An infinite series of triangles and conics with a common pole and polar.

By J. A. DONALDSON.

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I. An infinite family of triangles, having a common pole (determined by three fixed lines through it), and polar (determined by three fixed points on it), and an allied family of conics with imaginary double contact.
Construction for pole of a line with reference to a triangle.

II. A one-fold of such families with same pole and polar and associated in pairs which have their conics in common.

III. Proof that the one-fold of II. exhausts all the possible triangles with same pole and polar (determined by the three lines and points mentioned above).
Construction for any triangle,
(1) when given the pole and the three points on the polar;
(2) when given the polar and the three lines through the pole.

My thanks are due to Mr Nelson for his kind assistance in sections II. and III.

§ I.

The following piece of work is based on §23 of Miss Scott's "Modern Analytical Geometry." We use generalised trilinear coordinates, so that any point may be taken as the point $1:1:1$.

Take ABC as triangle of reference.

P is any point $(1:1:1)$.

AP, BP, and CP, cut BC, CA, and AB, in A′, B′, and C′ respectively.

AB and A′B′ intersect in S, BC and B′C′ in Q, and CA and C′A′ in R. Then Q, R, and S are collinear, and this line is

Call the point of concurrence of AP, BR, and CS, A_1.

" " " " " " BP, CS, " AQ, B_1.

" " " " " " CP, AQ, " BR, C_1.

A_1, B_1, and C_1 thus have as coordinates $(-1:1:1)$, $(1:-1:1)$, and $(1:1:-1)$ respectively.

Triangle ABC bears the same relation to $A_1B_1C_1$ that $A'B'C'$ bears to ABC; for A_1P, B_1P, and C_1P cut B_1C_1, C_1A_1, and A_1B_1 in A, B, and C respectively.

Therefore, *incidentally*, this gives a simple construction for triangle ABC when A', B', C', and P are given.

From what has already been proved, it is seen that

A_1B_1 and AB meet in S,
B_1C_1 " BC " " Q,
and C_1A_1 " CA " " R.

Hence, the polar of P with respect to triangle $A_1B_1C_1$ is determined by the same three points Q, R, and S.

It follows, obviously, that Q, R, and S determine the polar of P with respect to triangle $A'B'C'$. This may easily be verified analytically:—Call the point of intersection of $A'P$ and $B'C'$, A''. B' and C' are the points $(1:0:1)$ and $(1:1:0)$ respectively.

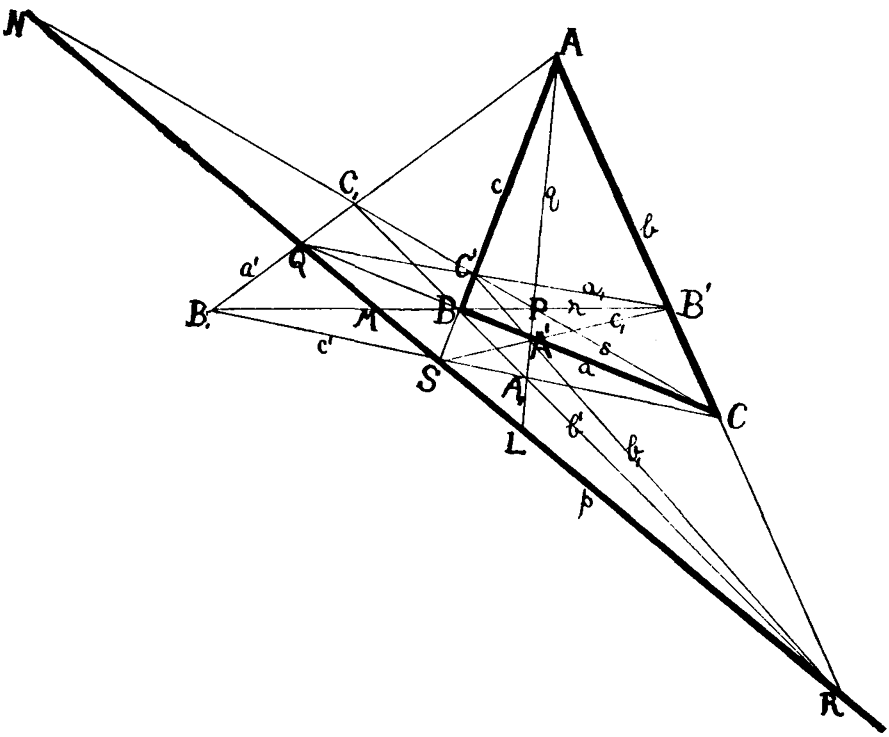
$\therefore$ $B'C'$ has equation $-x+y+z=0$.
Also $A'P$ has equation $y-z=0$.
$\therefore$ A'' has coordinates $2:1:1$;
Similarly B'' " " $1:2:1$;
and C'' " " $1:1:2$.
$\therefore$ $B''C''$ is the line $-3x+y+z=0$.

Hence $B''C''$ and $B'C'$ intersect in the point $(0:1:-1)$, *i.e.* the point Q. The rest follows from symmetry.

We can go on deriving triangles from $A'B'C'$ as we got $A'B'C'$ from ABC. Thus we get an infinite series of triangles which we will call the *descending series*.

Also, we may derive a triangle from $A_1B_1C_1$, as we got $A_1B_1C_1$ from ABC, and in this way we get an infinite series of triangles which we will call the *ascending series*. With respect to this doubly infinite series of triangles, the polar of P is determined by the three points Q, R, and S.

These triangles form a very particular kind of perspective, viz., all the A, B, and C vertices lie on the lines AP, BP, and CP defined to be the *polar* of P with respect to triangle ABC (see §23, Miss Scott).



QRS is easily found to have the equation, $x+y+z=0$.

Construct the lines AQ, BR, and CS, and let these three lines form the triangle $A_1B_1C_1$ as in figure.

Q, R, and S have as coordinates $(0:1:-1)$, $(-1:0:1)$, and $(1:-1:0)$ respectively.

Hence AQ is the line $\quad y+z=0$.

Similarly BR is the line $x+z=0$.

" CS " " " $x+y=0$.

We see that AP, BR, and CS are concurrent, for

$$(y-z)+(x+z)-(x+y)\equiv 0.$$

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respectively; all the AB, BC, and CA sides pass through the collinear points S, Q, and R respectively.

Also, from the symmetry of the signs of the coordinates of P, A_1, B_1, and C_1, it is obvious that if we start with any one as our fixed point, the other three points are the vertices of the first ascending triangle. This can easily be verified by following out the construction in the figure.

By Carnot's theorem, a conic can be described to touch BC, CA, and AB at A′, B′, and C′ respectively.

From symmetry its equation must be $\Sigma(x-y+z)(x+y-z)=0$, which reduces to $\Sigma x^2 - 2\Sigma yz = 0$. (1)

For a similar reason a conic can be described touching B_1C_1, C_1A_1, and A_1B_1 at A, B, and C respectively.

Its equation is $\Sigma yz = 0$. (2)

Conics (1) and (2) have double contact where they are cut by the line QRS, for (1) can be written

$$(x+y+z)^2 - 4(yz+zx+xy) = 0.$$

QRS is the polar of P with respect to both conics.

The pair of tangents from P to these conics has for equation $\Sigma x^2 - \Sigma yz = 0$. This equation splits up into the two imaginary lines

$$x - \frac{1-\sqrt{3}i}{2}y - \frac{1+\sqrt{3}i}{2}z = 0,$$

and
$$x - \frac{1+\sqrt{3}i}{2}y - \frac{1-\sqrt{3}i}{2}z = 0.$$

From what we have seen about the infinite derivation of triangles, it follows that (2) has double contact along QRS with the next ascending conic, *i.e.* the conic through A_1, B_1, and C_1, and touching B_2C_2, C_2A_2, and A_2B_2.

Also the conic (1) has double contact along QRS with the next descending conic.

This doubly infinite series of conics, each of which passes through the vertices of one triangle, and touches the sides of the consecutive ascending triangle, has a common polar QRS with respect to the point P, and they belong to a family of conics having double contact on the line QRS. Since the common tangents are imaginary, the conics cannot cut one another in any real points.

Thus, P and QRS are pole and polar respectively with respect to the infinity of triangles and conics.

We will now find the coordinates of the vertices, and the equations to the sides of the triangles, and the equations to the conics.

Descending Series.

All the A vertices lie on the line $AP(y=z)$, and $A^{(m+1)}$ is the intersection of $B^{(m)}C^{(m)}$ with $y=z$.

Suppose $A^{(m)}$ has the coordinates $\lambda:\mu:\mu$. The equation to $B^{(m)}C^{(m)}$ is

$$\begin{vmatrix} x & y & z \\ \mu & \lambda & \mu \\ \mu & \mu & \lambda \end{vmatrix} = 0;$$

i.e. $-(\lambda+\mu)x+\mu y+\mu z=0$.

$A^{(m+1)}$ is the point of intersection of $B^{(m)}C^{(m)}$ with $y=z$; *i.e.* $A^{(m+1)}$ is the point $2\mu:\lambda+\mu:\lambda+\mu$.

By giving λ and μ the values already found, we get the following table for the calculation of the general values :—

A°	$1:0:0$	$B^\circ C^\circ$	$x=0$
A'	$0:1:1$	$B'C'$	$-x+y+z=0$
A''	$2:1:1$	$B''C''$	$-3x+y+z=0$
A'''	$2:3:3$	$B'''C'''$	$-5x+3y+3z=0$
$A^{(4)}$	$6:5:5$	$B^{(4)}C^{(4)}$	$-11x+5y+5z=0$
$A^{(5)}$	$10:11:11$	$B^{(5)}C^{(5)}$	$-21x+11y+11z=0$
$A^{(6)}$	$22:21:21$	$B^{(6)}C^{(6)}$	$-43x+21y+21z=0$
$A^{(7)}$	$42:43:43$	$B^{(7)}C^{(7)}$	$-85x+43y+43z=0$
$A^{(8)}$	$86:85:85$	$B^{(8)}C^{(8)}$	$-171x+85y+85z=0$

To get the general expressions we must take every second vertex.

Vertex.	x coordinate.	Difference.
A°	1	
A''	2	$1=2^0$
$A^{(4)}$	6	$4=2^2$
$A^{(6)}$	22	$16=2^4$
$A^{(8)}$	86	$64=2^6$

This suggests that the x coordinate of $A^{(2m}$ is

$$1+1+2^2+2^4+\ldots+2^{2m-2}=\frac{2^{2m}+2}{3}.$$

$\therefore$ y and z coordinates are $\dfrac{2^{2m}+2}{3}-1=\dfrac{2^{2m}-1}{3}$.

Similarly we get the coordinates of $A^{(2m+1)}$, and knowing the vertices we at once have the equations to the sides.

The general expressions are :—

$$A^{(n)},\ \frac{2^n+(-1)^n 2}{3}:\frac{2^n-(-1)^n}{3}:\frac{2^n-(-1)^n}{3}.$$

$$B^{(n)}C^{(n)},\ -\frac{2^{n+1}+(-1)^n}{3}x+\frac{2^n-(-1)^n}{3}y+\frac{2^n-(-1)^n}{3}z=0.$$

These expressions are correct for they satisfy the conditions of *Mathematical Induction.*

We retain the dominator 3, as in this form, the above expressions are in their simplest form for the following reasons :—

$$\frac{2^n+(-1)^n 2}{3} \text{ and } \frac{2^n-(-1)^n}{3}$$

are integers and differ by 1, and are therefore prime to each other.

$$\frac{2^{n+1}+(-1)^n}{3}+\frac{2^n-(-1)^n}{3}=\frac{2^{n+1}+2^n}{3}=2^n.$$

Hence the coefficients of $B^{(n)}C^{(n)}$ are prime to each other, since neither is divisible by 2.

We get the equations to the conics in quite a simple way as follows :—Any conic through the vertices of a triangle whose sides have the equations $L=0$, $M=0$, $N=0$, is of the form

$$\alpha MN+\beta NL+\gamma LM=0,$$

where α, β and γ are numerical coefficients.

In our case, the sides of the triangles have symmetrical equations, and since, by Carnot's theorem, a conic can be described through the vertices of any one of our triangles, and touching the sides of the consecutive ascending triangle, this conic must have a symmetrical equation; *i.e.* $\alpha=\beta=\gamma$, since L, M, and N, are symmetrical.

Hence the conic through the vertices of $A^{(n)}B^{(n)}C^{(n)}$ and touching the sides of $A^{(n-1)}B^{(n-1)}C^{(n-1)}$ is

$$\Sigma(ax-by+az)(ax+ay-bz)=0,$$

where $$a \equiv \frac{2^n - (-1)^n}{3}, \text{ and } b \equiv \frac{2^{n+1} + (-1)^n}{3}.$$

This equation reduces to

$$-\frac{2^{2n} - 1}{3}\Sigma x^2 + \frac{2^{2n} + 2}{3}\Sigma yz = 0, \qquad (a)$$

and we call the conic the n^{th} *descending conic.*

The family (a) have the same pair of tangents from the point P, and P has the common polar QRS with respect to the family. Hence they have double contact on the line QRS.

The fact that all the equations are of the form $k(\Sigma x)^2 + l\Sigma yz = 0$, makes the double contact property evident.

Ascending Series.

The construction is simple. A_nQ, B_nR, C_nS are the sides of triangle $A_{n+1}B_{n+1}C_{n+1}$; *i.e.* A_nQ is the line $B_{n+1}C_{n+1}$, and A_{n+1} is the point of intersection of AP and B_nR.

If A_n is the point $-\lambda : \mu : \mu$, A_nQ or $B_{n+1}C_{n+1}$ has the equation

$$\begin{vmatrix} x & y & z \\ -\lambda & \mu & \mu \\ 0 & 1 & -1 \end{vmatrix} = 0;$$

i.e. $2\mu x + \lambda y + \lambda z = 0$.

A_{n+1} is the intersection of $C_{n+1}A_{n+1}$ and AP;

i.e. of $\lambda x + 2\mu y + \lambda z = 0$, and $y = z$.

Hence A_{n+1} is the point $-(2\mu + \lambda) : \lambda : \lambda$.

By giving λ and μ the values already found, we get the following table:—

A_0	$-1 : 0 : 0$	B_0C_0	$x = 0$
A_1	$-1 : 1 : 1$	B_1C_1	$y + z = 0$
A_2	$-3 : 1 : 1$	B_2C_2	$2x + y + z = 0$
A_3	$-5 : 3 : 3$	B_3C_3	$2x + 3y + 3z = 0$
A_4	$-11 : 5 : 5$	B_4C_4	$6x + 5y + 5z = 0$
A_5	$-21 : 11 : 11$	B_5C_5	$10x + 11y + 11z = 0$
A_6	$-43 : 21 : 21$	B_6C_6	$22x + 21y + 21z = 0$
A_7	$-85 : 43 : 43$	B_7C_7	$42x + 43y + 43z = 0$
A_8	$-171 : 85 : 85$	B_8C_8	$86x + 85y + 85z = 0$

We take every second vertex, as in the descending series, and by using the principle of *Mathematical Induction* we get the following general expressions :—

$$A_n, \quad -\frac{2^{n+1}+(-1)^n}{3} : \frac{2^n-(-1)^n}{3} : \frac{2^n-(-1)^n}{3}.$$

$$B_nC_n, \quad \frac{2^n+(-1)^n 2}{3}x + \frac{2^n-(-1)^n}{3}y + \frac{2^n-(-1)^n}{3}z = 0.$$

The conic through the vertices of $A_{n-1}B_{n-1}C_{n-1}$ and touching the sides of $A_nB_nC_n$ is

$$\Sigma(ax+by+az)(ax+ay+bz)=0,$$

where $\quad a \equiv \dfrac{2^{n-1}-(-1)^{n-1}}{3}, \text{ and } b \equiv \dfrac{2^{n-1}+(-1)^{n-1}2}{3};$

i.e. $$\frac{2^{2n-2}-1}{3}\Sigma x^2 + \frac{2^{2n-1}+1}{3}\Sigma yz = 0. \qquad (\beta)$$

We call this conic the n^{th} *ascending conic.*

Every member of the family (β) has double contact with every member of the family (a) where they are cut by the line QRS, and the double contact is imaginary.

They have no real points of intersection, and cannot cut the line QRS in any real points.

When $n=\infty$:—

$A^{(n)}$ becomes $1:1:1$; *i.e.* P.

$B^{(n)}C^{(n)}$ becomes $-2x+y+z=0$; *i.e.* PQ.

(a) reduces to $\Sigma x^2 - \Sigma yz = 0$; *i.e.* the pair of tangents from P to the family.

A_n becomes $-2:1:1$; *i.e.* L.

B_nC_n becomes $x+y+z=0$; *i.e.* QRS.

(β) reduces to $\Sigma x^2 + 2\Sigma yz = 0$; *i.e.* the line QRS taken twice over.

Hence the descending limits are :—

of the vertices – the point P,

of the sides – the lines PQ, PR, and PS,

and of the conics – the pair of imaginary tangents to the family.

The ascending limits are :—

of the vertices - the points L, M, and N,

of the sides - the line QRS,

and of the conics - the line QRS taken twice over, or the two imaginary points of intersection of the conics with QRS.

$A^{(n)}$ is the point $2^n + (-1)^n 2 : 2^n - (-1)^n : 2^n - (-1)^n$.

If now for n we substitute $-n$, we get

$$\frac{1}{2^n} + (-1)^n 2 : \frac{1}{2^n} - (-1)^n : \frac{1}{2^n} - (-1)^n,$$

i.e. $\quad -\{2^{n+1} + (-1)^n\} : 2^n - (-1)^n : 2^n - (-1)^n,$

i.e. the point A_n.

Similarly, if we substitute $-n$ for n in the equation of the line $B^{(n)}C^{(n)}$, we get the equation of B_nC_n.

Again, the n^{th} descending conic has an equation of the form $f(x, y, z, n) = 0$, and is the symmetrical conic through the vertices of $A^{(n)}B^{(n)}C^{(n)}$, say $\chi(n) \times \psi(n) + \psi(n) \times \phi(n) + \phi(n) \times \chi(n) = 0$.

Hence the conic $f(x, y, z, -n) = 0$,

i.e. $\quad \chi(-n) \times \psi(-n) + \psi(-n) \times \phi(-n) + \phi(-n) \times \chi(-n) = 0,$

is the symmetrical conic through the vertices of $A_nB_nC_n$, *i.e.* the $(n+1)^{\text{th}}$ ascending conic.

Thus it is sufficient to have only the descending (or ascending) expressions, and to give n the values $0, \pm 1, \pm 2, \ldots \pm \infty$, but it is interesting to keep the two sets of expressions on account of their connection with the dual correspondence which we are now to consider.

Dual.

Take $l : m : n$ as the line coordinates of the line $lx + my + nz = 0$, and then $fx + gy + hz = 0$ (line coordinates) is the equation to the point $f : g : h$.

We name lines in the dual construction by the corresponding small letters, and ap means the point of intersection of the two straight lines a and p, just as AP means the line joining A and P.

Point Construction.	Line Construction.
Given triangle ABC and P.	Given triangle abc and p.
AP cuts BC in A′, etc.	ap and bc lie on a', etc.
B′C′ cuts BC in Q, etc.	$b'c'$ and bc lie on q, etc.
Q, R, and S are collinear.	q, r, and s are concurrent.
BR cuts CS in A_1, etc.	br and cs lie on a_1, etc.

If we take abc the same triangle as ABC, and the line QRS as $p(1:1:1)$, the line construction gives us exactly the same figure as the point construction. We have merely put in the small letters in the same figure.

The analysis is quite the same, and everything is interpreted dually.

From the figure we see that:—

aq is A′, and AQ is a'; *i.e.* a' is the line B_1C_1;

BR and CS fix A_1; $\therefore$ br and cs fix a_1; *i.e.* a_1 is the line B′C′;

a_1q is A″, and A_1Q is a''; *i.e.* a'' is the line B_2C_2;

B_1R and C_1S fix A_2: $\therefore$ b_1r and c_1s fix a_2; *i.e.* a_2 is the line B″C″.

Proceeding in this way we see that the figures coincide as follows:—

$$\begin{aligned} A_n &\text{ is the same as } b^{(n)}c^{(n)}, \\ A^{(n)} &\quad ,, \quad ,, \quad ,, \quad ,, \quad b_nc_n, \\ B_nC_n &\quad ,, \quad ,, \quad ,, \quad ,, \quad a^{(n)}, \\ B^{(n)}C^{(n)} &\quad ,, \quad ,, \quad ,, \quad ,, \quad a_n. \end{aligned}$$

We have also the following correspondence:—To the n^{th} descending conic through $A^{(n)}$, $B^{(n)}$, and $C^{(n)}$ and touching $B^{(n-1)}C^{(n-1)}$, etc., corresponds the conic touching $a^{(n)}$, $b^{(n)}$, and $c^{(n)}$, and through the points $b^{(n-1)}c^{(n-1)}$, etc.; *i.e.* touching B_nC_n, etc., and through the points A_{n-1}, B_{n-1}, and C_{n-1}; *i.e.* the n^{th} ascending conic.

Point Construction (corresponds to)	Line Construction.
n^{th} descending conic and triangle.	n^{th} ascending conic and triangle.
n^{th} ascending conic and triangle.	n^{th} descending conic and triangle.

The above coincidence and correspondence are more easily verified by the analysis in the following way:—

To $$A^{(n)}, \left(\frac{2^n+(-1)^n 2}{3} : \frac{2^n-(-1)^n}{3} : \frac{2^n-(-1)^n}{3}\right)$$

corresponds $a^{(n)}$ with the same line coordinates; *i.e.* $a^{(n)}$ has the point equation

$$\frac{2^n+(-1)^n 2}{3}x + \frac{2^n-(-1)^n}{3}y + \frac{2^n-(-1)^n}{3}z = 0,$$

which is the equation to B_nC_n.

In the same way we can prove the coincidence of the other lines and points in the two constructions.

To the n^{th} descending conic

$$-\frac{2^{2n}-1}{3}\Sigma x^2+\frac{2^{2n}+2}{3}\Sigma yz=0,\ \text{(i) (point coordinates)}$$

corresponds the same line equation.

But (i) and $\dfrac{2^{2n-2}-1}{3}\Sigma x^2+\dfrac{2^{2n-1}+1}{3}\Sigma yz=0$ are related to one another in the same way as

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,\quad \text{(point)}$$

and $$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0,\quad \text{(line)}$$

where A, B, C, F, G, and H are the signed minors of the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

$\therefore$ (i) interpreted as a *line* equation is the n^{th} ascending conic. Hence the correspondence of the conics is established.

The line construction also gives us the following *construction for the pole of a line with respect to a triangle* :—

Let ABC be the triangle and QRS the line.

BC, CA and AB cut QRS in the points Q, R, and S respectively. BR and CS intersect in A_1, etc. AA_1, BB_1, CC_1 are concurrent, and their point of intersection is P, the required pole.

§ II.

Since all the above triangles have their vertices on the lines AP, BP and CP, we proceed to find if there are other triangles with their vertices on these lines, and such that the polar of P with respect to them is determined by the same three points, Q, R, and S.

Suppose we take a triangle ${}_\lambda A\,{}_\mu B\,{}_\nu C$ whose vertices have the coordinates $\lambda:1:1$, $1:\mu:1$, $1:1:\nu$ respectively. ${}_\mu B$, ${}_\nu C$ and Q must be collinear;

i.e. $$\begin{vmatrix} 1 & \mu & 1 \\ 1 & 1 & \nu \\ 0 & 1 & -1 \end{vmatrix}=0,\ \text{which reduces to } \mu=\nu.$$

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$\therefore$ The first conditions are $\lambda=\mu=\nu$; *i.e.* the sides of the triangle ${}_\lambda A\,{}_\lambda B\,{}_\lambda C$ with vertices $\lambda:1:1$, $1:\lambda:1$, $1:1:\lambda$, respectively pass through Q, R, and S. In order that the polar of P with respect to this triangle may be determined by Q, R, and S, we must show in addition that ${}_\lambda B'\,{}_\lambda C'$ passes through Q, etc.

${}_\lambda A'$ is the intersection of ${}_\lambda B\,{}_\lambda C$ and ${}_\lambda AP$;

i.e. of $y=z$, and $\begin{vmatrix} x & y & z \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix}=0$;

i.e. $\dfrac{2}{\lambda+1}:1:1.$

$\therefore$ The vertices of ${}_\lambda A'\,{}_\lambda B'\,{}_\lambda C'$ are

$$\frac{2}{\lambda+1}:1:1,\ 1:\frac{2}{\lambda+1}:1, \text{ and } 1:1:\frac{2}{\lambda+1},$$

and it is a triangle of the series named above.

$\therefore$ The polar of P with respect to the triangle ${}_\lambda A\,{}_\lambda B\,{}_\lambda C$ is determined by the three points Q, R, and S.

We can evidently derive an ascending and a descending series of triangles from ${}_\lambda A\,{}_\lambda B\,{}_\lambda C$ as in I., and thus we have an infinity of these series, and they must be mutually exclusive.

If we call the descending series $\lambda, \lambda', \lambda'', \ldots$ and the ascending series $\lambda, \lambda_1, \lambda_2 \ldots$,

then $$\lambda'=\frac{2}{\lambda+1},\ \lambda''=\frac{2}{\lambda'+1}=\frac{2}{\dfrac{2}{\lambda+1}+1}, \text{ etc.,}$$

and $$\lambda_1=\frac{2}{\lambda}-1,\ \lambda_2=\frac{2}{\lambda_1}-1=\frac{2}{\dfrac{2}{\lambda}-1}-1, \text{ etc.,}$$

and we can express the vertices of the n^{th} ascending or descending triangle as a continued fraction involving λ.

We can by means of the following table find the n^{th} value of the fractions, and thus get $\lambda^{(n)}$ and λ_n.

${}_\lambda A^\circ$	$\lambda+0 : 0\lambda+1 : 0\lambda+1$
${}_\lambda A'$	$0\lambda+2 : \lambda+1 : \lambda+1$
${}_\lambda A''$	$2\lambda+2 : \lambda+3 : \lambda+3$
${}_\lambda A'''$	$2\lambda+6 : 3\lambda+5 : 3\lambda+5$
${}_\lambda A^{(4)}$	$6\lambda+10 : 5\lambda+11 : 5\lambda+11$
${}_\lambda A^{(5)}$	$10\lambda+22 : 11\lambda+21 : 11\lambda+21$

When we compare the coefficients of λ and the constant terms with the table of descending vertices given in I. and test by Mathematical Induction, we get the general expressions :—

$$\begin{aligned} {}_\lambda A^{(n)} \text{ is } & [\{2^n+(-1)^n 2\}\lambda+\{2^{n+1}-(-1)^n 2\}] \\ & :[\{2^n-(-1)^n\}\lambda+\{2^{n+1}+(-1)^n\}] \\ & :[\{2^n-(-1)^n\}\lambda+\{2^{n+1}+(-1)^n\}]. \end{aligned}$$

Also if ${}_\lambda A$ is $\lambda:1:1$,

$${}_\lambda B\, {}_\lambda C \text{ is } -(\lambda+1)x+y+z=0,$$

and we get the equation to ${}_\lambda B^{(n)}\, {}_\lambda C^{(n)}$

$$-[\{2^{n+1}+(-1)^n\}\lambda+\{2^{n+2}-(-1)^n\}]x+[\{2^n-(-1)^n\}\lambda+\{2^{n+1}+(-1)^n\}]y$$
$$+[\{2^n-(-1)^n\}\lambda+\{2^{n+1}+(-1)^n\}]z=0.$$

$$\lambda^{(n)}=\frac{x \text{ coordinate of } {}_\lambda A^{(n)}}{y \text{ coordinate of } {}_\lambda A^{(n)}}.$$

For λ_n we have the following table :—

${}_\lambda A_0$	$-\lambda+0 : 0\lambda-1 : 0\lambda-1$
${}_\lambda A_1$	$-\lambda+2 : \lambda-0 : \lambda-0$
${}_\lambda A_2$	$-3\lambda+2 : \lambda-2 : \lambda-2$
${}_\lambda A_3$	$-5\lambda+6 : 3\lambda-2 : 3\lambda-2$
${}_\lambda A_4$	$-11\lambda+10 : 5\lambda-6 : 5\lambda-6$

When we compare this table with the corresponding table in I., we find that ${}_\lambda A_n$ is the point

$$\begin{aligned} & [-\{2^{n+1}+(-1)^n\}\lambda+\{2^{n+1}-(-1)^n 2\}] \\ & :[\{2^n-(-1)^n\}\lambda-\{2^n+(-1)^n 2\}] \\ & :[\{2^n-(-1)^n\}\lambda-\{2^n+(-1)^n 2\}], \end{aligned}$$

and ${}_\lambda B_n\, {}_\lambda C_n$ is the line

$$[\{2^n+(-1)^n 2\}\lambda-\{2^n-(-1)^n 4\}]x+[\{2^n-(-1)^n\}\lambda-\{2^n+(-1)^n 2\}]y$$
$$+[\{2^n-(-1)^n\}\lambda-\{2^n+(-1)^n 2\}]z=0.$$

$$\lambda_n=\frac{x \text{ coordinate of } {}_\lambda A_n}{y \text{ coordinate of } {}_\lambda A_n}.$$

When we substitute $-n$ for n in the coordinates of ${}_\lambda A^{(n)}$, and in the equation to ${}_\lambda B^{(n)}\, {}_\lambda C^{(n)}$, we get the coordinates of ${}_\lambda A_n$ and the equation to ${}_\lambda B_n\, {}_\lambda C_n$ respectively.

Hence it is sufficient to have only the descending (or ascending) expressions, and to give n the values $0, \pm 1, \pm 2, \ldots \pm \infty$.

As in I., we find that the n^{th} λ-descending conic which passes through the vertices of ${}_\lambda A^{(n)}\,{}_\lambda B^{(n)}\,{}_\lambda C^{(n)}$ and touches the sides of ${}_\lambda A^{(n-1)}\,{}_\lambda B^{(n-1)}\,{}_\lambda C^{(n-1)}$ has the equation

$$-\{(2^{2n}-1)\lambda^2+(2^{2n+2}+2)\lambda+(2^{2n+2}-1)\}\Sigma x^2$$

$$+\{(2^{2n}+2)\lambda^2+(2^{2n+2}-4)\lambda+(2^{2n+2}+2)\}\Sigma yz=0, \qquad (\gamma)$$

and the whole λ-system of conics is got by giving n the values $0,\ \pm 1,\ \pm 2,\ \ldots \pm\infty$.

The family (γ) includes (α) and (β), and is the complete double contact family $\Sigma x^2+k\Sigma yz=0$. $\qquad (\delta)$

When we put $n=\pm\infty$ in the λ-expressions, we find the same descending and ascending limits in every λ-series, as we found in I.

This must be the case, since all the limiting values in I. are fixed points and lines of the system.

If we put $n=0$ in (γ), we get $(2\lambda+1)\Sigma x^2-(\lambda^2+2)\Sigma xy=0$, (γ') as the equation to the conic through the points ${}_\lambda A, {}_\lambda B, {}_\lambda C$ and touching the sides of ${}_\lambda A_1\,{}_\lambda B_1\,{}_\lambda C_1$, and either (γ) or (γ') shows us that each conic of the family may belong to two λ-groups.

The series in I. corresponds to $\lambda=\infty$, if we take ABC as the primary member, or to $\lambda=0$ if we take $A'B'C'$ as the primary.

So when we put $\lambda=\infty$ all our λ-expressions are the same as the corresponding expressions in I.; or if we put $\lambda=0$, the n^{th} λ-descending expression is the same as the m^{th} descending expression in I. where $m=n+1$.

As an example of the two values of λ we may take

$$\Sigma x^2-\Sigma yz=0 \quad \text{(pair of imaginary tangents).}$$

This gives $\dfrac{\lambda^2+2}{2\lambda+1}=1$; *i.e.* $\lambda=1$ twice; *i.e.* we are at the limiting descending position, for the point P corresponds to $\lambda=1$.

The ascending and descending limiting conics corresponding to $k=2$ and $k=-1$ in (δ) are the only two for which the two values of λ are equal.

If we take the conic $\Sigma yz=0$, we get $2\lambda+1=0$.

The solutions are $\lambda=\infty$ and $\lambda=-\frac{1}{2}$. $\lambda=\infty$ gives us the conic circumscribing ABC and inscribed in $A_1B_1C_1$, as we found in I.

$\lambda = -\frac{1}{2}$ means that this conic also circumscribes the triangle whose vertices are $-\frac{1}{2}:1:1$, etc., and is inscribed in its first ascending triangle with vertices $-5:1:1$, etc.,

$$\left(\lambda_1 = \frac{2}{\lambda} - 1 \; ; \; \lambda = -\tfrac{1}{2} \; ; \; \therefore \; \lambda_1 = -5\right).$$

Suppose we take the n^{th} descending conic derived from the original triangle with vertices $(\lambda : 1 : 1)$, etc., and see what is the nature of the second triangle which has the same conic.

Let $(X_1 : 1 : 1)$, etc., and $(X_2 : 1 : 1)$, etc., be the two triangles which are inscribed in the conic, and whose first ascending derived triangles circumscribe it.

X_1 and X_2 are the solutions of the equation

$$\frac{X^2+2}{2X+1} = \frac{(2^{2n}+2)\lambda^2 + (2^{2n+2}-4)\lambda + (2^{2n+2}+2)}{(2^{2n}-1)\lambda^2 + (2^{2n+2}+2)\lambda + (2^{2n+2}-1)} \; ;$$

$$\begin{aligned} \textit{i.e.} \quad & \{(2^{2n}-1)\lambda^2 + (2^{2n+2}+2)\lambda + (2^{2n+2}-1)\}X^2 \\ & - \{(2^{2n+1}+4)\lambda^2 + (2^{2n+3}-8)\lambda + (2^{2n+3}+4)\}X \\ & + \{(2^{2n}-4)\lambda^2 + (2^{2n+2}+8)\lambda + (2^{2n+2}-4)\} = 0. \end{aligned}$$

We know that one solution is

$$X_1 = \frac{\{2^n + (-1)^n 2\}\lambda + \{2^{n+1} - (-1)^n 2\}}{\{2^n - (-1)^n\}\lambda + \{2^{n+1} + (-1)^n\}}. \qquad (a)$$

$\therefore$ the other solution is

$$X_2 = \frac{\{2^n - (-1)^n 2\}\lambda + \{2^{n+1} + (-1)^n 2\}}{\{2^n + (-1)^n\}\lambda + \{2^{n+1} - (-1)^n\}}. \qquad (b)$$

All the (a) solutions belong to the λ-family.

The first descending triangle derived from X_2 is given by X_2' where $X_2' = \dfrac{2}{X_2+1}$, and we find that X_2' is got by substituting $(n+1)$ for n in X_2.

$\therefore$ *all the* (b) *solutions belong to one family.*

Put $n=0$ in (a) and (b), and we find that X_1 and X_2 are the n^{th} descending triangles derived from λ and $\dfrac{-\lambda+4}{2\lambda+1}$.

Call the two families λ_1 and λ_2. $\quad \therefore \; \lambda_2 = \dfrac{-\lambda_1+4}{2\lambda_1+1}$;

$$\textit{i.e.} \quad 2\lambda_1\lambda_2 + (\lambda_1+\lambda_2) - 4 = 0. \qquad (c)$$

This is simply the involutary relation connecting the two roots λ_1 and λ_2 of the equations $\dfrac{\lambda^2+2}{2\lambda+1} = -k$, where k is an arbitrary parameter.

The double points of the involution are $\lambda_1 = \lambda_2 = 1$ and $\lambda_1 = \lambda_2 = -2$, which are the limiting cases of both families, as we have seen before.

So the two families whose initial triangles have the vertices $(\lambda_1 : 1 : 1)$, etc., and $(\lambda_2 : 1 : 1)$, etc., respectively, where λ_1 and λ_2 are connected by the relation (c), have their n^{th} descending (and n^{th} ascending) conics in common; *i.e.* they have all their conics in common.

Compare this with the theorem:—If the vertices of two triangles lie on a conic, their sides touch another conic, and conversely.

We might have suspected, without analysis, that the families would be associated in pairs, for the three lines AP, BP, and CP cut each conic of the family in 6 points, and the above analysis shows that these 6 points are the vertices of a λ_1 and a λ_2 triangle where λ_1 and λ_2 have the same meaning as in (c).

§ III.

We will now show that the triangles of II. include all the possible triangles with respect to which the polar of P is determined by Q, R, and S. This amounts to showing that when P, Q, R, and S are given, the lines AP, BP, and CP are fixed.

On referring to § 1, the equations to AC, AB, AL, and AQ are $y=0$, $z=0$, $y-z=0$, and $y+z=0$ respectively.

$\therefore$ A(RSLQ) is a harmonic pencil; *i.e.* (RSLQ) is a harmonic range.

Similarly, (SQMR) and (QRNS) are harmonic ranges.

$\therefore$ Since Q, R and S are fixed, L, M, and N are fixed points; *i.e.* AP, BP, and CP are fixed and unique.

Construction for any triangle ABC of the series, when P, Q, R and S are given.

Construct L, M, and N so that (QRNS), (RSLQ), and (SQMR) are harmonic ranges.

Take A any point on LP (for λ may have any value).

AS and AR cut MP and NP in B and C respectively.

Dually, it follows that when p, q, r and s are given, Q, R and S are unique, and that $(rslq)$, etc., are harmonic pencils (L is the point pq, so l is the line PQ).

Hence (MNQL), (NLRM) and (LMSN) are harmonic ranges.

L, M, and N are the points pq, pr and ps.

Hence, *construction for any triangle ABC of series when p, q, r and s are given.*

Construct Q, R and S so that (MNQL), etc., are harmonic ranges. If A is any point on LP, AS and AR cut MP and NP in B and C respectively.

The following additional proof of uniqueness of AP, BP, and CP may be of some interest.

We must look for triangles such that when P, Q, R, and S are referred to these, it is possible to choose our coordinates so that P, Q, R, and S are the points $(1:1:1)$, $(0:1:-1)$, $(-1:0:1)$, and $(1:-1:0)$ respectively.

Let P, Q, R, and S have the cartesian coordinates $(0, 0)$, $(\lambda, 1)$, $(\mu, 1)$, and $(\nu, 1)$ (arbitrary choice of axes and scale).

Let the transformation from cartesian to trilinear coordinates be

$$\xi = a_1x + b_1y + c_1,$$
$$\eta = a_2x + b_2y + c_2,$$
$$\text{and} \quad \zeta = a_3x + b_3y + c_3.$$

For P, $\xi : \eta : \zeta = 1 : 1 : 1$, $x = 0$, and $y = 0$;

$$\therefore \; c_1 = c_2 = c_3 = c, \text{ say}.$$

For Q, $\xi = 0$, $\eta = -\zeta$, $x = \lambda$, $y = 1$;

$$\therefore \; a_1\lambda + b_1 + c = 0,$$
$$\text{and} \quad a_2\lambda + b_2 + c = -(a_3\lambda + b_3 + c),$$
$$\text{etc.}$$

$$i.e. \quad a_1\lambda + b_1 + c = 0.$$
$$a_2\mu + b_2 + c = 0.$$
$$a_3\nu + b_3 + c = 0.$$
$$(a_2 + a_3)\lambda + (b_2 + b_3) + 2c = 0.$$
$$(a_3 + a_1)\mu + (b_3 + b_1) + 2c = 0.$$
$$(a_1 + a_2)\nu + (b_1 + b_2) + 2c = 0.$$

We easily get

$$\frac{a_1}{\mu - \nu} = \frac{a_2}{\nu - \lambda} = \frac{a_3}{\lambda - \mu} = \rho \quad \text{(say)}.$$

$$\therefore \quad b_1 = -c - \rho\lambda(\mu - \nu).$$

$$b_2 = -c - \rho\mu(\nu - \lambda).$$

$$b_3 = -c - \rho\nu(\lambda - \mu).$$

ρ and c are arbitrary.

In cartesians let A be (x_1, y_1).

In trilinears A is $1 : 0 : 0$.

$$\therefore \quad \rho(\nu - \lambda)x_1 - \rho\mu(\nu - \lambda)y_1 - cy_1 + c = 0,$$

$$\text{and} \quad \rho(\lambda - \mu)x_1 - \rho\nu(\lambda - \mu)y_1 - cy_1 + c = 0.$$

Subtracting and re-arranging, we get

$$\frac{y_1}{x_1} = \frac{2\lambda - \mu - \nu}{\lambda\mu + \lambda\nu - 2\mu\nu} = \text{constant}. \quad \text{(AP)}$$

$\therefore$ AP, BP, and CP are fixed.

The equation for AP, when converted into trilinears by the above transformation, reduces to $\eta = \zeta$, as in I.

Hence if P, Q, R, and S be given, q, r, and s are determined; and, dually, if p, q, r, and s be given, Q, R, and S are determined.