrary prime in A and suppose P_1 and P_2 are primes in A' over $\mathfrak{p}$. Let M be any maximal ideal of A' containing P_1 and let $\mathfrak{m} = M \cap A$. By Going Up there exists a maximal ideal $M' \supset P_2$ lying over $\mathfrak{m}$. Since (1) above holds for $\mathfrak{m}$ we have $M = M' \supset P_2$. Thus every maximal ideal containing P_1 also contains P_2 and hence $P_2 \subset P_1$. Similarly, $P_1 \subset P_2$ so that $P_1 = P_2$.

Now let $\mathfrak{p} \subset A$ be an arbitrary prime and let P be the unique prime in A' over $\mathfrak{p}$. Then $A/\mathfrak{p} \subset A'/P$ is a finite integral extension of domains with quotient fields $\kappa(\mathfrak{p})$ and $\kappa(P)$, respectively. Since the induced morphism of varieties is a homeomorphism we must have $[\kappa(P):\kappa(\mathfrak{p})] = 1$ by (2.1). Thus the canonical map $\kappa(\mathfrak{p}) \to \kappa(P)$ is an isomorphism. Hence $A' = {}^+_BA$.

Clearly, if $f \in B$ then $f \in A'$ if and only if $f(y_1) = f(y_2)$ whenever $\pi(y_1) = \pi(y_2)$ where $\pi: Y = \mathrm{Var}(B) \to X$ is the induced morphism.

We should point out that this result was first indicated by observing that this is precisely how one defines the holomorphic functions on the weak normalization of a complex space. The following sequence of results shows that the algebraic notion is entirely analogous to the complex space concept. In order to present the material in a unified fashion we will present some results that are probably well known by many.

Remark 2.3. A third characterization of ${}^+_BA$ where $A \subset B$ is a finite integral extension of affine rings is as follows. Let $A' = \{b \in B \mid b \otimes 1 - 1 \otimes b \text{ is nilpotent in } B \otimes_A B\}$. Let $\pi: \mathrm{Var}(B) \to \mathrm{Var}(A)$ be the induced morphism. Then $b \in A' \Leftrightarrow b \otimes 1 - 1 \otimes b$ is in every maximal ideal of $B \otimes_A B \Leftrightarrow b(y_1) = b(y_2)$ whenever $\pi(y_1) = \pi(y_2) \Leftrightarrow b \in {}^+_BA$. In particular, the Lipschitz-saturation $A^*_{B,k}$ of A in B relative to $k \subset A \subset B$ (cf. [12]) is contained in ${}^+_BA$.

Definitions 2.4. The sheaf of c-regular functions on a variety X, denoted by $\mathcal{O}^c_X$, is defined as follows: If $U \subset X$ is open, then $\Gamma(U, \mathcal{O}^c_X)$ consists of all continuous k-valued functions on U which are regular at the nonsingular points of U.

We say that X is weakly normal at $x \in X$ if $\mathcal{O}_{X,x} = \mathcal{O}^c_{X,x}$. X is weakly normal if $\mathcal{O}_X = \mathcal{O}^c_X$.

If Y is a variety we shall always let $S(Y)$ denote the singular locus of Y.

Lemma 2.5. *A normal variety is weakly normal.*

Proof. It suffices to assume that $X = \mathrm{Var}(A)$ is affine and show that $A = \Gamma(X, \mathcal{O}_X) = \Gamma(X, \mathcal{O}^c_X)$. Suppose that $\varphi: X \to k$ is continuous and $\varphi|_{X-S(X)}$

https://doi.org/10.1017/S0027763000019279 Published online by Cambridge University Press

is regular. If I is the ideal of A defining $S(X)$, then ht $I \geq 2$ since A is normal. Since A satisfies condition S_2 of Serre and I is reduced we must have I-depth $A \geq 2$. Then looking at the long exact sequence of cohomology with support on $S(X)$ we see that the restriction map $A = \Gamma(X, \mathcal{O}_X) \to \Gamma(X - S(X), \mathcal{O}_X)$ is an isomorphism (see [11], Example 2.3 (p. 212), Examples 3.3–3.4 (p. 217).). Hence there exists a regular function f on X such that $f|_{X-S(X)} = \varphi|_{X-S(X)}$. Since $\varphi - f$ is globally continuous and vanishes on the dense open subset $X - S(X)$ of X, it must be identically zero. Thus $\varphi = f$ is regular on X.

PROPOSITION 2.6. *A continuous k-valued function on a variety X regular on some dense open subset of X is c-regular on X.*

Proof. Suppose $\varphi: X \to k$ is continuous and regular on the dense open subset U of X. Let $x \in X$ be a non-singular point. Let V be an irreducible affine open neighborhood of x. Then φ is regular on $V \cap U$. So there exist regular functions f, g on V with $g \neq 0$ such that $\varphi|_{V_g} = f/g|_{V_g}$. Then $g\varphi - f$ is zero on V_g so that $g\varphi = f$ on V and in particular $g_x\varphi_x = f_x$ as germs of continuous functions. We claim that this implies g_x divides f_x in $\mathcal{O}_{X,x}$.

Since x is a non-singular point $\mathcal{O}_{X,x}$ is a UFD. Factor g_x in $\mathcal{O}_{X,x}$ so that $g_x = \prod p_j^{s_j}$ and each germ p_j is prime in $\mathcal{O}_{X,x}$. Since $\prod p_j^{s_j}\varphi_x = f_x$ we can find an affine open neighborhood W of x in V such that $p_1\Gamma(W, \mathcal{O}_X)$ is prime and $\prod p_j^{s_j}\varphi|_W = f|_W$. So f vanishes on $W(p_1)$ and hence $p_1f_1 = f$ for some $f_1 \in \Gamma(W, \mathcal{O}_X)$. By induction on $\sum s_j$ we may conclude that $g_xh_x = f_x$ for some $h_x \in \mathcal{O}_{X,x}$. Then $g_x(h_x - \varphi_x) = 0$ and hence $h = \varphi$ in some neighborhood of x.

(2.7) Suppose $X = \mathrm{Var}(A)$ is affine. Let B denote the normalization of X, $\tilde{X} = \mathrm{Var}(B)$ and $\pi: \tilde{X} \to X$ the projection. Suppose $\varphi \in \Gamma(X, \mathcal{O}_X^c)$. Then by (2.6) $\varphi \circ \pi$ is c-regular on $\tilde{X}$ and hence regular. By (2.2) we have $\varphi \circ \pi \in {}^+A$. Conversely if $f \in {}^+A$ then f is regular on $\tilde{X}$ and is constant on the fibres of π. Hence f induces a continuous function on X which is regular off the singular locus of X. Now if S is any multiplicative subset of A then ${}^+(S^{-1}A) = S^{-1}({}^+A)$ by (1.6). Hence for an arbitrary variety X the sheaf $\mathcal{O}_X^c$ is coherent. In particular, the set $W(X)$ of non-weakly normal points is a closed subset of X.

COROLLARY 2.8. *Suppose that X is weakly normal and that $f: Y \to X$ is a finite birational morphism. If f is a homeomorphism then f is an* *isomorphism (of varieties).*

Proof. Suppose that f is a homeomorphism. Let $U = \mathrm{Var}(A)$ be any affine open in X so that $f^{-1}(U) = \mathrm{Var}(B)$ is again affine. Then $f^{\#}: A \to B$ is an inclusion of affine rings and induces an isomorphism on the respective total quotient rings. By abuse, let us assume $A \subset B$. Then A is weakly normal in B (since A is weakly normal) and B is contained in the normalization of A. But f is a homeomorphism so that every regular function on $f^{-1}(U)$ agrees on the fibres of f. Hence $A = B$ by (2.2). Thus f is an isomorphism.

THEOREM 2.9. *Let X be a variety. Then there exists an essentially unique pair (X^w, π) consisting of a weakly normal variety X^w together with a finite birational morphism $\pi: X^w \to X$ which is a homeomorphism. By essentially unique we mean that if (X_1, π_1) is any other such pair, then there is a unique morphism $\eta: X_1 \to X^w$ such that $\pi \circ \eta = \pi_1$ and η is an isomorphism.*

Proof. Let $\varphi: \tilde{X} \to X$ be the normalization of X. Define an equivalence relation $\mathscr{R}$ on $\tilde{X}$ by $x_1 \underset{\mathscr{R}}{\sim} x_2$ if and only if $\varphi(x_1) = \varphi(x_2)$. Let $X^w = \tilde{X}/\mathscr{R}$ and let $\rho: \tilde{X} \to X^w$ be the projection. Let $\pi: X^w \to X$ be the (uniquely determined) continuous map such that $\pi \circ \rho = \varphi$. Clearly π is a homeomorphism. Define a sheaf of k-valued functions $\mathscr{O}_{X^w}$ on X^w as follows. For $U \subset X^w$ open let $\Gamma(U, \mathscr{O}_{X^w}) = \{f \circ \pi \mid f \in \Gamma(\pi(U), \mathscr{O}_X^c)\}$.

By our earlier remarks, if $\mathrm{Var}(A) = V$ is an affine open in X and $\varphi^{-1}(V) = \mathrm{Var}(B)$, then $(\pi^{-1}(V), \mathscr{O}_{X^w}|_{\pi^{-1}(V)}) \simeq \mathrm{Var}({}^{+}_{B}A)$. Hence π is finite and birational.

To see that X^w is separated suppose $\alpha, \beta: Y \to X^w$ are morphisms. Then $\{y \in Y \mid \alpha(y) = \beta(y)\} = \{y \in Y \mid \pi \circ \alpha(y) = \pi \circ \beta(y)\}$ is closed in Y. We also note that X^w is weakly normal. For if $U \subset X^w$ is open and f is c-regular on U then $f \circ \pi^{-1}$ is c-regular on $\pi(U)$, where π^{-1} is the topological inverse of π. Hence $f = f \circ \pi^{-1} \circ \pi$ is regular on U.

Finally, suppose (X_1, π_1) is another such pair. Then there exists a unique continuous map $\eta: X_1 \to X^w$ such that $\pi \circ \eta = \pi_1$. Since X_1 is weakly normal, η must carry regular functions on X^w to regular functions on X_1; i.e. η is a morphism. Then η is an isomorphism by (2.8).

The pair (X^w, π) constructed in (2.9) is called the weak normalization of X. We could have defined it by mimicing the construction of the normalization of X. That is, cover X by affine opens $\mathrm{Var}(A_i)$ and show that the affine varieties $\mathrm{Var}\,({}^+A_i)$ can be glued to obtain a variety X^w with the specified properties. However, the function theoretic approach enables us to quickly obtain results which seem obscure from the algebraic point of view. Towards this end we prove the following lemma.

LEMMA 2.10. *Let X be a variety. A k-valued function φ on X is c-regular if and only if its graph Γ_φ is closed in $X \times A^1$.*

Proof. Suppose that $\Gamma_\varphi = \{(x, \varphi(x)) \mid x \in X\}$ is closed in $X \times A^1$.

Since any proper closed subset of A^1 is a finite set of points, to see that φ is continuous it suffices to show that $\varphi^{-1}(\alpha)$ is closed for each $\alpha \in k$. Now $\Gamma_\varphi \cap X \times \{\alpha\} \to X$ is a closed mapping. Hence $\varphi^{-1}(\alpha) = p_1(\Gamma_\varphi \cap X \times \{\alpha\})$ is closed in X and φ is continuous.

To see that φ is c-regular it suffices to assume that $X = \mathrm{Var}\,(A)$ is a non-singular irreducible affine variety. Thus A is a regular ring and $X \times A^1 \simeq \mathrm{Var}\,(A[t])$ where $A[t]$ is again regular (see [13], (17.J), p. 126.) By assumption,

$$X \xrightarrow{1 \times \varphi} X \times A^1$$

is a closed mapping and hence Γ_φ is an irreducible closed subset of $X \times A^1$ of codimension one (since Γ_φ is homeomorphic to X.). Let $P \subset A[t]$ be the height one prime defining Γ_φ and let $\mathfrak{p} = P \cap A$. Since $PA_\mathfrak{p}[t]$ is a height one prime in the UFD $A_\mathfrak{p}[t]$ it must be principal. Hence replacing X by a non-empty affine open we may and shall assume there exists a prime F in $A[t]$ such that Γ_φ is the zero set of F in $X \times A^1$.

Say $F = a_0 + a_1 t + \cdots + a_r t^r$, $a_i \in A$, $a_r \neq 0$. Then the discriminant $D(F)$ of F is non-zero in A (since $\mathrm{ch}\, k = 0$) and if $D(F)(x) \neq 0$ $(x \in X)$, then $F_x = a_0(x) + a_1(x)t + \cdots + a_r(x)t^r$ has r distinct roots in k. Since Γ_φ meets each fibre $\{x\} \times A^1$ in precisely one point we must have $r = 1$. Hence $\varphi = -a_0/a_1$ on X_{a_1}. Thus by (2.6) φ is regular on X.

Conversely, suppose that $\varphi\colon X \to A^1$ is c-regular. Let $\pi\colon \tilde{X} \to X$ be the normalization of X. Then $\varphi \circ \pi\colon \tilde{X} \to A^1$ is regular. Hence $\Gamma_{\varphi\circ\pi}$ is closed in $\tilde{X} \times A^1$. Since $\pi \times 1\colon \tilde{X} \times A^1 \to X \times A^1$ is a closed mapping and $\Gamma_\varphi = \pi \times 1(\Gamma_{\varphi\circ\pi})$ we see that Γ_φ is closed in $X \times A^1$.

COROLLARY 2.11. *If $f\colon X \to Y$ is a morphism of varieties and (X^w, τ), (Y^w, μ) are the weak normalizations of X and Y respectively, then there is a unique morphism $f^w\colon X^w \to Y^w$ such that $\mu \circ f^w = f \circ \tau$.*

Proof. Let f^w be the unique continuous map such that $\mu \circ f^w = f \circ \tau$. We wish to see that f^w carries regular functions on Y^w to regular functions on X^w. For this it suffices to show that f carries c-regular functions on Y to c-regular functions on X. Let $U \subset Y$ be open and $\varphi\colon U \to k$ be c-regular. Then Γ_φ is a closed subset of $U \times A^1$. Since $g = f|_{f^{-1}(U)} \times 1\colon f^{-1}(U) \times A^1 \to U \times A^1$ is a morphism, $g^{-1}(\Gamma_\varphi)$ is closed in $f^{-1}(U) \times A^1$. Now $g^{-1}(\Gamma_\varphi) = \Gamma_{\varphi \circ f}$ so that $\varphi \circ f$ is c-regular on $f^{-1}(U)$ by (2.9).

COROLLARY 2.12. *If φ is c-regular on X and $Y \subset X$ is closed, then $\varphi|_Y$ is c-regular on Y.*

Proof. Since $\Gamma_\varphi|_Y = \Gamma_\varphi \cap Y \times A^1$ and $Y \times A^1$ is closed in $X \times A^1$ it follows that $\Gamma_\varphi|_Y$ is closed in $Y \times A^1$. Hence $\varphi|_Y$ is c-regular by (2.10).

COROLLARY 2.13. *Suppose X and Y are weakly normal varieties. Then $X \times Y$ is weakly normal.*

Proof. It suffices to assume that X and Y are affine varieties and show that $\Gamma(X \times Y, \mathcal{O}_{X\times Y}) = \Gamma(X \times Y, \mathcal{O}^c_{X\times Y})$.

Let $\varphi\colon X \times Y \to k$ be c-regular and let $U = X - S(X)$, $V = Y - S(Y)$ so that $\varphi|_{U\times V}$ is regular. Let $\{U_i\}_{i=1}^r$ and $\{V_j\}_{j=1}^s$ be affine open covers of U and V respectively. For each $x \in X$ let $\varphi_x\colon Y \to k$ be defined by $\varphi_x(y)$ $\varphi(x, y)$. Since $\varphi|_{\{x\}\times Y}$ is c-regular by (2.12) and Y is weakly normal we see that φ_x is regular on Y. Similarly for each $y \in Y$ define a regular function $\varphi^y\colon X \to k$ by $\varphi^y(x) = \varphi(x, y)$.

For a fixed i, consider the collection $\{\varphi_x\}_{x\in U_i}$ of regular functions on Y. We wish to see it spans a finite dimensional subspace of $\Gamma(Y, \mathcal{O}_Y)$. Since $V = \bigcup_{j=1}^s V_j$ is dense in Y and the restriction map $\Gamma(Y, \mathcal{O}_Y) \to \Gamma(V, \mathcal{O}_Y)$ is injective it suffices to show that $\{\varphi_x|_V\}_{x\in U}$ spans a finite dimensional subspace of $\Gamma(V, \mathcal{O}_Y)$. Now for each j, $U_i \times V_j$ is affine and $\varphi|_{U_i\times V_j} \in A_i \otimes B_j$, where $A_i = \Gamma(U_i, \mathcal{O}_X)$, $B_j = \Gamma(V_j, \mathcal{O}_Y)$. Hence $\{\varphi_x|_{V_j}\}_{x\in U}$ spans a finite dimensional subspace of B_j. As the canonical map $\Gamma(V, \mathcal{O}_Y) \to B_1 \times \cdots \times B_s$ is injective we may conclude that $\{\varphi_x|_V\}_{x\in U_i}$ spans a finite dimensional subspace of $\Gamma(V, \mathcal{O}_Y)$. Hence $\{\varphi_x\}_{x\in U_i}$ spans a finite dimensional subspace of $\Gamma(Y, \mathcal{O}_Y)$. Since i was arbitrary

$$\{\varphi_x\}_{x\in U} = \bigcup_{i=1}^r \{\varphi_x\}_{x\in U_i}$$

spans a finite dimensional subspace $\mathscr{V}$ of $\Gamma(Y, \mathcal{O}_Y)$.

For $y \in Y$ let $\tau_y\colon \mathscr{V} \to k$ be evaluation at y. By a lemma of Palais

([17], Lemma 3.1) there exist $y_1, \cdots, y_n$ in Y such that $\tau_{y_1}, \cdots, \tau_{y_n}$ span $\mathscr{V}^*$ and if $\xi_1, \cdots, \xi_n$ is the dual basis for $\mathscr{V}$ then $\varphi|_{U\times Y} = \sum_{i=1}^n \varphi^{y_i} \otimes \xi_i$. But $\varphi - \sum_{i=1}^n \varphi^{y_i} \otimes \xi_i$ is globally defined and vanishes on the dense open set $U \times Y$ of $X \times Y$ and hence must be identically 0. Thus $\varphi = \sum_{i=1} \varphi^{y_i} \otimes \xi_i$ and φ is regular on $X \times Y$.

Let $\mathscr{C}_X$ denote the sheaf of continuous k-valued functions on a variety X. An interesting way to view the germs of c-regular functions at a point is given by the following:

PROPOSITION 2.14. *$\mathscr{O}^c_{X,x}$ is the integral closure of $\mathscr{O}_{X,x}$ in $\mathscr{C}_{X,x}$.*

Proof. We first assume that $X = \operatorname{Var}(A)$ is irreducible, normal and affine and that a continuous function $\varphi: X \to k$ is integral over A. Let K denote the quotient field of A and let $n = [K(\varphi): K]$.

Let $F(t)$ be the minimal polynomial of φ over K. Since A is normal, $F(t) \in A[t]$. We claim that $\deg F = n = 1$.

Let $Y = \operatorname{Var}(A[\varphi])$ and let $\pi: Y \to X$ be the morphism induced by the inclusion $A \subset A[\varphi]$. By (2.1) we can find a non-empty open set $U \subset X$ such that for each $x \in U$ the fibre $\pi^{-1}(x)$ consists of n distinct points. Let $x \in U$ and let $\{y_1, \cdots, y_n\} = \pi^{-1}(x)$. Then there is an element $b \in A[\varphi]$ such that $b(y_1), \cdots, b(y_n)$ are distinct. But $b = a_0 + a_1\varphi + \cdots + a_m\varphi^m$, $a_i \in A$, $m < n$ and $\varphi(y_i) = \varphi(x)$ each $i = 1, \cdots, n$ so that $b(y_i) = a_0(x) + a_1(x)\varphi(x) + \cdots + a_m(x)\varphi(x)^m$ each $i = 1, \cdots, n$. Hence $n = 1$ and there exists an $\alpha \in A$ such that $\alpha + \varphi = 0$. Then $\varphi \in A$. Thus A is integrally closed in $\Gamma(X, \mathscr{C}_X)$.

In the general case, let X be an arbitrary variety and let $\pi: \tilde{X} \to X$ be the normalization of X. Suppose that $x \in X$ and $f_x \in \mathscr{C}_{X,x}$ is integral over $\mathscr{O}_{X,x}$. Then there is an affine open neighborhood $U = \operatorname{Var}(A)$ of x such that $f: U \to k$ is continuous and integral over A. Then $f \circ \pi: \pi^{-1}(U) \to k$ is continuous and integral over $\tilde{A} = \Gamma(\pi^{-1}(U), \mathscr{O}_{\tilde{X}})$. Now $\pi^{-1}(U)$ is an affine open in $\tilde{X}$ and its irreducible components are disjoint and hence open so that $f \circ \pi$ is regular on each component by the preceding remarks and hence $f \circ \pi$ is regular on $\pi^{-1}(U)$. Hence f is c-regular on U and $f_x \in \mathscr{O}^c_{X,x}$.

We now give some criteria for determining when a union of weakly normal varieties is again weakly normal. The first result (2.18) is based on a Mayer-Vietoris sequence for ideals in a ring and is proven for complex spaces in [2]. Proposition (2.19) generalizes that result.

Let I and J be ideals in a ring A so that we have an exact sequence

https://doi.org/10.1017/S0027763000019279 Published online by Cambridge University Press

of A-modules:

$$0 \longrightarrow A/I \cap J \xrightarrow{\alpha} A/I \oplus A/J \xrightarrow{\beta} A/I + J \longrightarrow 0 \tag{2.15}$$

where

$$\alpha(a + I \cap J) = (a + I, a + J)$$

and

$$\beta(a + I, a' + J) = (a - a' + (I + J)) .$$

Now suppose that X is a variety such that $X = V_1 \cup V_2$, where V_1 and V_2 are closed subvarieties. Consider the sequence of sheaves:

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\varphi} \mathcal{O}_{V_1} \oplus \mathcal{O}_{V_2} \xrightarrow{\psi} \mathcal{O}_{V_1 \cap V_2} \longrightarrow 0 \tag{2.16}$$

where $\varphi(f) = (f|_{V_1}, f|_{V_2})$ and $\psi(g, h) = g|_{V_1 \cap V_2} - h|_{V_1 \cap V_2}$. Clearly $\psi \circ \varphi = 0$ but the sequence (2.16) is not in general exact. Let $\mathscr{I}_1$ and $\mathscr{I}_2$ be the sheaves of ideals defining V_1 and V_2 respectively.

LEMMA 2.17. *The sequence* (2.16) *is exact if and only if* $\mathscr{I}_1 + \mathscr{I}_2 = \mathscr{I}_{V_1 \cap V_2}$ *if and only if* $\mathscr{I}_1 + \mathscr{I}_2$ *is a reduced sheaf of ideals.*

Proof. Suppose $U \subset X$ is an affine open and let

$$A = \Gamma(U, \mathcal{O}_X) , \quad I_1 = \Gamma(U, \mathscr{I}_1) , \quad I_2 = \Gamma(U, \mathscr{I}_2) , \quad I = \Gamma(U, \mathscr{I}_{V_1 \cap V_2}) .$$

Then we have a commutative diagram.

$$\begin{array}{ccccccccc}
0 \longrightarrow & A/I_1 \cap I_2 & \xrightarrow{\alpha} & A/I_1 \oplus A/I_2 & \xrightarrow{\beta} & A/I_1 + I_2 & \longrightarrow 0 \\
& \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \gamma_3} & \\
0 \longrightarrow & A/I_1 \cap I_2 & \xrightarrow{\varphi} & A/I_1 \oplus A/I_2 & \xrightarrow{\psi} & A/I & \longrightarrow 0
\end{array}$$

where the first two vertical arrows are identity maps and the last is the canonical map. As we remarked earlier the top row is exact. If the bottom row is exact then γ_3 is an isomorphism by the 5-Lemma and if γ_3 is an isomorphism then the bottom row must be exact. Since U was arbitrary the result follows. As $I = \sqrt{I_1 + I_2}$ this is equivalent to the condition that $\mathscr{I}_1 + \mathscr{I}_2$ be a reduced sheaf of ideals.

PROPOSITION 2.18. *Let* $X = V_1 \cup V_2$ *where* V_1 *and* V_2 *are closed subvarieties.*

(1) *If* X *is weakly normal the complex* (2.16) *is exact.*

(2) *If* V_1 *and* V_2 *are weakly normal, then* X *is weakly normal if and only if* (2.16) *is exact.*

Proof. (1) Let f_i be regular on V_i $(i = 1, 2)$ and suppose that $f_1|_{V_1 \cap V_2} = f_2|_{V_1 \cap V_2}$. Let $\pi : \tilde{X} \to X$ be the normalization of X and let $\tilde{V}_i = \pi^{-1}(V_i)$ $(i = 1, 2)$. Then $\tilde{X} = \tilde{V}_1 \cup \tilde{V}_2$ and $f_1 \circ \pi|_{\tilde{V}_1 \cap \tilde{V}_2} = f_2 \circ \pi|_{\tilde{V}_1 \cap \tilde{V}_2}$. Hence there exists a unique regular function h on $\tilde{X}$ such that $h|_{\tilde{V}_i} = f_i \circ \pi$ $(i = 1, 2)$. Then h agrees on the fibres of π and hence $h = f \circ \pi$ for some regular function f on X. Then $f|_{V_i} = f_i$ $(i = 1, 2)$.

(2) Now let us assume that V_1 and V_2 are weakly normal and that the sequence (2.16) is exact. Let $U \subset X$ be open and suppose that $\varphi : U \to k$ is c-regular. Then $\varphi_i = \varphi|_{U \cap V_i}$ is c-regular on $U \cap V_i$ $(i = 1, 2)$ by (2.12) and hence is regular. Since (2.16) is exact this implies that φ is regular on U.

Proposition 2.19. *Let* $X = X_1 \cup \cdots \cup X_n$ *where each* X_i *is a closed subvariety and suppose that* X_i *is weakly normal for each* i. *Further assume that* $X_i \cap X_j = Y$ *whenever* $i \neq j$. *Then* X *is weakly normal if and only if* $\mathscr{I}_Y = \mathscr{I}_{X_i} + \mathscr{I}_{X_1 \cup \cdots \cup X_{i-1}}$ $(i = 2, \cdots, n)$.

Proof. We proceed by induction on n, the case $n = 1$ being trivial.

Suppose $X = X_1 \cup \cdots \cup X_n$ where $n > 1$ and each X_i is weakly normal and $X_i \cap X_j = Y$ whenever $i \neq j$. Then $X = (X_1 \cup \cdots \cup X_{n-1}) \cup X_n$. By the induction hypothesis $X_1 \cup \cdots \cup X_{n-1}$ is weakly normal if and only if $\mathscr{I}_Y = \mathscr{I}_{X_i} + \mathscr{I}_{X_1 \cup \cdots \cup X_{i-1}}$ $(i = 2, \cdots, n-1)$. Combining this with (2.18) we see that X is weakly normal if and only if

$$\mathscr{I}_Y = \mathscr{I}_{X_i} + \mathscr{I}_{X_1 \cup \cdots \cup X_{i-1}} \qquad (i = 2, \cdots, n)\,.$$

Remark 2.20. In [16] Orecchia proves a related result. Namely, suppose that A is a noetherian reduced ring with minimal primes $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$. Suppose, in addition, that the ideals

$$\mathfrak{p}_i + \bigcap_{i \neq j \leq h} \mathfrak{p}_j \qquad (i, h = 1, \cdots, n)$$

are unmixed of constant height t. Then A is seminormal in $C = A/\mathfrak{p}_1 \times \cdots \times A/\mathfrak{p}_n$ if and only if $\mathfrak{p}_i + \bigcup_{j \neq i} \mathfrak{p}_j$ is a radical ideal $(i = 1, \cdots, n)$ if and only if $(A : C)$ is a radical ideal of A.

Lemma 2.21. *Let* $X = X_1 \cup X_2$ *be a union of closed subvarieties and suppose that* $Y = X_1 \cap X_2$ *is weakly normal. Then* X *is weakly normal if and only if each* X_i *is weakly normal and* $\mathscr{I}_Y = \mathscr{I}_{X_1} + \mathscr{I}_{X_2}$.

Proof. One part of the assertion follows from (2.18). Suppose then that X and $Y = X_1 \cap X_2$ are weakly normal so that $\mathscr{I}_Y = \mathscr{I}_{X_1} + \mathscr{I}_{X_2}$ and the sequence (2.16)

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{\varphi} \mathscr{O}_{X_1} \oplus \mathscr{O}_{X_2} \xrightarrow{\psi} \mathscr{O}_{X_1 \cap X_2} \longrightarrow 0$$

is exact (cf. (2.18)). Let $U \subset X$ be an affine open and suppose $\varphi: U \cap X_1 \to k$ is c-regular. Then $\varphi|_{U \cap Y}$ is c-regular by (2.12) and hence is regular on $U \cap Y$. So there exists a regular function f defined on U such that $f|_{U\cap Y} = \varphi|_{U \cap Y}$. Then the pair $(\varphi, f|_{U \cap X_2})$ defines a c-regular function on U by (2.10) and since X is weakly normal there exists a regular function g on U such that $g|_{U \cap X_1} = \varphi$ and $g|_{U \cap X_2} = f|_{U \cap X_2}$. Hence φ is regular on $U \cap X_1$ and since U was an arbitrary affine open X_1 is weakly normal. Similarly, X_2 is weakly normal.

(2.22) Recall that if $x \in X$ the Zariski tangent space $T_{X,x}$ is defined by $T_{X,x} = \mathrm{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k)$ where $\mathfrak{m}_x$ is the germs of regular functions vanishing at x. We shall identify $T_{X,x}$ with $\mathrm{Der}_k(\mathscr{O}_{X,x}, k)$ via the canonical isomorphism. If $Y \subseteq X$ is a closed subvariety and $\mathscr{I}_Y$ is the ideal sheaf defining Y, then for a point $x \in Y$ we have $\mathrm{Der}_k(\mathscr{O}_{Y,x}, k)$ equal to the subspace of $\mathrm{Der}_k(\mathscr{O}_{X,x}, k)$ consisting of all derivations vanishing on $\mathscr{I}_{Y,x}$. So we have a natural inclusion $T_{Y,x} \subseteq T_{X,x}$.

PROPOSITION 2.23. *Let $X = X_1 \cup \cdots \cup X_n$ be a union of closed subvarieties and assume that $X_i \cap X_j = Y$ whenever $i \neq j$ and Y is weakly normal.*

(1) *X is weakly normal if and only if each X_i is weakly normal and $\mathscr{I}_{X_i} + \mathscr{I}_{X_1 \cup \cdots \cup X_{i-1}} = \mathscr{I}_Y$ $(i = 2, \cdots, n)$.*

(2) *Suppose in addition that Y is non-singular. Then X is weakly normal if and only if each X_i is weakly normal and $T_{Y,x} = T_{X_i,x} \cap T_{X_1 \cup \cdots \cup X_{i-1},x}$ for all $x \in Y$ $(i = 2, \cdots, n)$.*

Proof. The first assertion follows from (2.21) and the induction argument of (2.19).

Suppose then that Y is non-singular. It suffices to see that for any i between 2 and n, $\mathscr{I}_{X_i} + \mathscr{I}_{X_1 \cup \cdots \cup X_{i-1}} = \mathscr{I}_Y$ if and only if $T_{Y,x} = T_{X_i,x} \cap T_{X_1 \cup \cdots \cup X_{i-1},x}$ for all $x \in Y$. Let $x \in Y$. Then $T_{Y,x} = \mathrm{Der}_k(\mathscr{O}_{X,x}/\mathscr{I}_{Y,x}, k)$ and $T_{X_i,x} \cap T_{X_1 \cup \cdots \cup X_{i-1},x} = \mathrm{Der}_k(\mathscr{O}_{X,x}/\mathscr{I}_{X_i,x}, k) \cap \mathrm{Der}_k(\mathscr{O}_{X,x}/\mathscr{I}_{X_1 \cup \cdots \cup X_{i-1},x}, k) \cong \mathrm{Der}_k(\mathscr{O}_{X,x}/\mathscr{I}_{X_i,x} + \mathscr{I}_{X_1 \cup \cdots \cup X_{i-1},x}, k)$. Now $\sqrt{\mathscr{I}_{X_i,x} + \mathscr{I}_{X_1 \cup \cdots \cup X_{i-1},x}} = \mathscr{I}_{Y,x}$ and $\mathscr{O}_{X,x}/\mathscr{I}_{Y,x}$ is a regular local ring. Then $\mathscr{I}_{X_i,x} + \mathscr{I}_{X_1 \cup \cdots \cup X_{i-1},x} = \mathscr{I}_{Y,x}$ if and

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only if $C = \mathcal{O}_{X,x}/\mathcal{I}_{X_i,x} + \mathcal{I}_{X_1\cup\cdots\cup X_{i-1},x}$ is regular if and only if $\dim_k(T_{X_i,x} \cap T_{X_1\cup\cdots\cup X_{i-1},x}) = (\text{Krull}) \dim C$ if and only if $T_{X_i,x} \cap T_{X_1\cup\cdots\cup X_{i-1},x} = T_{Y,x}$.

At this point we would like to establish the correspondence between the sheaf of c-regular functions on a complex algebraic variety and the sheaf of c-holomorphic functions on the associated complex analytic space. First we need to establish some notation (see [11] for a more detailed description).

If $(X, \mathcal{O}_X)$ denotes a complex algebraic variety we let $(X_h, \mathcal{O}_{X_h})$ denote the associated complex analytic space so that $\mathcal{O}_{X_h}$ is the sheaf of holomorphic functions on X_h. We let $f\colon (X_h, \mathcal{O}_{X_h}) \to (X, \mathcal{O}_X)$ denote the canonical morphism of locally ringed spaces. Since the point sets of X and X_h are identical we will often identify them. When we say a subset of X is open (resp. closed) we mean with respect to the usual topology. When we say a subset of X is Z-open (resp. Z-closed) we mean with respect to the Zariski topology. Let $\mathcal{O}^c_{X_h}$ denote the sheaf of c-holomorphic functions on X_h as defined in the introduction.

PROPOSITION 2.24. *With notation as above, the canonical morphism* $f^*\mathcal{O}^c_X \to \mathcal{O}^c_{X_h}$ *is an isomorphism. In particular, X is weakly normal at a point x if and only if X_h is weakly normal at x.*

Proof. If U is open and V is any Z-open containing U then we have a map of $\boldsymbol{C}$-algebras:

$$\Gamma(V, \mathcal{O}^c_X) \longrightarrow \Gamma(U, \mathcal{O}^c_{X_h}) .$$

Since if $\varphi\colon V \to \boldsymbol{C}$ is c-regular then its graph Γ_φ is Z-closed in $V \times \boldsymbol{C}$ and hence Γ_φ is an analytic subset of $V \times \boldsymbol{C}$ so that φ is c-holomorphic on V (see [2], Remark 1.4) and $\varphi|_U$ is c-holomorphic on U.

Hence we have a canonical map:

$$\Gamma(V, \mathcal{O}^c_X) \underset{\Gamma(V, \mathcal{O}_X)}{\otimes} \Gamma(U, \mathcal{O}_{X_h}) \longrightarrow \Gamma(U, \mathcal{O}^c_{X_h})$$

Thus there is a canonical morphism:

$$f^*\mathcal{O}^c_X \longrightarrow \mathcal{O}^c_{X_h} .$$

Suppose $x \in X$ is arbitrary. We would like to see that $(f^*\mathcal{O}^c_X)_x \to \mathcal{O}^c_{X_h,x}$ is an isomorphism. Since this question is local in nature we may assume that $X \subset \boldsymbol{C}^n$ is affine and embedding $\boldsymbol{C}^n$ in $\boldsymbol{P}^n$ we have $x \in X \subset \bar{X}$ where $\bar{X}$ is a projective variety and X is an affine Z-neighborhood of x. Now

$\mathcal{O}^c_{\bar{X}_h}$ is a coherent sheaf on $\bar{X}_h$ so that by twisting up if necessary we may and shall assume that $\mathcal{O}^c_{\bar{X}_h,x} = \mathcal{O}^c_{X_h,x}$ is generated by global meromorphic functions on $\bar{X}_h$. Suppose φ is such a generator. Then φ is a rational function on $\bar{X}$ and there is some open neighborhood U of x on which is c-holomorphic.

Let $A = \mathcal{O}_{X,x}$, $B = \mathcal{O}_{X_h,x}$ so that we have $A \subset B$ is a faithfully flat map of noetherian local rings (see [18]). Then φ is in the total ring of quotients of A and $B[\varphi] = A[\varphi] \otimes_A B$ is a finite B-module. By faithful flatness, if $1, \varphi, \cdots, \varphi^r$ generate $B[\varphi]$ as a B-module they will generate $A[\varphi]$ as an A-module. Hence φ is integral over A.

Let $\pi: \tilde{X} \to X$ be the normalization of X. Replacing X by an affine Z-neighborhood of x and cutting down U if necessary, we may and shall assume that there is a regular function f on $\tilde{X}$ such that $f|_{\pi^{-1}(U)} = \varphi \circ \pi|_{\pi^{-1}(U)}$.

Let $F: \tilde{X} \times \tilde{X} \to \boldsymbol{C}$ be defined by $F(y, z) = f(y) - f(z)$. Let Y denote the Z-closed subset of $\tilde{X} \times \tilde{X}$ consisting of all points (y, z) such that $\pi(y) = \pi(z)$. Then F vanishes on $Y \cap \pi^{-1}(U) \times \pi^{-1}(U)$. Hence if Y_1 denotes the union of those irreducible components of Y which meet $\pi^{-1}(U) \times \pi^{-1}(U)$ we have $F|_{Y_1} = 0$. Moreover, if Y_2 denotes the union of the remaining components then $(x, x) \notin \pi \times \pi(Y_2)$ so if $\Delta: X \to X \times X$ is the diagonal map $V = \Delta^{-1}(X \times X - \pi \times \pi(Y_2))$ is a Z-neighborhood of x. Then whenever $y, z \in \pi^{-1}(V)$ and $\pi(y) = \pi(z)$ we have $f(y) = f(z)$. Thus there exists a c-regular function g on V such that $g \circ \pi = f|_{\pi^{-1}(V)}$. Hence $g|_{U \cap V} = \varphi|_{U \cap V}$ and $\varphi_x \in \mathcal{O}^c_{X,x}$.

Then the $\mathcal{O}_{X_h,x}$-module generators of $\mathcal{O}^c_{X_h,x}$ are in $\mathcal{O}^c_{X,x}$ so that we have a surjection.

$$\mathcal{O}^c_{X,x} \underset{\mathcal{O}_{X,x}}{\otimes} \mathcal{O}_{X_h,x} \longrightarrow \mathcal{O}^c_{X_h,x} .$$

But this map is clearly injective (since $\mathcal{O}_{X_h,x}$ is flat over $\mathcal{O}_{X,x}$) and hence $(f^*\mathcal{O}^c_X)_x \to \mathcal{O}^c_{X_h,x}$ is an isomorphism.

In particular, X is weakly normal at x if and only if $\mathcal{O}_{X,x} = \mathcal{O}^c_{X,x}$ if and only if

$$\mathcal{O}_{X_h,x} = \mathcal{O}^c_{X,x} \underset{\mathcal{O}_{X,x}}{\otimes} \mathcal{O}_{X_h,x} = \mathcal{O}^c_{X_h,x}$$

if and only if X_h is weakly normal at x.

https://doi.org/10.1017/S0027763000019279 Published online by Cambridge University Press

§ 3. Some examples

Using the results of section 2 we offer some methods of construction of weakly normal varieties which arise from the complex space theory. The method of (3.1) was first observed by Mochizuki [14] in the case where X is nonsingular and was generalized in [2]. The constructions of (3.2) and (3.4) also appear in [2]. We include only the affine version of the multicross here. We will further investigate this generic type singularity for weakly normal spaces in a future paper.

As before, all varieties are taken over an algebraically closed field k of characteristic 0.

PROPOSITION 3.1. *Let $F = (f_1, \cdots, f_n): X \to A^n$ be a morphism and assume that X is weakly normal. Then the variety $Y = X \times \{0\} \cup \Gamma_F$ is weakly normal if and only if $(f_1, \cdots, f_n)\mathcal{O}_X$ is a reduced sheaf of ideals.*

Proof. Since $X \times \{0\}$ and Γ_F are both isomorphic to X and hence are weakly normal we can appeal to (2.18). Then Y is weakly normal if and only if $\mathcal{I}_{X\times\{0\}} + \mathcal{I}_{\Gamma_F}$ is a reduced sheaf of ideals. Let $y_1, \cdots, y_n$ be affine coordinates on A^n. Then

$$\mathcal{I}_{X\times\{0\}} = (y_1, \cdots, y_n)\mathcal{O}_{X\times A^n} \quad \text{and} \quad \mathcal{I}_{\Gamma_F} = (y_1 - f_1, \cdots, y_n - f_n)\mathcal{O}_{X\times A^n}$$

so that $\mathcal{I}_{X\times\{0\}} + \mathcal{I}_{\Gamma_F} = (y_1, \cdots, y_n, f_1, \cdots, f_n)\mathcal{O}_{X\times A^n}$. But this is a reduced ideal sheaf if and only if $(f_1, \cdots, f_n)\mathcal{O}_X$ is a reduced ideal sheaf and the assertion follows.

Suppose that $W_1, \cdots, W_p$ are linearly disjoint linear subspaces of A^n and let $W = W_1 \cup \cdots \cup W_p$. In the complex space theory W is said to be a normal crossing at the origin. By a homogeneous change of coordinates we may assume that

$$W_i = \{x \mid x_{i,j} = 0, j = 1, \cdots, s_i\} \qquad \text{where } x_{1,1}, \cdots, x_{p,s_p}, x_{p+1}, \cdots, x_q$$

are coordinates for A^n.

PROPOSITION 3.2. *The normal crossing $W = W_1 \cup \cdots \cup W_p$ is weakly normal.*

Proof. Let $W^{(r)} = W_1 \cup \cdots \cup W_r$ $(1 \le r \le p)$ and assume that $W_i = \{x \mid x_{i,j} = 0, j = 1, \cdots, s_i\}$. We proceed by induction on r to show that if $I^{(r)}$ is the ideal defining $W^{(r)}$ then $I^{(r)} = (x_{1,j_1} \cdots x_{r,j_r} \mid 1 \le j_i \le s_i, 1 \le i \le r)$ and $W^{(r)}$ is weakly normal. The assertion is trivial for $r = 1$ so assume that $2 \leq r \leq p$ and that $W^{(r-1)}$ is weakly normal with $I^{(r-1)}$ as described above.

Then $I^{(r)} = I^{(r-1)} \cap (x_{r,k} \mid 1 \leq k \leq s_r)$ is a homogeneous ideal. Suppose that a homogeneous polynomial f of degree d is in $I^{(r)}$. Then

$$f = \sum_{|\alpha|=d} a_{(\alpha)} x_{1,1}^{\alpha_{1,1}} \cdots x_{p,s_p}^{\alpha_{p,s_p}} x_{p+1}^{\alpha_{p+1}} \cdots x_q^{\alpha_q}$$

where $|\alpha| = \alpha_{1,1} + \cdots + \alpha_{p,s_p} + \alpha_{p+1} + \cdots + \alpha_q$. Since $f \in (x_{r,k} \mid 1 \leq k \leq s_r)$ we must have $a_{(\alpha)} = 0$ whenever $\sum_{k=1}^{s_r} \alpha_{r,k} = 0$. Since $f \in I^{(r-1)}$ we must have $a_{(\alpha)} = 0$ whenever $\sum_{j=1}^{s_i} \alpha_{i,j} = 0$ some $i = 1, \cdots, r-1$. Thus

$$f \in (x_{1,j_1} \cdots x_{r,j_r} \mid 1 \leq j_i \leq s_i, 1 \leq i \leq r)$$

as desired. Since $I^{(r)}$ is homogeneous,

$$I^{(r)} = (x_{1,j_1} \cdots x_{r,j_r} \mid 1 \leq j_i \leq s_i, 1 \leq i \leq r)\ .$$

Now $X^{(r-1)}$ is weakly normal by the induction hypothesis. Since $I^{(r-1)} + (x_{r,k} \mid 1 \leq k \leq s_r)$ is a reduced ideal, $W^{(r)} = W^{(r-1)} \cup W_r$ is weakly normal by (2.17) and (2.18).

Definition 3.3. Let $I = \{T_1, \cdots, T_p\}$ be a collection of disjoint subsets of $\{1, \cdots, n\}$. Let $T_i' = \bigcup_{j \neq i} T_j$ $(i = 1, \cdots, p)$ and let

$$V_i = \{x \in A^n \mid x_\alpha = 0\ \forall\ \alpha \in T_i'\}\ .$$

The variety $V_I = V_1 \cup \cdots \cup V_p$ is said to be a multicross of type I.

Proposition 3.4. *The multicross V_I of type I is weakly normal.*

Proof. This is a direct consequence of (2.19). For let $I_i = (x_\alpha \mid \alpha \in T_i')$ $(i = 1, \cdots, p)$, $J = (x_\alpha \mid \alpha \in \bigcup_{i=1}^{p} T_i)$. Then $I_i + \bigcap_{j \neq i} I_j = J$ $(i = 1, \cdots, p)$, since $(x_\beta \mid \beta \in T_i) \subset \bigcap_{j \neq i} I_j$.

Proposition 3.5. *Let $\pi: A^n \to A^{n+p}$ be defined by $\pi(u_1, \cdots, u_{n-1}, v) = (u_1, \cdots, u_{n-1}, v^2, u_1 v, \cdots, u_p v)$ where $1 \leq p \leq n-1$. Then $V = \pi(A^n)$ is weakly normal.*

Proof. Since $\pi: A^n \to V$ is the normalization of V it suffices by (2.2) to see that if $f(u, v)$ agrees on the fibres of π then f comes from a regular function on V. Let $B = k[u_1, \cdots, u_{n-1}, v]$ and let A denote the k-subalgebra generated by $u_1, \cdots, u_{n-1}, v^2, u_1 v, \cdots, u_p v$.

Suppose $f \in B$ agrees on the fibres of π. Write $f = \sum_{i=0}^{m} g_i(u) v^i$, $g_i \in k[u_1, \cdots, u_{n-1}]$. Then $f(0, \alpha_{p+1}, \cdots, \alpha_n, \beta) = f(0, \alpha_{p+1}, \cdots, \alpha_n, -\beta)$ all $\alpha_i, \beta \in k$ implies that

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$$\sum g_i(0, u_{p+1}, \cdots, u_n)v^i = \sum (-1)^i g_i(0, u_{p+1}, \cdots, u_n)v^i$$

in $k[u_{p+1}, \cdots, u_n, v]$ and hence $g_i(0, u_{p+1}, \cdots, u_n) = 0$ whenever i is odd. But then

$$f = \sum [g_i(u) - g_i(0, u_{p+1}, \cdots, u_n)]v^i + \sum_{i \text{ even}} g_i(0, u_{p+1}, \cdots, u_n)v^i$$

is in A since $g_i(u) - g(0, u_{p+1}, \cdots, u_n) \in (u_1, \cdots, u_p)B \subseteq A$.

EXAMPLE 3.6. The union of non-weakly normal components can be weakly normal.

In A^4, let $H = \{x \mid x_1 = x_2 = 0\}$ and let $W = f(A^2)$ where $f(u, v) = (u, uv, v^2, v^3)$. Consider $V = H \cup W$. Let $\mathfrak{P}_1$ and $\mathfrak{P}_2$ denote the ideals defining H and W respectively so that $I = \mathfrak{P}_1 \cap \mathfrak{P}_2$ is the ideal defining V. Then $A = k[x_1, x_2, x_3, x_4]/I$ is the affine coordinate ring of V and its normalization $B = k[x_3, x_4] \times k[x_1, x_2/x_1]$ is the product of two polynomial rings over k.

Suppose an element (f, g) of B lies in ${}^+A$, i.e., $f(\alpha^2, \alpha^3) = g(0, \alpha)$ for all $\alpha \in k$.

Then g lies in the k-subalgebra of $k[x_1, x_2/x_1]$ generated by $x_1, x_2, x_3 = (x_2/x_1)^2$ and $x_4 = (x_2/x_1)^3$ and we may consider (f, g) as an element of $A/\mathfrak{P}_1 \times A/\mathfrak{P}_2$. Recall the exact sequence of (2.15)

$$0 \longrightarrow A \xrightarrow{\alpha} A/\mathfrak{P}_1 \times A/\mathfrak{P}_2 \xrightarrow{\beta} A/\mathfrak{P}_1 + \mathfrak{P}_2 \longrightarrow 0 .$$

Now $\mathfrak{P}_1 + \mathfrak{P}_2 = (x_1, x_2, x_3^3 - x_4^2)$ is the ideal defining $W \cap H$ and since $f(\alpha^2, \alpha^3) = g(0, \alpha)$ for all $\alpha \in k$ we have (with the appropriate identifications) $f - g \in \mathfrak{P}_1 + \mathfrak{P}_2$. Hence $(f, g) = \alpha(h)$ for some $h \in A$ and A is weakly normal.

However, W is not weakly normal. Its affine coordinate ring is the k-subalgebra of the polynomial ring $k[u, v]$ generated by u, uv, v^2 and v^3 and $k[u, v]$ is its normalization. Since $k[u, uv, v^2, v^3] \subset k[u, v]$ fails to satisfy condition 3 of (1.4), W is not weakly normal.

EXAMPLE 3.7. If B is the normalization of a weakly normal affine ring A, then B/A can have embedded primes.

Let u, x, y, z, w be transcendentals over k, let C denote the polynomial ring $k[u, x, y, z, w]$ and let $\mathfrak{p}_1 = (x^2z - w^2, u - w)$, $\mathfrak{p}_2 = (u, x, y)$. Let $A = C/\mathfrak{p}_1 \cap \mathfrak{p}_2$. Since $\mathfrak{p}_1 + \mathfrak{p}_2 = (u, x, y, w)$ is prime, $C/\mathfrak{p}_1$ is weakly normal by (3.5) and $C/\mathfrak{p}_2$ is normal, A is weakly normal by (2.19). Then $B = k[x, y, w/x] \times k[z, w]$ is the normalization of A and $(A{:}\,B) = (\bar{x}, \bar{u}, \bar{w})A = ((x, 0), (x(w/x), 0), (x(w/x), w))B = (x) \times k[z, w] \cap k[x, y, w/x] \times (w)$. Notice

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that the prime ideal $P = k[x, y, w/x] \times (w)$ contracts to $\mathfrak{p} = (\bar{u}, \bar{x}, \bar{y}, \bar{w})A$ and this properly contains $(\bar{x}, \bar{u}, \bar{w})A$. In lieu of the exact sequence of A-modules:

$$0 \longrightarrow A/(A: B) \longrightarrow B/(A: B) \longrightarrow B/A \longrightarrow 0$$

we see that $\mathfrak{p}$ is an embedded prime of B/A. One sees that $\mathfrak{p} = (A: (0, 1))$ and $\bar{y}(0, 1) = (0, 0) \in A$ while $\bar{y} \notin (A: B)$.

More generally, suppose $X = \mathrm{Var}\,(A)$ is a weakly normal affine variety with normalization $\pi: \tilde{X} = \mathrm{Var}\,(B) \to X$. Suppose that X has two irreducible components W and Z, that Z is normal, $W \cap Z$ is irreducible and properly contained in some component of $N(W)$, the non-normal points of W. Then $N(X) = N(W)$. Let g be the regular function on $\tilde{X}$ such that $g|_{\pi^{-1}(W)} = 0$ and $g|_{\pi^{-1}(Z)} = 1$ and let h be a regular function on X such that h vanishes on $W \cap Z$ but not on any component of $N(W)$. Then $(h \circ \pi)g$ is c-regular and hence regular on X but h is not in any minimal prime of $(A: B)$. Hence B/A has embedded primes.

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