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THE DECOMPOSITION OF PERMUTATION MODULE FOR INFINITE CHEVALLEY GROUPS, II

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Abstract. Let $\mathbf{G}$ be a connected reductive algebraic group over an algebraically closed field $\Bbbk$ and $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$. In this paper, we completely determine the composition factors of the permutation module $\mathbb{F}[\mathbf{G}/\mathbf{B}]$ for any field $\mathbb{F}$.

§1. Introduction

Let $\mathbf{G}$ be a connected reductive algebraic group over an algebraically closed field $\Bbbk$ and $\mathbf{B}$ be an Borel subgroup of $\mathbf{G}$. We will identify $\mathbf{G}$ with $\mathbf{G}(\Bbbk)$ and $\mathbf{B}$ with $\mathbf{B}(\Bbbk)$. Let $\mathbb{F}$ be another field and all the representations are over $\mathbb{F}$. Now we just regard $\mathbf{G}/\mathbf{B}$ as a quotient set and consider the vector space $\mathbb{F}[\mathbf{G}/\mathbf{B}]$, which has a basis of the left cosets of $\mathbf{B}$ in $\mathbf{G}$. With left multiplication of the group $\mathbf{G}$, $\mathbb{F}[\mathbf{G}/\mathbf{B}]$ is an $\mathbb{F}\mathbf{G}$-module, which is isomorphic to $\mathbb{F}\mathbf{G}\otimes_{\mathbb{F}\mathbf{B}}\mathrm{tr}$, where tr denotes the one-dimensional trivial $\mathbf{B}$-module. The permutation module $\mathbb{F}[\mathbf{G}/\mathbf{B}]$ was studied in [2] and [3] when $\Bbbk=\bar{\mathbb{F}}_q$, where $\bar{\mathbb{F}}_q$ is the algebraically closure of finite field $\mathbb{F}_q$ of q elements. In their determination of the composition factors of $\mathbb{F}[\mathbf{G}/\mathbf{B}]$, the proofs make essential use of the fact that $\bar{\mathbb{F}}_q$ is a union of finite fields.

The Steinberg module St is the socle of $\mathbb{F}[\mathbf{G}/\mathbf{B}]$, and the irreducibility of St has been proved by Xi (see [8]) in the case $\Bbbk=\bar{\mathbb{F}}_q$, and char $\mathbb{F}=0$ or char $\bar{\mathbb{F}}_q$. Later, Yang removed this restriction on char $\mathbb{F}$ and proved the irreducibility of Steinberg module for any field $\mathbb{F}$ in [9] (also in the case $\Bbbk=\bar{\mathbb{F}}_q$). Recently, Putman and Snowden showed that when $\Bbbk$ is an infinite field (not necessary to be algebraically closed), then the Steinberg representation of $\mathbf{G}$ is always irreducible for any field $\mathbb{F}$ (see [6]). Their work inspires the idea of the determination of the composition factors of $\mathbb{F}[\mathbf{G}/\mathbf{B}]$ for general case in this paper. We will construct a filtration of submodules for $\mathbb{F}[\mathbf{G}/\mathbf{B}]$ whose subquotients are denoted by E_J (indexed by the subsets of the set I of simple reflections). The main theorem is as follows:

THEOREM 1.1. *Let $\mathbb{F}$ be any field. All $\mathbb{F}\mathbf{G}$-modules E_J are irreducible and pairwise nonisomorphic. Moreover, the $\mathbb{F}\mathbf{G}$-module $\mathbb{F}[\mathbf{G}/\mathbf{B}]$ has exactly $2^{|I|}$ composition factors, each occurring with multiplicity one.*

It is well known that the flag variety $\mathbf{G}/\mathbf{B}$ plays a very important role in the representation theory. So the decomposition of $\mathbb{F}[\mathbf{G}/\mathbf{B}]$ may have many applications in other areas such as algebraic geometry and number theory.

This paper is organized as follows: Section 2 contains some notations and preliminary results. In particular, we study the properties of the subquotient modules E_J of $\mathbb{F}[\mathbf{G}/\mathbf{B}]$. In Section 3, we list some properties of the unipotent radical $\mathbf{U}$ of $\mathbf{B}$ and study the self-enclosed subgroup of $\mathbf{U}$, which is useful in the later discussion. Section 4 gives the nonvanishing

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property of the augmentation. In the last section, we will prove that all the $\mathbb{F}\mathbf{G}$-modules E_J are irreducible for any fields $\Bbbk$ and $\mathbb{F}$.

§2. Preliminaries

As in the introduction, $\mathbf{G}$ is a connected reductive algebraic group over an algebraically closed field $\Bbbk$ and $\mathbf{B}$ is a Borel subgroup. Let $\mathbf{T}$ be a maximal torus contained in $\mathbf{B}$, and $\mathbf{U} = R_u(\mathbf{B})$ be the unipotent radical of $\mathbf{B}$. We identify $\mathbf{G}$ with $\mathbf{G}(\Bbbk)$ and do likewise for various subgroups of $\mathbf{G}$ such as $\mathbf{B}, \mathbf{T}, \mathbf{U} \cdots$. We denote by $\Phi = \Phi(\mathbf{G}; \mathbf{T})$ the corresponding root system, and by Φ^+ (resp. Φ^-) the set of positive (resp. negative) roots determined by $\mathbf{B}$. Let $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ be the corresponding Weyl group. We denote by $\Delta = \{\alpha_i \mid i \in I\}$ the set of simple roots and by $S = \{s_i := s_{\alpha_i} \mid i \in I\}$ the corresponding simple reflections in W. For each $\alpha \in \Phi$, let $\mathbf{U}_\alpha$ be the root subgroup corresponding to α and we fix an isomorphism $\varepsilon_\alpha : \Bbbk \to \mathbf{U}_\alpha$ such that $t\varepsilon_\alpha(c)t^{-1} = \varepsilon_\alpha(\alpha(t)c)$ for any $t \in \mathbf{T}$ and $c \in \Bbbk$. For any $w \in W$, let $\mathbf{U}_w$ (resp. $\mathbf{U}'_w$) be the subgroup of $\mathbf{U}$ generated by all $\mathbf{U}_\alpha$ with $w(\alpha) \in \Phi^-$ (resp. $w(\alpha) \in \Phi^+$). For any $J \subset I$, let W_J be the corresponding standard parabolic subgroup of W and w_J be the longest element in W_J. For a subgroup H of $\mathbf{G}$ and $g \in \mathbf{G}$, let $H^g = g^{-1}Hg$.

The permutation module $\mathbb{F}[\mathbf{G}/\mathbf{B}]$ is isomorphic to the induced module $\mathbb{M}(\mathrm{tr}) = \mathbb{F}\mathbf{G} \otimes_{\mathbb{F}\mathbf{B}} \mathrm{tr}$. Now let $\mathbf{1}_{tr}$ be a nonzero element of tr. For convenience, we abbreviate $x \otimes \mathbf{1}_{\mathrm{tr}} \in \mathbb{M}(\mathrm{tr})$ to $x\mathbf{1}_{\mathrm{tr}}$. Each element $\varphi \in \mathrm{End}_{\mathbb{F}\mathbf{G}}(\mathbb{M}(\mathrm{tr}))$ is determined by $\varphi(\mathbf{1}_{tr})$. Note that $\varphi(\mathbf{1}_{tr})$ is a $\mathbf{B}$-stable vector. Thus we have $\varphi(\mathbf{1}_{tr}) = \lambda \mathbf{1}_{tr}$ for some $\lambda \in \mathbb{F}$, which implies that $\mathrm{End}_{\mathbb{F}\mathbf{G}}(\mathbb{M}(\mathrm{tr})) \cong \mathbb{F}$. In particular, the $\mathbb{F}\mathbf{G}$-module $\mathbb{M}(\mathrm{tr})$ is indecomposable.

For any $w \in W$, let $\dot{w}$ be a representative of w. For any $t \in \mathbf{T}$ and $n \in N_{\mathbf{G}}(\mathbf{T})$, we have $nt\mathbf{1}_{\mathrm{tr}} = n\mathbf{1}_{\mathrm{tr}}$. Thus $w\mathbf{1}_{\mathrm{tr}} = \dot{w}\mathbf{1}_{\mathrm{tr}}$ is well-defined. For any $J \subset I$, we set

$$\eta_J = \sum_{w \in W_J} (-1)^{\ell(w)} w\mathbf{1}_{\mathrm{tr}},$$

where $\ell(w)$ is the length of w. Let $\mathbb{M}(\mathrm{tr})_J = \mathbb{F}\mathbf{G}\eta_J$. It was proved in [8, Prop. 2.3] that $\mathbb{M}(\mathrm{tr})_J = \mathbb{F}\mathbf{U}W\eta_J$. For $w \in W$, we set

$$\mathscr{R}(w) = \{i \in I \mid ws_i < w\}.$$

For any subset $J \subset I$, we let

$$X_J = \{x \in W \mid x \text{ has minimal length in } xW_J\}.$$

Proposition 2.1. *For any $J \subset I$, the $\mathbb{F}\mathbf{G}$-module $\mathbb{M}(\mathrm{tr})_J$ has the form*

$$\mathbb{M}(\mathrm{tr})_J = \sum_{w \in X_J} \mathbb{F}\mathbf{U}w\eta_J = \sum_{w \in X_J} \mathbb{F}\mathbf{U}_{w_J w^{-1}} w\eta_J,$$

and the set $\{uw\eta_J \mid w \in X_J, u \in \mathbf{U}_{w_J w^{-1}}\}$ forms a basis of $\mathbb{M}(\mathrm{tr})_J$.

Proof. First, it is easy to see that $\mathbb{M}(\mathrm{tr})_J = \mathbb{F}\mathbf{U}W\eta_J = \mathbb{F}\mathbf{U}X_J\eta_J$ since $y\eta_J = (-1)^{\ell(y)}\eta_J$ for any $y \in W_J$. Let $w \in X_J$. For any $\gamma \in \Phi^+$ such that $w_J w^{-1}(\gamma) \in \Phi^+$, we have $x^{-1}w^{-1}(\gamma) \in \Phi^+$ for any $x \in W_J$. For $u \in \mathbf{U}_\gamma$ and $x \in W_J$, we get

$$uwx\mathbf{1}_{\mathrm{tr}} = wx(x^{-1}w^{-1}uwx)\mathbf{1}_{\mathrm{tr}} = wx\mathbf{1}_{\mathrm{tr}},$$

since $x^{-1}w^{-1}uwx \in \mathbf{U}$. In particular, we get $\mathbf{U}w\eta_J = \mathbf{U}_{w_J w^{-1}} w\eta_J$. Then we obtain the first part.

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In the following, we show that $\{uw\eta_J \mid w \in X_J, u \in \mathbf{U}_{w_J w^{-1}}\}$ forms a basis of $\mathbb{M}(\mathrm{tr})_J$. It is enough to prove that this set is linearly independent. Suppose this set is linearly dependent, then there exist $f_{u,w} \in \mathbb{F}$ (not all zero) such that

$$\sum_{w \in X_J} \sum_{u \in \mathbf{U}_{w_J w^{-1}}} f_{u,w} uw\eta_J = 0. \tag{2.1}$$

Let $z \in X_J$ whose length is maximal such that $f_{u_0,z} \neq 0$ for some $u_0 \in \mathbf{U}_{w_J z^{-1}}$. Substitute $\eta_J = \sum_{x \in W_J} (-1)^{\ell(x)} x \mathbf{1}_{\mathrm{tr}}$ in the equation (2.1). According to the Bruhat decomposition, the set $\{uw\mathbf{1}_{\mathrm{tr}} \mid w \in W, u \in \mathbf{U}_{w^{-1}}\}$ is linearly independent in $\mathbb{M}(\mathrm{tr})$. Then we have $\sum_{u \in \mathbf{U}_{w_J z^{-1}}} f_{u,z} uzw_J \mathbf{1}_{\mathrm{tr}} = 0$. So we get $f_{u,z} = 0$ for all $u \in \mathbf{U}_{w_J z^{-1}}$, which is a contradiction. The proposition is proved. ∎

For any $i \in I$, set $\mathbf{U}^*_{\alpha_i} = \mathbf{U}_{\alpha_i} \backslash \{id\}$, where id is the neutral element of $\mathbf{U}$. For the convenience of later discussion, we give some details about the expression of the element $\dot{s}_i u_i w \eta_J$, where $u_i \in \mathbf{U}^*_{\alpha_i}$ and $w \in X_J$. For each $u_i \in \mathbf{U}^*_{\alpha_i}$, we have

$$\dot{s}_i u_i \dot{s}_i = f_i(u_i) \dot{s}_i h_i(u_i) g_i(u_i),$$

where $f_i(u_i), g_i(u_i) \in \mathbf{U}^*_{\alpha_i}$, and $h_i(u_i) \in \mathbf{T}$ are uniquely determined. Moreover, if we regard f_i as a morphism on $\mathbf{U}^*_{\alpha_i}$, then f_i is a bijection. The following lemma is very useful in the later discussion. Its proof can be found in the proof of [8, Prop. 2.3] and we omit it.

LEMMA 2.2. *Let $u_i \in \mathbf{U}^*_{\alpha_i}$, with the notation above, then we have*
(a) *If $ww_J \leq s_i w w_J$, then $\dot{s}_i u_i w \eta_J = s_i w \eta_J$.*
(b) *If $s_i w \leq w$, then $\dot{s}_i u_i w \eta_J = f_i(u_i) w \eta_J$.*
(c) *If $w \leq s_i w$ but $s_i w w_J \leq w w_J$, then $\dot{s}_i u_i w \eta_J = (f_i(u_i) - 1) w \eta_J$.*

Following [8, 2.6], we define

$$E_J = \mathbb{M}(\mathrm{tr})_J / \mathbb{M}(\mathrm{tr})'_J,$$

where $\mathbb{M}(\mathrm{tr})'_J$ is the sum of all $\mathbb{M}(\mathrm{tr})_K$ with $J \subsetneq K$. We denote by C_J the image of η_J in E_J. For each $w \in W$, let

$$h_w = \sum_{y \leq w} (-1)^{\ell(w) - \ell(y)} P_{y,w}(1) y \in \mathbb{F}W,$$

where $P_{y,w}$ are Kazhdan-Lusztig polynomials (see [5, Th. 1.1]). The set $\{h_w \mid w \in W\}$ is a basis of $\mathbb{F}W$. We set

$$Y_J = \{w \in X_J \mid \mathscr{R}(ww_J) = J\}.$$

LEMMA 2.3. *Let $J \subset I$. Then each one of the following sets is a basis of $\mathbb{F}W h_{w_J}$:*
(a) $\{w h_{w_J} \mid w \in X_J\}$;
(b) $\{h_{ww_J} \mid w \in X_J\}$;
(c) $\{y h_{w_J} \mid y \in Y_J\} \cup \{h_{xw_J} \mid x \in X_J \backslash Y_J\}$.

Proof. (a) By [5, Lem. 2.6(vi)], we see that

$$h_{w_J} = (-1)^{\ell(w_J)} \sum_{y \in W_J} (-1)^{\ell(y)} y \in \mathbb{F}W.$$

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It is clear that $wh_{w_J} = (-1)^{\ell(w)} h_{w_J}$ for any $w \in W_J$. So we have $\mathbb{F}Wh_{w_J} = \mathbb{F}X_J h_{w_J}$. Now suppose that there exist $a_w \in \mathbb{F}$ (not all zero) such that $\sum_{w \in X_J} a_w w h_{w_J} = 0$. Let $z \in X_J$ whose length is maximal such that $a_z \neq 0$. Substitute h_{w_J} and we get $a_z w w_J = 0$ in $\mathbb{F}W$. So $a_z = 0$, which is a contraction. Therefore, $\{wh_{w_J} \mid w \in X_J\}$ is a basis of $\mathbb{F}Wh_{w_J}$.

(b) By [4, Lem. 2.8(c)], for $x \in X_J$, we have

$$h_{xw_J} = xh_{w_J} + \sum_{w \in X_J, w < x} b_w w h_{w_J}, \quad b_w \in \mathbb{F}. \tag{2.2}$$

Using induction on $\ell(x)$ we see that

$$xh_{w_J} = h_{xw_J} + \sum_{w \in X_J, w < x} b'_w h_{ww_J}, \quad b'_w \in \mathbb{F}. \tag{2.3}$$

Thus (b) is proved by (a).

(c) We claim that for any $w \in X_J$, wh_{w_J} is a linear combination of the elements in $\{yh_{w_J} \mid y \in Y_J\} \cup \{h_{xw_J} \mid x \in X_J \backslash Y_J\}$. If $\ell(w) = 0$, then the claim is obvious. Now assume that the claim is true for $z \in X_J$ with $\ell(z) < \ell(w)$. If $w \in Y_J$, then the claim is clear. If $w \in X_J \backslash Y_J$, using formula (2.3) and induction hypothesis, we see that the claim is true. Now (c) is proved. ∎

Proposition 2.4. *For $J \subset I$, we have*

$$E_J = \sum_{w \in Y_J} \mathbb{F}\mathbf{U}_{w_J w^{-1}} w C_J,$$

and the set $\{uwC_J \mid w \in Y_J, u \in \mathbf{U}_{w_J w^{-1}}\}$ forms a basis of E_J.

Proof. For $w \in W$, we set $h'_w = h_w \mathbf{1}_{\mathrm{tr}} \in \mathbb{M}(\mathrm{tr})$. Thus, $h'_{w_J} = (-1)^{\ell(w_J)} \eta_J$ for any $J \subset I$ by [5, Lem. 2.6(vi)]. According to Lemma 2.3 (c), we get

$$\mathbb{M}(\mathrm{tr})_J = \sum_{w \in X_J} \mathbb{F}\mathbf{U} w \eta_J = \sum_{w \in Y_J} \mathbb{F}\mathbf{U} w \eta_J + \sum_{x \in X_J \backslash Y_J} \mathbb{F}\mathbf{U} h'_{xw_J}.$$

We claim that $\mathbb{M}(\mathrm{tr})'_J = \sum_{x \in X_J \backslash Y_J} \mathbb{F}\mathbf{U} h'_{xw_J}$. For $x \in X_J \setminus Y_J$, we see that $\mathscr{R}(xw_J) = K$ for some $K \supsetneq J$. Thus $xw_J = yw_K$ for some $y \in X_K$. By Lemma 2.3 (b), we have $h'_{xw_J} = h'_{yw_K} \in \mathbb{F}W\eta_K$ which implies $\sum_{x \in X_J \backslash Y_J} \mathbb{F}\mathbf{U} h'_{xw_J} \subseteq \mathbb{M}(\mathrm{tr})'_J$. On the other hand, we see that $X_K \subseteq X_J \backslash Y_J$ for any $K \supsetneq J$. Therefore we get $\mathbb{M}(\mathrm{tr})_K \subseteq \sum_{x \in X_J \backslash Y_J} \mathbb{F}\mathbf{U} h'_{xw_J}$ for any $K \supsetneq J$. The claim is proved and we get

$$E_J = \mathbb{M}(\mathrm{tr})_J / \mathbb{M}(\mathrm{tr})'_J = \sum_{w \in Y_J} \mathbb{F}\mathbf{U} w C_J.$$

It is not difficult to see that $\mathbf{U} w C_J = \mathbf{U}_{w_J w^{-1}} w C_J$ for any $w \in Y_J$. Thus, we obtain the first part.

Now we show that the set $\{uwC_J \mid w \in Y_J, u \in \mathbf{U}_{w_J w^{-1}}\}$ is a basis of E_J. It is enough to prove that this set is linearly independent. Suppose that this set is linearly dependent.

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Then there exist $f_{u,w} \in \mathbb{F}$ (not all zero) such that

$$\sum_{w \in Y_J} \sum_{u \in \mathbf{U}_{w_J w^{-1}}} f_{u,w} u w C_J = 0.$$

Noting that $E_J = \mathbb{M}(\mathrm{tr})_J / \mathbb{M}(\mathrm{tr})'_J$, we have

$$\sum_{w \in Y_J} \sum_{u \in \mathbf{U}_{w_J w^{-1}}} f_{u,w} u w \eta_J \in \mathbb{M}(\mathrm{tr})'_J.$$

Without loss of generality, we assume that $u_0 = id$ for some $z \in Y_J$ with $f_{u_0,z} \neq 0$. Note that the $\mathbf{T}$-fixed subspace of $\mathbb{M}(\mathrm{tr})_J$ is $\sum_{w \in X_J} \mathbb{F} w \eta_J$. Since $z\eta_J$ is a $\mathbf{T}$-stable vector and $\mathbb{M}(\mathrm{tr})'_J = \sum_{x \in X_J \setminus Y_J} \mathbb{F}\mathbf{U} h'_{x w_J}$, it is not difficult to see that $z\eta_J$ is a linear combination of the following set

$$\{w\eta_J \mid w \in Y_J, w \neq z\} \cup \{h'_{x w_J} \mid x \in X_J \setminus Y_J\}.$$

This is a contradiction by Lemma 2.3 (c). The proposition is proved. ∎

PROPOSITION 2.5. *[8, Prop. 2.7] If J and K are different subsets of I, then E_J and E_K are not isomorphic.*

By the definition of E_J, there exists a filtration of submodules for $\mathbb{F}[\mathbf{G}/\mathbf{B}]$ whose subquotients are E_J ($J \subset I$). In the following of this paper, we prove the irreducibility of E_J for any $J \subset I$. Combining Proposition 2.5, we get Theorem 1.1.

§3. Self-enclosed subgroups

This section contains some preliminaries and properties of unipotent groups that are useful in later discussion. As before, let $\mathbf{U}$ be the unipotent radical of the Borel subgroup $\mathbf{B}$. For any $w \in W$, we set

$$\Phi_w^- = \{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^-\}, \quad \Phi_w^+ = \{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^+\}.$$

As before, $\mathbf{U}_w$ (resp. $\mathbf{U}'_w$) is the subgroup of $\mathbf{U}$ generated by all $\mathbf{U}_\alpha$ with $\alpha \in \Phi_w^-$ (resp. $\alpha \in \Phi_w^+$). The following properties are well known (see [1]).

(a) For $w \in W$ and any root $\alpha \in \Phi$, we have $\dot{w}\mathbf{U}_\alpha \dot{w}^{-1} = \mathbf{U}_{w(\alpha)}$;

(b) $\mathbf{U}_w$ and $\mathbf{U}'_w$ are subgroups of $\mathbf{U}$, and we have $\dot{w}\mathbf{U}'_w \dot{w}^{-1} \subset \mathbf{U}$;

(c) The multiplication map $\mathbf{U}_w \times \mathbf{U}'_w \to \mathbf{U}$ is a bijection;

(d) Let $\Phi^+ = \{\delta_1, \delta_2, \ldots, \delta_m\}$. Then $\mathbf{U} = \mathbf{U}_{\delta_1}\mathbf{U}_{\delta_2}\ldots\mathbf{U}_{\delta_m}$ and each element $u \in \mathbf{U}$ is uniquely expressible in the form $u = u_1 u_2 \ldots u_m$ with $u_i \in \mathbf{U}_{\delta_i}$;

(e) (*Commutator relations*) Given two positive roots α and β, there exist a total ordering on Φ^+ and integers $c_{\alpha\beta}^{mn}$ such that

$$[\varepsilon_\alpha(a), \varepsilon_\beta(b)] := \varepsilon_\alpha(a)\varepsilon_\beta(b)\varepsilon_\alpha(a)^{-1}\varepsilon_\beta(b)^{-1} = \prod_{m,n>0} \varepsilon_{m\alpha+n\beta}(c_{\alpha\beta}^{mn} a^m b^n),$$

for all $a, b \in \Bbbk$, where the product is over all integers $m, n > 0$ such that $m\alpha + n\beta \in \Phi^+$, taken according to the chosen ordering.

As before, let $\Phi^+ = \{\delta_1, \delta_2, \ldots, \delta_m\}$ and for an element $u \in \mathbf{U}$, we have $u = x_1 x_2 \ldots x_m$ with $x_i \in \mathbf{U}_{\delta_i}$. If we choose another order of Φ^+ and write $\Phi^+ = \{\delta'_1, \delta'_2, \ldots, \delta'_m\}$, we get another expression of u such that $u = y_1 y_2 \dots y_m$ with $y_i \in \mathbf{U}_{\delta_i'}$. If $\delta_i = \delta_j' = \alpha$ is a simple root, by the commutator relations of root subgroups, we get $x_i = y_j$ which is called the $\mathbf{U}_\alpha$-component of u. Noting that the simple roots are $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and each $\gamma \in \Phi^+$ can be written as $\gamma = \sum_{i=1}^{n} k_i \alpha_i$, we denote by $\mathrm{ht}(\gamma) = \sum_{i=1}^{n} k_i$ the height of γ. It is easy to see that $\prod_{\mathrm{ht}(\gamma) \geq s} \mathbf{U}_\gamma$ is a subgroup of $\mathbf{U}$ for any fixed integer $s \in \mathbb{N}$ by the commutator relations of root subgroups.

Given an order "$\prec$" on Φ^+, we list all the positive roots $\delta_1, \delta_2, \dots, and\ \delta_m$ with respect to this order such that $\delta_i \prec \delta_j$ when $i < j$. For any $u \in \mathbf{U}$, we have a unique expression in the form $u = u_1 u_2 \dots u_m$ with $u_i \in \mathbf{U}_{\delta_i}$. Let X be a subset of $\mathbf{U}$, we denote by

$$X \cap_\prec \mathbf{U}_{\delta_k} = \{u_k \in \mathbf{U}_{\delta_k} \mid \text{there exists } u \in X \text{ such that } u = u_1 u_2 \dots u_k \dots u_m\}.$$

It is easy to see that $X \cap \mathbf{U}_{\delta_k} \subseteq X \cap_\prec \mathbf{U}_{\delta_k}$. Now let H be a subgroup of $\mathbf{U}$, and we say that a subgroup $H \subset \mathbf{U}$ is self-enclosed with respect to the order "$\prec$" if

$$H \cap_\prec \mathbf{U}_{\delta_k} = H \cap \mathbf{U}_{\delta_k} \text{ for any } k = 1, 2, \dots, m.$$

If H is self-enclosed with respect to any order on Φ^+, then we say that H is a self-enclosed subgroup of $\mathbf{U}$.

Let H be a self-enclosed subgroup of $\mathbf{U}$. For each $\gamma \in \Phi^+$, we set $H_\gamma = H \cap \mathbf{U}_\gamma$. Then we have $H = H_{\delta_1} H_{\delta_2} \dots H_{\delta_m}$. For $w \in W$, set $H_w = H \cap \mathbf{U}_w$. Then it is easy to see that H_w is also a self-enclosed subgroup, and we have $H_w = \prod_{\gamma \in \Phi_w^-} H_\gamma$.

EXAMPLE 3.1. Suppose $\Bbbk = \bar{\mathbb{F}}_q$ and $\{\delta_1, \delta_2, \dots, \delta_m\}$ are all the positive roots such that $\mathrm{ht}(\delta_1) \leq \mathrm{ht}(\delta_2) \leq \dots \leq \mathrm{ht}(\delta_m)$. Assume that $\mathbf{U}$ is defined over $\mathbb{F}_q$ and let U_{q^a} be the set of $\mathbb{F}_{q^a}$-points of $\mathbf{U}$. Given $a_1, a_2, \dots, a_m \in \mathbb{N}$ such that a_i is divisible by a_j for any $i < j$, we set

$$H = U_{\delta_1, q^{a_1}} U_{\delta_2, q^{a_2}} \dots U_{\delta_m, q^{a_m}}.$$

Then it is not difficult to check that H is a self-enclosed subgroup of $\mathbf{U}$.

Now let H be a subgroup of $\mathbf{U}$. Let $\mathbf{V}$ be a subgroup of $\mathbf{U}$ which has the form $\mathbf{V} = \mathbf{U}_{\beta_1} \mathbf{U}_{\beta_2} \dots \mathbf{U}_{\beta_k}$. We let

$$\mathbf{U} = \bigcup_{x \in L} x\mathbf{V} \quad \text{and} \quad \mathbf{U} = \bigcup_{y \in R} \mathbf{V} y,$$

where L (resp. R) is a set of the left (resp. right) coset representatives of $\mathbf{V}$ in $\mathbf{U}$. Then we define the following two sets:

$$H_\mathbf{V} = \{v \in \mathbf{V} \mid \text{there exists } u \in H \text{ such that } u = xv \text{ for some } x \in L\},$$

$$_\mathbf{V}H = \{v \in \mathbf{V} \mid \text{there exists } u \in H \text{ such that } u = vy \text{ for some } y \in R\}.$$

PROPOSITION 3.2. *Let H be a self-enclosed subgroup of $\mathbf{U}$. Let $\mathbf{V}$ be a subgroup of $\mathbf{U}$ with the form $\mathbf{V} = \mathbf{U}_{\beta_1} \mathbf{U}_{\beta_2} \dots \mathbf{U}_{\beta_k}$, where $\beta_1, \beta_2, \dots, \beta_k \in \Phi^+$. Then we have*

$$H_\mathbf{V} = {}_\mathbf{V}H = H \cap \mathbf{V}.$$

Proof. We just prove that $H_{\mathbf{V}} = H \cap \mathbf{V}$. It is clear that $H \cap \mathbf{V} \subset H_{\mathbf{V}}$. Noting that $\mathbf{V}$ is a subgroup of $\mathbf{U}$, we denote

$$\mathbf{U} = \mathbf{U}_{\gamma_1}\mathbf{U}_{\gamma_2}\dots\mathbf{U}_{\gamma_l}\mathbf{U}_{\beta_1}\mathbf{U}_{\beta_2}\dots\mathbf{U}_{\beta_k}.$$

Let $v \in H_{\mathbf{V}}$. Thus, there exists $h \in H$ such that $h = xv$ for some $x \in L$. We write

$$h = x_{\gamma_1}x_{\gamma_2}\dots x_{\gamma_l}v_{\beta_1}v_{\beta_2}\dots v_{\beta_k}, \quad x_{\gamma_i} \in \mathbf{U}_{\gamma_i}, v_{\beta_j} \in \mathbf{U}_{\beta_j}.$$

Since H is self-enclosed, we see that $v_{\beta_j} \in H \cap \mathbf{U}_{\beta_j}$ which implies that $v \in H \cap \mathbf{V}$. Therefore, we get $H_{\mathbf{V}} = H \cap \mathbf{V}$. Similarly, we have ${}_{\mathbf{V}}H = H \cap \mathbf{V}$. The proposition is proved. ∎

Now we consider the special case that $\Bbbk$ is a field of positive characteristic p. In this case, it is well known that all the finitely generated subgroups of $\mathbf{U}$ are finite p-groups. We have the following lemma.

LEMMA 3.3. *Let X be a finite subset of $\mathbf{U}$. There exists a finite p-subgroup H of $\mathbf{U}$ such that $H \supseteq X$ and H is self-enclosed.*

Proof. Let $\Phi^+ = \{\delta_1, \delta_2, \dots, \delta_m\}$ such that $\mathrm{ht}(\delta_1) \le \mathrm{ht}(\delta_2) \le \dots \le \mathrm{ht}(\delta_m)$. For each $1 \le k \le m$, we set $X_k = X \cap_{\prec} \mathbf{U}_{\delta_k}$. Let H_1 be the subgroup of $\mathbf{U}_{\delta_1}$, which is generated by X_1. Now we define the subgroup H_k by recursive step. Suppose that $H_1, H_2, \dots, H_{k-1}$ are defined, we set

$$Y_k = \langle H_1, H_2, \dots, H_{k-1}\rangle \cap_{\prec} \mathbf{U}_{\delta_k},$$

and let H_k be the subgroup of $\mathbf{U}_{\delta_k}$, which is generated by X_k and Y_k. Now we have a series of subgroups $H_1, H_2, \dots, H_m$ and then we set $H = \langle H_1, H_2, \dots, H_m\rangle$, which is a finitely generated subgroup of $\mathbf{U}$. Thus H is a finite p-subgroup of $\mathbf{U}$, which contains X by its construction. Moreover, it is not difficult to check that H is a self-enclosed of $\mathbf{U}$ using the commutator relations of root subgroups. ∎

It is easy to verify that the intersection of two self-enclosed subgroups of $\mathbf{U}$ is also self-enclosed. For a finite subset X of $\mathbf{U}$, there exists a minimal self-enclosed subgroup V containing X. In this case, we also say that V is the self-enclosed subgroup generated by X.

§4. Nonvanishing property of the augmentation

In this section, we fix a subset $J \subset I$. By Proposition 2.4, we have

$$E_J = \bigoplus_{w \in Y_J} \mathbb{F}\mathbf{U}_{w_J w^{-1}} w C_J,$$

as $\mathbb{F}$-vector space. For each $w \in Y_J$, we denote by

$$\mathfrak{P}_w : E_J \to \mathbb{F}\mathbf{U}_{w_J w^{-1}} w C_J,$$

the projection of vector spaces and by

$$\epsilon_w : \mathbb{F}\mathbf{U}_{w_J w^{-1}} \dot{w} C_J \to \mathbb{F},$$

the augmentation (restricting on w) which takes the sum of the coefficients with respect to the natural basis, i.e., for $\xi = \sum_{x \in \mathbf{U}_{w_J w^{-1}}} a_x x w C_J$, we set $\epsilon_w(\xi) = \sum_{x \in \mathbf{U}_{w_J w^{-1}}} a_x$. Now we

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denote by

$$\epsilon = \bigoplus_{w \in Y_J} \epsilon_w \mathfrak{P}_w : E_J \to \mathbb{F}^{|Y_J|},$$

the augmentation on E_J.

When considering the irreducibility of Steinberg module, the nonvanishing property of the augmentation is very crucial (see [9, Lem. 2.5] and [6, Prop. 1.6]). In this section, we show that the non-vanishing property also holds for the augmentation ϵ defined above. Firstly we have the following lemma.

LEMMA 4.1. *Let $\xi \in E_J$ be a nonzero element. Then there exists $g \in \mathbf{G}$ such that $\mathfrak{P}_e(g\xi)$ is nonzero.*

Proof. By Proposition 2.4, $\xi \in E_J$ has the following expression

$$\xi = \sum_{w \in Y_J} \sum_{x \in \mathbf{U}_{w_J w^{-1}}} a_{w,x} x w C_J.$$

Then there exists an element $h \in W$ with minimal length such that $a_{h,x} \neq 0$ for some $x \in \mathbf{U}_{w_J h^{-1}}$, which implies that $\mathfrak{P}_h(\xi)$ is nonzero. When $h = e$, the lemma is proved. Now suppose that $\ell(h) \geq 1$, so there is a simple reflection s such that $\sigma = sh < h$. Without loss of generality, we can assume that $a_{h,id} \neq 0$. We claim that either $\mathfrak{P}_\sigma(\dot{s}\xi)$ is nonzero or $\mathfrak{P}_\sigma(\dot{s}y\xi)$ is nonzero for some $y \in \mathbf{U}_s$.

If $\mathfrak{P}_\sigma(\dot{s}\xi) = 0$, then according to Lemma 2.2, there exists at least one element $v \in Y_J$, which satisfies the following condition

$$(\spadesuit)\ sv \notin Y_J \text{ and } \mathfrak{P}_\sigma(svC_J) \neq 0.$$

The subset of Y_J whose elements satisfy this condition is also denoted by $\spadesuit$. Thus, $\mathfrak{P}_\sigma(\dot{s}\xi) = 0$ tells us that

$$\mathfrak{P}_\sigma(\dot{s} \cdot \mathfrak{P}_h(\xi)) + \mathfrak{P}_\sigma(\dot{s} \cdot \sum_{v \in \spadesuit} \mathfrak{P}_v(\xi)) = 0.$$

In particular, we get $\mathfrak{P}_\sigma(\dot{s} \cdot \sum_{v \in \spadesuit} \mathfrak{P}_v(\xi)) \neq 0$. Since $\mathbf{U}$ is infinite, there exists infinitely many $y \in \mathbf{U}_s$ such that the $\mathbf{U}_s$-component of yx is nontrivial for any x with $a_{h,x} \neq 0$. For such an element y, we get $\mathfrak{P}_\sigma(\dot{s} \cdot \mathfrak{P}_h(y\xi)) = 0$ by Lemma 2.2 (b).

On the other hand, for $v \in \spadesuit$ and $a_{v,x} \neq 0$, we see that the $\mathbf{U}_s$-component of x is trivial, i.e., $x \in \mathbf{U}'_s$. Note that $\mathbf{U}_{w_J\sigma^{-1}s} = (\mathbf{U}_{w_J\sigma^{-1}})^s \cdot \mathbf{U}_s$ and $\mathbf{U}'_{w_J\sigma^{-1}} = (\mathbf{U}'_{w_J\sigma^{-1}s})^s \cdot \mathbf{U}_s$. Then we can write

$$x = n(x)p(x), \quad \text{where } n(x) \in (\mathbf{U}_{w_J\sigma^{-1}})^s \text{ and } p(x) \in \mathbf{U}'_{w_J\sigma^{-1}s}.$$

Since this expression is unique, we can regard $p(-)$ and $n(-)$ as functions on $\mathbf{U}'_s$. We let $yx = \omega_y(x)y$, where $\omega_y(x) \in \mathbf{U}'_s$. Using the commutator relations of root subgroups, we can choose y such that $n(\omega_y(x')) \neq n(\omega_y(x))$ unless $n(x) = n(x')$ since there are only finitely many $x's$ satisfying $a_{v,x} \neq 0$. Therefore, if we write

$$\mathfrak{P}_\sigma(\dot{s} \cdot \sum_{v \in \spadesuit} \mathfrak{P}_v(\xi)) = \sum b_{\sigma,x} n(x)^{\dot{s}} \sigma C_J \neq 0,$$

it is not difficult to see that

$$\mathfrak{P}_\sigma(\dot{s}\cdot\sum_{v\in\spadesuit}\mathfrak{P}_v(y\xi)) = \sum b_{\sigma,x}n(\omega_y(x))^{\dot{s}}\sigma C_J,$$

which is also nonzero. Therefore,

$$\mathfrak{P}_\sigma(\dot{s}y\xi) = \mathfrak{P}_\sigma(\dot{s}\cdot\sum_{v\in\spadesuit}\mathfrak{P}_v(y\xi)) \neq 0.$$

By the argument above, we can do induction on the length of h and thus the lemma is proved. ∎

The nonvanishing property of the augmentation ϵ on E_J is as follows:

PROPOSITION 4.2. *Let $\xi \in E_J$ be a nonzero element. Then there exists $g \in \mathbf{G}$ such that $\epsilon(g\xi)$ is nonzero.*

Proof. By Lemma 4.1, we can assume that $\mathfrak{P}_e(\xi)$ is nonzero. For

$$\xi = \sum_{w\in Y_J}\sum_{x\in\mathbf{U}_{w_J w^{-1}}} a_{w,x}xwC_J \in E_J,$$

we say that ξ satisfies the condition $\heartsuit_h$ if $\sum_{x\in\mathbf{U}'_h} a_{e,x} \neq 0$ for some $h \in W_J$. We prove the following claim: if ξ satisfies the condition $\heartsuit_h$ for some $h \in W_J$, then there exists $g \in \mathbf{G}$ such that $\epsilon_e\mathfrak{P}_e(g\xi)$ is nonzero.

We prove this claim by induction on the length of h. If $h = e$, then it is obvious that $\epsilon_e\mathfrak{P}_e(\xi) = \sum_{x\in\mathbf{U}_{w_J}} a_{e,x}$ which is already nonzero. We assume that the claim is valid for any $h \in W_J$ with $\ell(h) \leq m$. Now let $h \in W_J$ with $\ell(h) = m+1$ such that $\sum_{x\in\mathbf{U}'_h} a_{e,x} \neq 0$. We have $h = \tau s$ for some $s \in \mathscr{R}(h)$. Then $\mathbf{U}_h = \mathbf{U}^s_\tau \cdot \mathbf{U}_s$ and $\mathbf{U}'_\tau = (\mathbf{U}'_h)^s \cdot \mathbf{U}_s$ by definition. Now our aim is to show that there exists $g \in \mathbf{G}$ such that $g\xi$ satisfies the condition $\heartsuit_\tau$.

First, we prove that the element $\dot{s}\cdot\mathfrak{P}_e(\xi)$ satisfies the condition $\heartsuit_\tau$. Since $\mathbf{U}_{w_J} = \mathbf{U}'_h\mathbf{U}_h = \mathbf{U}'_h\mathbf{U}^s_\tau\mathbf{U}_s$, each element $x \in \mathbf{U}_{w_J}$ has a unique expression

$$x = x'_h x_\tau x_s, \quad x'_h \in \mathbf{U}'_h, x_\tau \in \mathbf{U}^s_\tau, x_s \in \mathbf{U}_s.$$

We just need to consider the coefficients of $a_{e,x}$ with $\dot{s}x'_h x_\tau \dot{s}^{-1} \in \mathbf{U}'_\tau$, which implies that $x_\tau = id$. For the case $x_s \neq id$, using Lemma 2.2 (c), we have

$$\dot{s}xC_J = x''_h\dot{s}x_sC_J = x''_h(f(x_s)-1)C_J,$$

where $x''_h = \dot{s}x'_h\dot{s}^{-1} \in (\mathbf{U}'_h)^s \subset \mathbf{U}'_\tau$ and $f(x_s) \in \mathbf{U}_s$. Therefore if we write

$$\dot{s}\cdot\mathfrak{P}_e(\xi) = \sum_{x\in\mathbf{U}_{w_J}} b_{e,x}xC_J,$$

then $\sum_{x\in\mathbf{U}'_\tau} b_{e,x} = -\sum_{x\in\mathbf{U}'_h} a_{e,x} \neq 0$. Thus $\dot{s}\cdot\mathfrak{P}_e(\xi)$ satisfies the condition $\heartsuit_\tau$.

Now we consider $\dot{s}\xi$ and if $\dot{s}\xi$ satisfies the condition $\heartsuit_\tau$, we are done. Otherwise, there exists at least one element $v \in Y_J$ which satisfies the following condition

$$(\clubsuit): \; sv \notin Y_J \text{ and } \mathfrak{P}_e(svC_J) \neq 0.$$

The subset of Y_J whose elements satisfy this condition is also denoted by $\clubsuit$. With this setting, $\dot{s}\cdot\mathfrak{P}_e(\xi)+\mathfrak{P}_e(\dot{s}\cdot\sum_{v\in\clubsuit}\mathfrak{P}_v(\xi))$ does not satisfy the condition $\heartsuit_\tau$, which implies that $\mathfrak{P}_e(\dot{s}\cdot\sum_{v\in\clubsuit}\mathfrak{P}_v(\xi))$ satisfies the condition $\heartsuit_\tau$ since we have proved that $\dot{s}\cdot\mathfrak{P}_e(\xi)$ satisfies the condition $\heartsuit_\tau$. Since $\mathbf{U}$ is infinite, we can choose an element $y \in \mathbf{U}_s$ such that the $\mathbf{U}_s$-component of yx is nontrivial for any x with $a_{e,x} \neq 0$. Then we consider the element $\dot{s}y\xi$. Using Lemma 2.2 (c), it is easy to see that $\dot{s}\cdot\mathfrak{P}_e(y\xi)$ does not satisfy the condition $\heartsuit_\tau$.

Now for $v \in \clubsuit$ and $x \in \mathbf{U}_{w_J v^{-1}}$ with $a_{v,x} \neq 0$, noting that the $\mathbf{U}_s$-component of x is trivial, we write

$$x = m(x)q(x), \quad \text{where } m(x) \in \mathbf{U}_{w_J s}, q(x) \in (\mathbf{U}'_{w_J})^s.$$

For $y \in \mathbf{U}_s$, using the commutator relations of root subgroups, we have

$$ym(x) = m_y(x)y, \quad \text{where } m_y(x) \in \mathbf{U}_{w_J s},$$

and

$$yq(x) = q_y(x)y, \quad \text{where } q_y(x) \in (\mathbf{U}'_{w_J})^s.$$

Since $\mathbf{U}'_\tau = (\mathbf{U}'_h)^s\mathbf{U}_s$, we get $m(x)^{\dot{s}} \in \mathbf{U}'_\tau$ if and only if $m(x) \in (\mathbf{U}'_h)^s$. Thus, $m_y(x)^{\dot{s}} \in \mathbf{U}'_\tau$ if and only if $m(x)^{\dot{s}} \in \mathbf{U}'_\tau$. Therefore, if we write

$$\mathfrak{P}_e(\dot{s}\cdot\sum_{v\in\clubsuit}\mathfrak{P}_v(\xi)) = \sum b_x m(x)^{\dot{s}} C_J,$$

it is not difficult to see that

$$\mathfrak{P}_e(\dot{s}\cdot\sum_{v\in\clubsuit}\mathfrak{P}_v(y\xi)) = \sum b_x m_y(x)^{\dot{s}} C_J.$$

Noting that $\mathfrak{P}_e(\dot{s}\cdot\sum_{v\in\clubsuit}\mathfrak{P}_v(\xi))$ satisfies the condition $\heartsuit_\tau$, we see that $\mathfrak{P}_e(\dot{s}\cdot\sum_{v\in\clubsuit}\mathfrak{P}_v(y\xi))$ satisfies the condition $\heartsuit_\tau$. Finally, there exists $g \in \mathbf{G}$ such that $g\xi$ satisfies the condition $\heartsuit_e$, which implies that $\epsilon_e\mathfrak{P}_e(g\xi)$ is nonzero. We have proved our claim.

Now we can assume that $a_{e,id} \neq 0$. Thus, the element ξ satisfies the condition $\heartsuit_{w_J}$. According to our claim, there exists $g \in \mathbf{G}$ such that $\epsilon_e\mathfrak{P}_e(g\xi)$ is nonzero. In particular, $\epsilon(g\xi)$ is nonzero and the proposition is proved. ∎

§5. Proof of the main theorem

In this section, we give the proof of Theorem 1.1. First, we deal with the cases: (1) char $\Bbbk = 0$ and (2) char $\Bbbk > 0$ and char $\Bbbk \neq$ char $\mathbb{F}$. For $J \subset I$, we show that any nonzero submodule M of E_J contains C_J, and hence $M = E_J$. In particular, E_J is irreducible for any $J \subset I$. Let $\xi \in M$ be a nonzero element with the following expression

$$\xi = \sum_{w\in Y_J}\sum_{x\in\mathbf{U}_{w_J w^{-1}}} a_{w,x} xwC_J \in M.$$

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By Proposition 4.2, we can assume that $\epsilon(\xi) \neq 0$. In the case (1) by [6, Prop. 5.4] and in the case (2) by [6, Prop. 6.7], we have

$$\sum_{w \in Y_J} \sum_{x \in \mathbf{U}_{w_J w^{-1}}} a_{w,x} w C_J \in M.$$

In particular, we see that

$$M \cap \sum_{w \in Y_J} \mathbb{F} w C_J \neq 0.$$

Noting that the discussion in the proof of [2, Claim 2] is still valid in our general setting, we see that E_J is irreducible for any $J \subset I$. The E_J's are pairwise nonisomorphic by Proposition 2.5.

It remains to consider the case char $\Bbbk = \text{char } \mathbb{F} = p > 0$. From now on, we assume that char $\mathbb{F} = \text{char } \Bbbk = p$. For any finite subset X of $\mathbf{G}$, let $\underline{X} := \sum_{x \in X} x \in \mathbb{F}\mathbf{G}$. The following lemma is easy to get and will be very useful in our later discussion.

Lemma 5.1. *Let P be a finite abelian p-group such that $P = H \times K$, where H, K are two subgroups of P. Let H' be a subgroup of P such that $|H'| = |H|$. Then $\underline{H'}\ \underline{K} = 0$ or $\underline{P}$.*

For a self-enclosed subgroup H of $\mathbf{U}$, set $H_\gamma = H \cap \mathbf{U}_\gamma$ as before for each $\gamma \in \Phi^+$. Let $\Phi^+ = \{\delta_1, \delta_2, \ldots, \delta_m\}$. We have

$$\underline{H} = \underline{H_{\delta_1}}\ \underline{H_{\delta_2}}\ \ldots\ \underline{H_{\delta_m}}.$$

Let $H_w = H \cap \mathbf{U}_w$. Then we have $H_w = \prod_{\gamma \in \Phi_w^-} H_\gamma$ and $\underline{H_w} = \prod_{\gamma \in \Phi_w^-} \underline{H_\gamma}$. The following two lemmas are very crucial in the later proof of Theorem 1.1.

Lemma 5.2. *Assume that char $\mathbb{F} = $ char $\Bbbk = p > 0$ and let M be a nonzero $\mathbb{F}\mathbf{G}$-submodule of E_J. Then there exist an element $w \in Y_J$ and a finite p-subgroup X of $\mathbf{U}_{w_J w^{-1}}$ such that $\underline{X} w C_J \in M$.*

Proof. Let ξ be a nonzero element of M which has the form

$$\xi = \sum_{w \in Y_J} \sum_{x \in \mathbf{U}_{w_J w^{-1}}} a_{w,x} x w C_J \in E_J.$$

By Lemma 3.3, there exists a self-enclosed finite p-subgroup V of $\mathbf{U}$, which contains all $x \in \mathbf{U}_{w_J w^{-1}}$ with $a_{w,x} \neq 0$. Then we have

$$\mathbb{F} V \xi \subset \bigoplus_{w \in Y_J} \mathbb{F} V_{w_J w^{-1}} w C_J,$$

as $\mathbb{F}V$-modules. Since $(\mathbb{F}V\xi)^V \neq 0$ by [7, Prop. 26] and noting that

$$\Big(\bigoplus_{w \in Y_J} \mathbb{F} V_{w_J w^{-1}} w C_J\Big)^V \subset \bigoplus_{w \in Y_J} \mathbb{F} \underline{V_{w_J w^{-1}}} w C_J,$$

there exists a nonzero element

$$\eta = \sum_{w \in Y_J} a_w \underline{V_{w_J w^{-1}}} w C_J \in \mathbb{F} V \xi \subset M.$$

Set $A(\eta) = \{w \in Y_J \mid a_w \neq 0\}$. If $|A(\eta)| = 1$, the lemma is proved.

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Now we assume that $|A(\eta)| \geq 2$. We set $\Phi(\eta) = \bigcup_{w \in A(\eta)} \Phi^-_{w_J w^{-1}}$. Let $\Phi(\eta) = \{\gamma_1, \gamma_2, \ldots, \gamma_d\}$ such that $\mathrm{ht}(\gamma_1) \leq \mathrm{ht}(\gamma_2) \leq \cdots \leq \mathrm{ht}(\gamma_d)$. Let s be the maximal integer such that $\gamma_s \notin \bigcap_{w \in A(\eta)} \Phi^-_{w_J w^{-1}}$. Let $y \in \mathbf{U}_{\gamma_s} \backslash V_{\gamma_s}$ and H be a self-enclosed finite p-subgroup of $\mathbf{U}_{\gamma_s} \mathbf{U}_{\gamma_{s+1}} \ldots \mathbf{U}_{\gamma_d}$ such that H contains $V_{\gamma_s} V_{\gamma_{s+1}} \ldots V_{\gamma_d}$ and y. Let X be the self-enclosed subgroup of $\mathbf{U}$ which is generated by H and V. Then it is easy to see that X has the following form

$$X = V_{\gamma_1} \ldots V_{\gamma_{s-1}} X_{\gamma_s} \ldots X_{\gamma_d},$$

where $X_{\gamma_k} = X \cap \mathbf{U}_{\gamma_k}$ for $s \leq k \leq d$. Denote by Ω_s a set of the left coset representatives of $V_{\gamma_s} V_{\gamma_{s+1}} \ldots V_{\gamma_d}$ in $X_{\gamma_s} X_{\gamma_{s+1}} \ldots X_{\gamma_d}$. For the $w \in Y_J$ such that $\gamma_s \in \Phi^-_{w_J w^{-1}}$, we have

$$\underline{\Omega_s}\ \underline{V_{w_J w^{-1}}} w C_J = \underline{X_{w_J w^{-1}}} w C_J.$$

For the $w \in Y_J$ such that $\gamma_s \notin \Phi^-_{w_J w^{-1}}$, we have $\underline{\Omega_s}\ \underline{V_{w_J w^{-1}}} w C_J = 0$ since char $\mathbb{F} = p$. Then we get

$$\eta' = \underline{\Omega_s}\ \eta = \sum_{w \in Y_J} b_w \underline{X_{w_J w^{-1}}} w C_J,$$

which satisfies that $|A(\eta')| < |A(\eta)|$, where $A(\eta') = \{w \in Y_J \mid b_w \neq 0\}$. Thus by the induction on the cardinality of $A(\eta)$, the lemma is proved. ∎

LEMMA 5.3. *Assume that char $\mathbb{F} =$ char $\Bbbk = p > 0$ and let M be a nonzero $\mathbb{F}\mathbf{G}$-submodule of E_J. If there exists a finite p-subgroup X of $\mathbf{U}_{w_J w^{-1} s}$ such that $\underline{X} sw C_J \in M$, where $sw \in Y_J$ and $sw > w$ (which implies that $w \in Y_J$), then there exists a finite p-subgroup H of $\mathbf{U}_{w_J w^{-1}}$ such that $\underline{H} w C_J \in M$.*

Proof. Using Lemma 3.3, we can assume that X is a self-enclosed subgroup of $\mathbf{U}_{w_J w^{-1} s}$. Since $\mathbf{U}_{w_J w^{-1} s} = \mathbf{U}_s (\mathbf{U}_{w_J w^{-1}})^s$, we can write $X = X_\alpha V$, where $V = X \cap (\mathbf{U}_{w_J w^{-1}})^s$ is also a self-enclosed subgroup of $(\mathbf{U}_{w_J w^{-1}})^s$. Thus, we have $\underline{X} = \underline{X_\alpha}\ \underline{V}$. In the following, we will prove that if $\underline{Y}\ \underline{V}\ sw C_J \in M$ for some finite subset Y of $\mathbf{U}_s$ and a self-enclosed subgroup V of $(\mathbf{U}_{w_J w^{-1}})^s$, then there exists a finite p-subgroup H of $\mathbf{U}_{w_J w^{-1}}$ such that $\underline{H} w C_J \in M$. Without loss of generality, we can assume that Y contains the neutral element of $\mathbf{U}_s$.

For each $u \in \mathbf{U}_\alpha \backslash \{id\}$, we have

$$\dot{s} u \dot{s} = f_\alpha(u) h_\alpha(u) \dot{s} g_\alpha(u),$$

where $f_\alpha(u), g_\alpha(u) \in \mathbf{U}_\alpha$ and $h_\alpha(u) \in \mathbf{T}$ are uniquely determined. Then

$$\dot{s} u \underline{V} sw C_J = f_\alpha(u) h_\alpha(u) \dot{s} g_\alpha(u) \dot{s}^{-1} \underline{V} sw C_J.$$

Without loss of generality, we can assume that the group V contains enough elements such that

$$g_\alpha(u) \dot{s}^{-1} \underline{V} sw C_J = \dot{s}^{-1} \underline{V} sw C_J,$$

for any $u \in Y \backslash \{id\}$. Indeed, we let

$$G_\alpha(X) = \{g_\alpha(u) \in \mathbf{U}_\alpha \mid u \in Y \backslash \{id\}\},$$

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and H be a self-enclosed subgroup which contains $G_\alpha(X)$ and $\dot{s}^{-1}V\dot{s}$. Then $H_{w_Jw^{-1}} = H \cap \mathbf{U}_{w_Jw^{-1}}$ is also a self-enclosed subgroup which contains $\dot{s}^{-1}V\dot{s}$. Then we can consider $\underline{Y}\ \underline{\dot{s}H_{w_Jw^{-1}}\dot{s}^{-1}}$ instead of $\underline{Y}\ \underline{V}$ from the beginning. Noting that $h_\alpha(u) \in \mathbf{T}$, we have

$$\dot{s}u\underline{V}swC_J = f_\alpha(u)h_\alpha(u)\underline{V}swC_J = f_\alpha(u)\underline{h_\alpha(u)Vh_\alpha(u)^{-1}}swC_J,$$

which implies that

$$\dot{s}\underline{Y}\ \underline{V}swC_J = \underline{\dot{s}V\dot{s}^{-1}}wC_J + \sum_{u\in Y\backslash\{id\}} f_\alpha(u)\underline{h_\alpha(u)Vh_\alpha(u)^{-1}}swC_J.$$

Now we let

$$\Phi^-_{w_Jw^{-1}} \cup \Phi^-_{w_Jw^{-1}s} = \{\beta_1 = \alpha, \beta_2, \ldots, \beta_m\},$$

such that $\mathrm{ht}(\beta_1) \le \mathrm{ht}(\beta_2) \le \cdots \le \mathrm{ht}(\beta_m)$. Since $sw \in Y_J$ and $sw > w$, we have $(\mathbf{U}_{w_Jw^{-1}})^s \neq \mathbf{U}_{w_Jw^{-1}}$ by [3, Cor. 2.2]. Now let r be the maximal integer such that $\beta_r \notin \Phi^-_{w_Jw^{-1}} \cap \Phi^-_{w_Jw^{-1}s}$ and $\beta_j \in \Phi^-_{w_Jw^{-1}} \cap \Phi^-_{w_Jw^{-1}s}$ for $j > r$. When $\beta_r \in \Phi^-_{w_Jw^{-1}} \backslash \Phi^-_{w_Jw^{-1}s}$, using Lemma 3.3, Lemma 5.1 and [3, Lem. 4.5], we can choose certain subgroup Ω_k of $\mathbf{U}_{\beta_k}$ for each $r \le k \le m$ such that

$$\underline{\Omega_r}\ \underline{\Omega_{r+1}} \ldots \underline{\Omega_m}\ f_\alpha(u)\underline{h_\alpha(u)Vh_\alpha(u)^{-1}}swC_J = 0,$$

for any $u \in Y\backslash\{id\}$ and

$$\underline{\Omega_r}\ \underline{\Omega_{r+1}} \ldots \underline{\Omega_m}\ \underline{\dot{s}V\dot{s}^{-1}}wC_J = \underline{\Omega}wC_J,$$

for some finite subgroup Ω of $\mathbf{U}_{w_Jw^{-1}}$. Then the lemma is proved in this case.

When $\beta_r \in \Phi^-_{w_Jw^{-1}s} \backslash \Phi^-_{w_Jw^{-1}}$, also by Lemmas 3.3, 5.1 and [3, Lem. 4.5], we can choose certain subgroup Γ_k of $\mathbf{U}_{\beta_k}$ for each $r \le k \le m$ such that there exists at least one $u \in Y\backslash\{id\}$ which satisfies

$$\underline{\Gamma_r}\ \underline{\Gamma_{r+1}} \ldots \underline{\Gamma_m}\ f_\alpha(u)\underline{h_\alpha(u)Vh_\alpha(u)^{-1}}swC_J = f_\alpha(u)\underline{\Gamma}swC_J,$$

where Γ is some finite subgroup of $(\mathbf{U}_{w_Jw^{-1}})^s$. On the other hand, these groups $\Gamma_r, \Gamma_{r+1}, \ldots, \Gamma_m$ also make

$$\underline{\Gamma_r}\ \underline{\Gamma_{r+1}} \ldots \underline{\Gamma_m}\ \underline{\dot{s}V\dot{s}^{-1}}wC_J = 0.$$

Therefore, we get $\sum_{x\in F} x\underline{\Gamma}swC_J \in M$ for some set F with $|F| < |Y|$ and some finite subgroup Γ of $(\mathbf{U}_{w_Jw^{-1}})^s$. Hence by the same discussion as before, we get another element $\sum_{y\in F'} y\underline{\Gamma'}swC_J \in M$ for some set F' with $|F'| < |F|$ and some finite subgroup Γ' of $(\mathbf{U}_{w_Jw^{-1}})^s$. Finally, we get an element $\underline{K}swC_J \in M$ for some finite subgroup K of $(\mathbf{U}_{w_Jw^{-1}})^s$. Thus, we have $\underline{K^{\dot{s}}}wC_J \in M$ and the lemma is proved. ∎

Finally, we prove the irreducibility of E_J in the case char $\mathbb{F}$ = char $\Bbbk = p > 0$ using the previous lemmas. Let M be a nonzero $\mathbb{F}\mathbf{G}$-submodule of E_J. Combining Lemmas 5.2 and 5.3, there exists a finite p-subgroup H of $\mathbf{U}_{w_J}$ such that $\underline{H}C_J \in M$. Similar to the arguments of [9, Lem. 2.5], we see that the sum of all coefficients of $\dot{w}_JxC_J$ in terms the basis $\{uC_J \mid u \in \mathbf{U}_{w_J}\}$ is zero when x is not the neutral element of $\mathbf{U}_{w_J}$. So if

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we write

$$\xi = w_J \underline{H} C_J = \sum_{x \in \mathbf{U}_{w_J}} a_x x C_J,$$

we have $\sum_{x \in \mathbf{U}_{w_J}} a_x = (-1)^{\ell(w_J)}$, which is nonzero. We consider the $\mathbb{F}\mathbf{U}_{w_J}$-module generated by ξ, and then using [6, Prop. 4.1], we see that $C_J \in M$. Therefore $M = E_J$, which implies the irreducibility of E_J for any $J \subset I$. All the $\mathbb{F}\mathbf{G}$-modules E_J are pairwise non-isomorphic by Proposition 2.5 and thus Theorem 1.1 is proved.

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References

[1] R. W. Carter, *Finite Groups of Lie Type: Conjugacy Classes and Complex Characters*, Pure Appl. Math., John Wiley and Sons, New York, 1985.

[2] X. Chen and J. Dong, *The permutation module on flag varieties in cross characteristic*, Math. Z. **293** (2019), 475–484.

[3] X. Chen and J. Dong, *The decomposition of permutation module for infinite Chevalley groups*, Sci. China Math **64** (2021), no. 5, 921–930.

[4] M. Geck, *Kazhdan–Lusztig cells and the Murphy basis*, Proc. Lond. Math. Soc. **93** (2006), no. 3, 635–665.

[5] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165–184.

[6] A. Putman and A. Snowden, *The Steinberg representation is irreducible*, Duke Math. J. **172** (2023), 775–808.

[7] J. P. Serre, *Linear Representations of Finite Groups*, GTM, 42, Springer-Verlag, New York–Heidelberg, 1977.

[8] N. Xi, *Some infinite dimensional representations of reductive groups with Frobenius maps*, Sci. China Math. **57** (2014), 1109–1120.

[9] R. Yang, *Irreducibility of infinite dimensional Steinberg modules of reductive groups with Frobenius maps*, J. Algebra **533** (2019), 17–24.

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