TESSELLATION-VALUED PROCESSES THAT ARE GENERATED BY CELL DIVISION

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Abstract

Processes of random tessellations of the Euclidean space $\mathbb{R}^d$, $d \geq 1$, are considered that are generated by subsequent division of their cells. Such processes are characterized by the laws of the life times of the cells until their division and by the laws for the random hyperplanes that divide the cells at the end of their life times. The STIT (STable with respect to ITerations) tessellation processes are a reference model. In the present paper a generalization concerning the life time distributions is introduced, a sufficient condition for the existence of such cell division tessellation processes is provided, and a construction is described. In particular, for the case that the random dividing hyperplanes have a Mondrian distribution—which means that all cells of the tessellations are cuboids—it is shown that the intrinsic volumes, except the Euler characteristic, can be used as the parameter for the exponential life time distribution of the cells.

Keywords: Stochastic geometry; stochastic process of random tessellations; STIT tessellation; crack pattern; Mondrian model; fragmentation

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1. Introduction

The study of random tessellations (or mosaics) is a substantial part of stochastic geometry. The Voronoi tessellations and Poisson hyperplane tessellations are classical and well-established. Motivated by modeling fracture or crack patterns as they are observed, for example, in geology, materials science, nanotechnology, or drying soil patterns [2, 5, 23, 30, 31]; also, space–time processes of tessellations are considered where the cells are consecutively divided. Recently, such models are also of growing interest in the context of machine learning, see [24] and the references therein.

Several of the approaches suggested so far have turned out to be cumbersome, if not unfeasible, with respect to theoretical investigations, and only simulation studies can be performed. In [21] the STIT (STable with respect to ITerations) tessellation process was introduced, which is a cell division process allowing for numerous theoretical results; see, for example, [15, 17, 18, 22, 26–28]. Due to its nice mathematical properties, it can be considered as a reference model for fracture patterns. On the other hand, some statistical goodness-of-fit checks indicate

the need for modification and adaption of the STIT model, see [12, 20]. This motivates the present paper.

Our work is essentially inspired by [3], where a systematic approach to a wide class of cell division processes is provided by introducing 'selection rules'—in the present paper adapted as 'life time distributions'—and 'division rules'.

In [4, 29] a theoretical base for generalizations of the STIT model was provided. In those papers the focus was on modifications of the division rules for the cells.

Our purpose is the construction of tessellation-valued cell division processes in the d-dimensional Euclidean space $\mathbb{R}^d$, where the life time distributions of the cells differ from those for the STIT model. We provide sufficient conditions and a rigorous proof of their existence, which is a critical issue for such models, and study some of their features.

The idea of the proof of Theorem 1 is based on the 'global construction' of STIT tessellations in [18]. For this reason, the models we consider all have the same division rule like the STIT tessellation, driven by a translation-invariant hyperplane measure. The new results of this paper concern a variety of life time distributions of the cells.

In Section 3 we start with a proof of the tail triviality of the zero-cell process of a Poisson hyperplane tessellation process. Because this process of zero-cells plays a central role in several proofs of cell division processes, this is a result of independent interest. Then, in Section 4, a sufficient condition for the existence of a class of cell division processes is provided together with their construction. Several more concrete results are shown in Section 5 for the particular case of the so-called Mondrian model where all the tessellation cells are cuboids. We conclude by describing a link between a particular cell division process in a bounded window and a fragmentation. Fragmentation models as introduced and studied in [1] (for a definition, see Section 1.1.3 therein) are such that at any time there is a countable class of particles of sizes $(s_i \colon i \in \mathbb{N})$ evolving independently. At the end of its life a particle of size s_i splits into a sequence of fragments of sizes $(s_{i,j} \colon j \in \mathbb{N})$ such that the law of the ratios of the sizes of the split fragments with respect to the size of the original fragment (that is, the law of $(s_{i,j}/s_i \colon j \in \mathbb{N})$) does not depend on the size s_i. The lifetime of a particle of size s_i is exponentially distributed with a parameter of order s_i^α, and α is called the index of self-similarity. In Section 5.3 we discuss conditions that a cell size function G must satisfy in order that a random tessellation resulting from an (L-G) (D-Λ) cell division process corresponds to such a fragmentation.

2. Notation

By $\mathbb{R}^d$ we denote the d-dimensional Euclidean space, $d \geq 1$, with the scalar product $\langle \cdot, \cdot \rangle$. The origin is o, and S^{d-1} is the unit sphere with the Borel σ-algebra $\mathcal{B}(S^{d-1})$. Denote by B_r the ball centered at the origin o with radius $r > 0$. The topological interior of a set $A \subseteq \mathbb{R}^d$ is int(A). The Lebesgue measure on $\mathbb{R}$ is denoted by λ. The set of the nonnegative integers is $\mathbb{N}_0$, $\mathbb{Z}$ denotes the set of integers, and $\mathbb{Z}_- := \{i \in \mathbb{Z} \colon i \leq 0\}$. The symbol $\overset{\mathrm{D}}{=}$ means the identity of distributions of two random variables. We use the abbreviation 'i.i.d.' for 'independent and identically distributed' random variables.

Let $\mathcal{H}$ denote the set of all hyperplanes in $\mathbb{R}^d$, endowed with the Borel σ-algebra associated with the Fell topology. For a hyperplane $h \in \mathcal{H}$ we use the parametrization $h(u, x) := \{y \in \mathbb{R}^d \colon \langle y, u \rangle = x\}$, $u \in S^{d-1}$, $x \in \mathbb{R}$. Note, that $h(u, x) = h(-u, -x)$, and thus all hyperplanes can be parametrized in two different ways. Denote by $[B] := \{h \in \mathcal{H} \colon h \cap B \neq \emptyset\}$ the set of all hyperplanes that intersect the set $B \subset \mathbb{R}^d$.

Let φ denote an even probability measure on S^{d-1}, which is a probability measure on S^{d-1} with $\varphi(A) = \varphi(-A)$ for all $A \in \mathcal{B}(S^{d-1})$.

Throughout this paper we use a translation-invariant measure Λ on $\mathcal{H}$ which satisfies

$$\int_{\mathcal{H}} f \, \mathrm{d}\Lambda = \int_{S^{d-1}} \int_{\mathbb{R}} f(h(u, r)) \, \lambda(\mathrm{d}r) \, \varphi(\mathrm{d}u) \tag{1}$$

for all nonnegative measurable functions $f \colon \mathcal{H} \to \mathbb{R}$. The probability measure φ is called the spherical directional distribution. We need a further assumption on φ which guarantees that the constructed structures are indeed tessellations with bounded cells, see [25, p. 486]:

$$\varphi \text{ is not concentrated on a great subsphere of } S^{d-1}. \tag{2}$$

A great subsphere is the intersection of S^{d-1} with a hyperplane through the origin. Condition (2) is equivalent to the assumption that there is no line in $\mathbb{R}^d$ with which φ-almost-all hyperplanes are parallel.

A polytope in $\mathbb{R}^d$ is the convex hull of a nonempty finite set of points. By $\mathcal{P}_d$ we denote the set of all d-dimensional polytopes in $\mathbb{R}^d$.

Definition 1. A *tessellation* of $\mathbb{R}^d$ is a countable set $T \subset \mathcal{P}_d$ of d-dimensional polytopes satisfying the following conditions:

(i) $\bigcup_{z \in T} z = \mathbb{R}^d$ (covering);

(ii) if $z, z' \in T$ and $z \neq z'$, then $\mathrm{int}(z) \cap \mathrm{int}(z') = \emptyset$ (disjoint interiors);

(iii) the set $\{z \in T : z \cap C \neq \emptyset\}$ is finite for all compact sets $C \subset \mathbb{R}^d$ (local finiteness).

The polytopes $z \in T$ are referred to as the *cells* of T.

Let $\mathbb{T}$ denote the set of all tessellations of $\mathbb{R}^d$, endowed with the usual σ-algebra, which is associated with the Borel σ-algebra for the Fell topology in the space of closed subsets of $\mathbb{R}^d$, using that the union of cell boundaries $\partial T := \bigcup_{z \in T} \partial z$, also referred to as the $(d-1)$-skeleton, of a tessellation T is a closed subset of $\mathbb{R}^d$. A random tessellation is a measurable mapping from a probability space to $\mathbb{T}$.

For more details on the definition of random tessellation we refer to [25].

In this paper, Poisson point processes are used extensively. For this, we refer to [9], and in particular Chapters 5 and 7 therein.

3. An auxiliary Poisson process of hyperplanes marked with birth times, and the process of zero-cells

Let Λ be a translation-invariant measure on the space $\mathcal{H}$ of hyperplanes in $\mathbb{R}^d$, satisfying (1) and (2). We denote by $\hat{X}^*$ a Poisson point process on $\mathcal{H} \times (0, \infty)$ with the intensity measure $\Lambda \otimes \lambda_+$, where λ_+ is the Lebesgue measure on $(0, \infty)$. If $(h, t) \in \hat{X}^*$, we interpret this as a hyperplane with birth time t. Furthermore, define $\hat{X}_t^* := \{(h, t') \in \hat{X}^* : t' \leq t\}$ and $\hat{X}_t := \{h \in \mathcal{H} : \text{there exists } (h, t') \in X_t^*\}$. Now we can introduce the process $(\tilde{z}_t^o, \ t > 0)$, where $\tilde{z}_t^o$ is the zero-cell (that is, the cell containing the origin o) of the Poisson hyperplane tessellation of $\mathbb{R}^d$ induced by $\hat{X}_t$. The process $(\tilde{z}_t^o, \ t > 0)$ is a pure jump Markov process, with $\tilde{z}_{t'}^o \subseteq \tilde{z}_t^o$ if $t < t'$. And, most importantly,

$$\bigcup_{t > 0} \tilde{z}_t^o = \mathbb{R}^d \quad \text{almost surely (a.s.).} \tag{3}$$

Denote by $(t_i, \ i \in \mathbb{Z})$ the ordered sequence of jump times of $(\tilde{z}_t^o, \ t > 0)$ with $t_0 < 1 < t_1$. Correspondingly, $(h_i, t_i) \in \hat{X}^*$ are the time-marked dividing hyperplanes causing a jump of $(\tilde{z}_t^o, \ t > 0)$.

Denote by

$$\left(\check{z}_{(i)}^{o}, \ i \in \mathbb{Z}\right) \tag{4}$$

the sequence of zero cells with $\check{z}_{(i)}^{o} := \check{z}_{t_i}^{o}$.

Consider the process $(\hat{z}_n, \ n \in \mathbb{N}_0)$ with $\hat{z}_n := \check{z}_{(-n)}^{o}$, and denote by $\mathcal{F}_n := \sigma(\hat{z}_n)$, $n \in \mathbb{N}_0$, the σ-algebra generated by $\hat{z}_n$. The tail-σ-algebra is then $\mathcal{F}_\infty := \bigcap_{m \in \mathbb{N}_0} \bigvee_{k \geq m} \mathcal{F}_k$, where $\bigvee$ stands for the σ-algebra generated by the union of the respective σ-algebras.

Lemma 1. *The tail-σ-algebra $\mathcal{F}_\infty$ of the process $(\hat{z}_n, \ n \in \mathbb{N}_0)$ is trivial, i.e. $\mathbb{P}(B) \in \{0, 1\}$ for all $B \in \mathcal{F}_\infty$.*

Proof. Let $B \in \mathcal{F}_\infty$ and $\varepsilon > 0$ be fixed. Then there are an $m \in \mathbb{N}$ and a $B_m \in \bigvee_{k \leq m} \mathcal{F}_k$ such that $\mathbb{P}(B \triangle B_m) < \varepsilon$ [6, Theorem D, Chapter III]. In the following we consider m and B_m as fixed.

The main idea of the proof is to construct times $\tilde{t} < t^*$, two polytopes $W' \subset W$ containing the origin o, and events $E, E_m, A, A_n, n > m$, such that

- the event $E \cap E_m$ implies that $W' \subseteq \hat{z}_m \subseteq W$, and that the jump time $t_{-m} \geq t^*$;

- the event $A \cap A_n$ implies that $W \subseteq \hat{z}_n$, and that the jump time $t_{-n} \leq \tilde{t}$.

For $W', W \in \mathcal{P}_d$, $W' \subset W$, with the origin o in the interior of W', define the time $S(W', W) := \inf\{t > 0: W' \subset \check{z}_t^o \subset W\}$, which is the time of first separation of the boundary of W' from the boundary of W by the boundary of the zero-cell, introduced in [14] as the 'encapsulation time'.

For $t^* \in (0, 1)$ and $r > 0$, define the intervals $a_i := ((i-1)r/m, ir/m)$ of length r/m and time intervals $\Delta_i := (t^* + (i-1)(1-t^*)/m, t^* + i(1-t^*)/m)$ of length $(1-t^*)/m$ between t^* and 1, $i \in \{1, \ldots, m\}$.

Now, for $m \in \mathbb{N}$ and $\varepsilon > 0$ choose a pair $(t^*, r) \in (0, 1) \times (0, \infty)$ such that, for the event

$$E_m := \{\hat{X}^* \cap (\{h(u, x) \in \mathcal{H}: x \in (-r, r), \ u \in S^{d-1}\} \times (0, t^*)) = \emptyset\}$$
$$\cap \bigcap_{i=1}^{m} \{\hat{X}^* \cap (\{h(u, x) \in \mathcal{H}: x \in a_i \cup (-a_i), \ u \in S^{d-1}\} \times \Delta_i) \neq \emptyset\},$$

we have

$$\mathbb{P}(E_m) = \exp(-t^* r)\left(1 - \exp\left(-\frac{2(1-t^*)r}{m^2}\right)\right)^m > 1 - \varepsilon.$$

This can be realized by choosing $r > -(m^2/2) \ln(1 - (1 - \varepsilon)^{1/2m})$ and then $t^* < -(1/2r) \ln(1 - \varepsilon)$, which guarantees that each of the two factors is greater than $\sqrt{1 - \varepsilon}$.

The event E_m means that the ball B_r is not intersected by a hyperplane of $\hat{X}^*$ until time t^*, and then in each of the m time intervals $\Delta_i := (t^* + (i-1)(1-t^*)/m, t^* + i(1-t^*)/m)$ of length $(1-t^*)/m$ between t^* and 1 there is at least one hyperplane of $\hat{X}^*$ with a distance to the origin in the interval $a_i := ((i-1)r/m, ir/m)$, $i \in \{1, \ldots, m\}$. This guarantees that there are at least m jumps of the zero-cell process in the time interval $(t^*, 1)$.

The pair (t^*, r) is now considered to be fixed. Choose $W' \in \mathcal{P}_d$ such that $\mathsf{B}_r \subset W'$. By [14, Lemma 5], there exists a $W \in \mathcal{P}_d$ with $W' \subset W$ such that, for the event $E := \{S(W', W) < t^*\}$, $\mathbb{P}(E) = \mathbb{P}(S(W', W) < t^*) > 1 - \varepsilon$.

From now on, W is also considered to be fixed. Furthermore, choose $\tilde{t} \in (0, t^*)$ such that, for the event $A := \{[W] \cap \hat{X}_{\tilde{t}} = \emptyset\}$, which means that at time $\tilde{t}$ the window W is contained in the

zero-cell, $\mathbb{P}(A) = \exp(-\tilde{t}\,\Lambda([W])) > 1 - \varepsilon$. Then, having fixed $\tilde{t}$, choose $n > m$ such that, for the event A_n, in the time interval $(\tilde{t}, 1)$ at most n hyperplanes of $\hat{X}^*$ intersect W,

$$\mathbb{P}(A_n) = \sum_{i=0}^{n} \frac{((1 - \tilde{t})\Lambda([W]))^i}{i!} \exp\left(-(1 - \tilde{t})\,\Lambda([W])\right) > 1 - \varepsilon.$$

Note that $A \cap A_n \subseteq \{W \subseteq \hat{z}_n\}$.

By the characteristic property of Poisson point processes, the restricted point processes on disjoint sets, namely $\hat{X}^* \cap ([W] \times (t^*, 1))$ and $\hat{X}^* \cap ([W]^c \times (0, \tilde{t}))$, are independent. Hence, for $B_m \in \bigvee_{k \leq m} \mathcal{F}_k$, $C \in \bigvee_{k \geq n} \mathcal{F}_k$, and with $D := E \cap E_m \cap A \cap A_n$, the events B_m and C are conditionally independent under the condition D, and thus

$$|\mathbb{P}(B_m \cap C) - \mathbb{P}(B_m)\mathbb{P}(C)| = |\mathbb{P}(B_m \cap C \cap D) + \mathbb{P}(B_m \cap C \cap D^c) - \mathbb{P}(B_m)\mathbb{P}(C)|$$
$$\leq |\mathbb{P}(B_m \cap D)\mathbb{P}(C \cap D)/\mathbb{P}(D) - \mathbb{P}(B_m)\mathbb{P}(C)| + \mathbb{P}(B_m \cap C \cap D^c).$$

With the construction and the assumptions on the events above, we have $\mathbb{P}(D) > 1 - 4\varepsilon$, and this yields $(\mathbb{P}(B_m) - 4\varepsilon)(\mathbb{P}(C) - 4\varepsilon) \leq \mathbb{P}(B_m \cap D)\mathbb{P}(C \cap D) \leq \mathbb{P}(B_m)\mathbb{P}(C)$. Furthermore, $(1 - 4\varepsilon)\mathbb{P}(B_m)\mathbb{P}(C) \leq \mathbb{P}(B_m)\mathbb{P}(C)\mathbb{P}(D) \leq \mathbb{P}(B_m)\mathbb{P}(C)$, and also $\mathbb{P}(B_m \cap C \cap D^c) \leq \mathbb{P}(D^c) < 4\varepsilon$. Hence, for $\varepsilon < 1/4$,

$$\frac{1}{\mathbb{P}(D)}|\mathbb{P}(B_m \cap D)\mathbb{P}(C \cap D) - \mathbb{P}(B_m)\mathbb{P}(C)\mathbb{P}(D)| + \mathbb{P}(B_m \cap C \cap D^c)$$
$$< \frac{1}{1 - 4\varepsilon} \max\{\mathbb{P}(B_m)\mathbb{P}(C)(1 - (1 - 4\varepsilon)), 4\varepsilon(\mathbb{P}(B_m) + \mathbb{P}(C)) - 16\varepsilon^2)\} + 4\varepsilon$$
$$\leq \frac{\max\{4\varepsilon, 8\varepsilon - 16\varepsilon^2\} + 4\varepsilon - 16\varepsilon^2}{1 - 4\varepsilon} \leq \frac{12\varepsilon}{1 - 4\varepsilon}.$$

Because $B \in \mathcal{F}_\infty \subseteq \bigvee_{k \geq n} \mathcal{F}_k$, we have $|\mathbb{P}(B_m \cap B) - \mathbb{P}(B_m)\mathbb{P}(B)| < 12\varepsilon/(1 - 4\varepsilon)$. Then, $\mathbb{P}(B \triangle B_m) < \varepsilon$ yields $|\mathbb{P}(B \cap B) - \mathbb{P}(B)\mathbb{P}(B)| < 12\varepsilon/(1 - 4\varepsilon) + 2\varepsilon$ for all $\varepsilon > 0$, and hence $P(B) = 0$ or 1. $\qquad\square$

The tail triviality of the zero-cell process is now used to show further 0–1 probabilities. The following assertion will be applied in the proof of Theorem 1.

Corollary 1. *Let Λ be a translation-invariant measure on the space of hyperplanes in $\mathbb{R}^d$, satisfying* (1) *and* (2)*, and let $G: \mathcal{P}_d \to [0, \infty)$ be a measurable functional on the set of d-dimensional polytopes. Then*

$$\mathbb{P}\left(\text{there exists } m \in \mathbb{Z}: \ \sum_{i \leq m} G\big(\hat{z}^o_{(i)}\big)^{-1} < \infty\right) \in \{0, 1\}. \tag{5}$$

Let $(\tau'_i,\ i \in \mathbb{Z})$ be a sequence of i.i.d. random variables, independent of $\hat{X}^*$. Then

$$\mathbb{P}\left(\text{there exists } m \in \mathbb{Z}: \ \sum_{i \leq m} G\big(\hat{z}^o_{(i)}\big)^{-1} \tau'_i < \infty\right) \in \{0, 1\}, \tag{6}$$

$$\mathbb{P}\left(\text{there exists } m \in \mathbb{Z}: \ \sum_{i \leq m} G\big(\hat{z}^o_{(i)}\big)^{-1} \tau'_i < \infty\right) = \mathbb{P}\left(\text{there exists } m \in \mathbb{Z}: \sum_{i \leq m} G\big(\hat{z}^o_{(i)}\big)^{-1} < \infty\right). \tag{7}$$

Proof. By the measurability of G, the event $\big\{$there exists $m \in \mathbb{Z}: \ \sum_{i \leq m} G\big(\hat{z}^o_{(i)}\big)^{-1} < \infty\big\}$ is an element of the tail-σ-algebra $\mathcal{F}_\infty$, and we can apply Lemma 1.

If $(\tau_i', \ i \in \mathbb{Z})$ is a sequence of i.i.d. random variables that is independent of $\hat{X}^*$, then this result can be extended straightforwardly to the product-σ-algebras $\mathcal{F}_n \otimes \mathcal{F}_n^\tau$, $n \in \mathbb{N}_0$, where $\mathcal{F}_n := \sigma(\hat{z}_n)$, $n \in \mathbb{N}_0$, and $\mathcal{F}_n^\tau := \sigma(\tau_n)$. This yields (6).

To prove (7), first note that almost sure non-explosion of the process $\left(\sum_{-j \le i \le m} G(\check{z}_{(i)}^o)^{-1} \tau_i', \ j \in \mathbb{N} \right)$ means that $\mathbb{P}\left(\sum_{i \le m} G(\check{z}_{(i)}^o)^{-1} \tau_i' = \infty \right) = 1$. It is well known in the theory of birth processes (see, for example, [7, Proposition 13.5]) that this is equivalent to $\mathbb{P}\left(\sum_{i \le m} G(\check{z}_{(i)}^o)^{-1} = \infty \right) = 1$, which means that the property of non-explosion only depends on the series of the expectations $G(\check{z}_{(i)}^o)^{-1}$ of the holding times of the respective states of the process. Hence, for the respective complements,

$$\mathbb{P}\left(\sum_{i \le m} G(\check{z}_{(i)}^o)^{-1} \tau_i' < \infty \right) = 0 \quad \Longleftrightarrow \quad \mathbb{P}\left(\sum_{i \le m} G(\check{z}_{(i)}^o)^{-1} < \infty \right) = 0.$$

Because, by (5) and (6), both probabilities can have the values 0 or 1 only, we obtain (7). $\square$

In [16], the rescaled time-stationary process of zero-cells $(a^t \check{z}_{a^t}^o, \ t \in \mathbb{R})$ for $a > 1$ was studied in more detail, and it was shown that this is a Bernoulli flow with infinite entropy.

4. Cell division processes

We consider a class of tessellation-valued Markov processes $(T_t, \ t > 0)$ which are characterized by the laws of the random life times of the cells and the laws for the division of the cells by a random hyperplane at the end of their life times. Furthermore, given the state of the process, conditional independence of the future development in different cells is assumed.

Definition 2. Let $\mathbb{Q} = (\mathbb{Q}_{[z]}, \ z \in \mathcal{P}_d)$, where $\mathbb{Q}_{[z]}$ is a probability measure on the set $[z]$ of hyperplanes which intersect z, and let $G \colon \mathcal{P}_d \to [0, \infty)$ be a measurable functional defined on the set of polytopes.

A random tessellation-valued process $(T_t, \ t > 0)$ is called an (L-*G*) (D-$\mathbb{Q}$) *cell division process* if, for all $t > 0$ and all cells $z \in T_t$:

(L-G) If $G(z) > 0$, then the cell z has a random life time which is exponentially distributed with parameter $G(z)$ and, given z, conditionally independent of $(T_{t'}, \ 0 < t' < t)$ and of all other cells of T_t. If $G(z) = 0$ then the life time is infinite, which means that z is never divided.

(D-$\mathbb{Q}$) At the end of its life time, z is divided by a random hyperplane h_z with the law $\mathbb{Q}_{[z]}$; also, given z, the hyperplane h_z is conditionally independent of $(T_{t'}, \ 0 < t' < t)$ and of all other cells of T_t, and independent of the life time of z.

It is not trivial to see whether such a cell division process $(T_t, \ t > 0)$ exists for given G and $\mathbb{Q}$, because there is no appropriate initial tessellation at time $t = 0$. Even if it is clear how to perform the cell division dynamics in a bounded polytope W (a window), a consistent extension to the whole space $\mathbb{R}^d$ is by no means straightforward.

Examples of interest for such rules can be related to translation-invariant hyperplane measures Λ which satisfy (1) and (2). We also have the following definitions:

(L-Λ) The life time of each cell z is exponentially distributed with parameter $\Lambda([z])$.

(L-V_d) The life time of each cell z is exponentially distributed with parameter $V_d(z)$, the volume of z.

(D-Λ) The law of the random hyperplane dividing a cell z is $\mathbb{Q}_{[z]} = \Lambda([z])^{-1}\Lambda(\,\cdot\cap[z])$, which is the probability measure on $[z]$ induced by Λ.

The STIT tessellation process driven by Λ is a cell division process with (L-Λ) and (D-Λ). In [19] it was shown that in the class of cell division processes defined above, only the (L-Λ) (D-Λ) model has the property of spatial consistency which is sufficient for its existence. It is therefore of interest to show the existence of further cell division processes without requiring spatial consistency.

4.1. A sufficient condition for the existence of a cell division process in $\mathbb{R}^d$

Let $(\mathcal{P}_d, \mathcal{B}(\mathcal{P}_d))$ be the measurable space of d-dimensional polytopes, where $\mathcal{B}(\mathcal{P}_d)$ is the Borel σ-algebra with respect to the Fell topology on the space of closed subsets of $\mathbb{R}^d$ [25]. A functional $G\colon \mathcal{P}_d \to [0, \infty)$ is called monotone if $G(z_1) \le G(z_2)$ for all $z_1, z_2 \in \mathcal{P}_d$ with $z_1 \subseteq z_2$. It is called translation invariant if $G(z) = G(z + x)$ for all $z \in \mathcal{P}_d$ and all $x \in \mathbb{R}^d$.

Now we address the problem of under which assumptions on Λ and G an (L-G) (D-Λ) cell division process $(T_t, \ t > 0)$ according to Definition 2 exists. If the whole space $\mathbb{R}^d$ is considered, there is no initial tessellation at time $t = 0$, and there is no time of 'first division'. Therefore, an idea is to use an auxiliary process of zero-cells which is defined for all times $t > 0$ and whose cells cover the whole $\mathbb{R}^d$ as $t \to 0$. Then, for any time $t > 0$ the future process inside the zero-cell at that time is constructed. The crucial issue is that these constructions launched at different times $t_1 < t_2$ in different zero-cells must be compatible, in the sense that the distribution of the cell division process after t_2 is identical for all $t_1 \le t_2$. Intuitively, if such a construction is used, it is essential that the sum of the life times of the 'ancestor' cells of a zero-cell at time $t > 0$ fits into the interval (0,1), i.e. the life times in the past are small enough. For a translation-invariant measure Λ a sufficient condition concerns the growth of the functional G. This is formalized in the following theorem.

Theorem 1. *Let Λ be a translation-invariant measure on the space of hyperplanes in $\mathbb{R}^d$, satisfying* (1) *and* (2), *and let $G\colon \mathcal{P}_d \to [0, \infty)$ be a measurable, monotone, and translation-invariant functional on the set of d-dimensional polytopes. If, for the random sequence $(\tilde{z}^o_{(i)}, \ i \in \mathbb{Z})$ defined in* (4),

$$\text{there exists } m \in \mathbb{Z} \text{ such that } \sum_{i \le m} G\big(\tilde{z}^o_{(i)}\big)^{-1} < \infty \tag{8}$$

holds a.s., then there exists a Markov tessellation-valued process $(T_t, \ t > 0)$ that is an (L-G) (D-Λ) cell division process according to Definition 2.

Proof. The idea of the proof is inspired by the global construction of STIT tessellations described in [18]. The key is a backward construction for $t \downarrow 0$ of the process $(z^o_t, \ t > 0)$ of zero-cells of $(T_t, \ t > 0)$.

The auxiliary process $(\hat{X}_t, \ t > 0)$ of Poisson hyperplane tessellations has exactly the (D-Λ) rule as the division law for its zero-cells. Hence, the sizes and shapes of the zero-cells of the Poisson hyperplane tessellations can be used for the construction of $(z^o_t, \ t > 0)$. What has to be changed is the 'clock'. The time axis has to be transformed such that the pure jump process of

zero-cells receives jump times according to (L-G), i.e. the life times of the respective cells are exponentially distributed, and the parameter is the functional value of G for these cells.

We use the random sequence $(\tilde{z}_{(i)}^{o},\ i \in \mathbb{Z})$ defined in (4) for the construction, and we assign appropriate holding times to it.

Let $(\tau_i',\ i \in \mathbb{Z})$ be a sequence of i.i.d. random variables, exponentially distributed with parameter 1; this sequence has to be independent of $\hat{X}^*$. Define $(\tau_i,\ i \in \mathbb{Z})$ as

$$\tau_i := \sum_{j \leq i-1} G(\tilde{z}_{(j)}^{o})^{-1}\, \tau_j', \qquad i \in \mathbb{Z}. \tag{9}$$

By (7) there is a.s. an $i \in \mathbb{Z}$ such that τ_i is finite if and only if (8) holds a.s. Hence, (8) is sufficient for the existence of the process $(z_t^o,\ t > 0)$ which is defined by

$$z_t^o := \tilde{z}_{(i)}^{o} \text{ for } \tau_i \leq t < \tau_{i+1}, \qquad i \in \mathbb{Z}. \tag{10}$$

And this also implies that $\lim_{j \to -\infty} \tau_j = 0$ a.s., and hence $\bigcup_{t>0} z_t^o = \mathbb{R}^d$ by (3).

Having shown the existence of the process $(z_t^o,\ t > 0)$ of zero-cells, the process $(T_t,\ t > 0)$ is completed as follows. Denote by cl the topological closure of a subset of $\mathbb{R}^d$. At each time τ_i, when the process $(z_t^o,\ t > 0)$ has a jump from $\tilde{z}_{(i-1)}^{o}$ to $\tilde{z}_{(i)}^{o}$, an (L-G) (D-Λ) cell division process according to Definition 2 is launched in the new separated cell $\mathrm{cl}\big(\tilde{z}_{(i-1)}^{o} \setminus \tilde{z}_{(i)}^{o}\big)$.

It remains to show that this actually yields a tessellation T_t for all $t > 0$, and that the process $(T_t,\ t > 0)$ is an (L-G) (D-Λ) cell division process.

Condition (ii) of Definition 1 is obviously satisfied. To show (i), consider an arbitrary point $x \in \mathbb{R}^d$. Then (3) implies that $x \in z_s^o$ for $0 < s < t_x$, where t_x is the time when x and o are separated by a hyperplane. Thus, the construction guarantees that, for all $t > 0$, any point $x \in \mathbb{R}^d$ belongs to the interior or to the boundary of a cell of T_t.

To prove property (iii) of Definition 1, consider a compact set $C \subset \mathbb{R}^d$ and a time $t_0 > 0$ such that $C \subset z_{t_0}^o$. By (3), such a time exists. Thus, for all $t \leq t_0$, there is exactly one cell in T_t that has a nonempty intersection with C. For $t > t_0$ the number of cells inside $z_{t_0}^o$ of the cell division process can be described by a birth process. By the monotonicity and translation invariance of G, the birth rates are dominated by $G(z_{t_0}^o) \times$ *number of cells in* $G(z_{t_0}^o)$. Hence, at any time $t > 0$, the number of cells intersecting C is a.s. finite.

Finally, to see that the process $(T_t,\ t > 0)$ is an (L-G) (D-Λ) cell division process, observe that the division rule for the zero-cell process is induced by the Poisson hyperplane process $\hat{X}$ and thus the (D-Λ) rule is realized. The definition in (10) yields that the holding time of the state $z_{(i)}^o$ is $\tau_{i+1} - \tau_i = G\big(\tilde{z}_{(i)}^{o}\big)^{-1}\tau_i'$, which shows that (L-$G$) is satisfied for the zero-cell process. For cells which are already separated from the zero-cell, the (L-G) and (D-Λ) rules are satisfied by the definition of the construction. $\square$

In the rest of this paper, we always mean by an (L-G) (D-Λ) process $(T_t,\ t > 0)$ the tessellation-valued process defined in the proof. Note that the proof does not imply that the distribution of the (L-G) (D-Λ) process is uniquely determined by G and Λ.

An example where the sufficient condition (8) is obviously not satisfied is $G(z) = c$ for all $z \in \mathcal{P}_d$ and some fixed $c > 0$. It is not known whether a cell division tessellation process exists for this G, but the simulations in [3] for the corresponding 'equally likely' case suggest that the cell division does not yield a tessellation according to Definition 1, because the local finiteness condition (iii) seems to be violated.

4.2. Stationarity in space

Now we show that the translation invariance of the hyperplane measure Λ and of the functional G are sufficient for the spatial stationarity of the constructed tessellations at any fixed time $t > 0$.

Theorem 2. *If the assumptions of Theorem* 1 *are satisfied then, for all $t > 0$, the random tessellation T_t described in the proof of the theorem is a spatially stationary (or homogeneous) tessellation.*

Proof. The proof can be sketched as follows. Because the hyperplane measure Λ defined in (1) is invariant under translations of $\mathbb{R}^d$, the distribution of the auxiliary Poisson process $\hat{X}^*$ is invariant under translations by $(x, 0) \in \mathbb{R}^d \times \{0\}$. Thus, denoting by $(\tilde{z}_{(i)}^o(\hat{X}^*), \ i \in \mathbb{Z})$ the process of zero-cells generated by $\hat{X}^*$,

$$\left(\tilde{z}_{(i)}^o(\hat{X}^* + (x, 0)), \ i \in \mathbb{Z}\right) \stackrel{\mathrm{D}}{=} \left(\tilde{z}_{(i)}^o(\hat{X}^*), \ i \in \mathbb{Z}\right) \qquad \text{for all } x \in \mathbb{R}^d.$$

This means invariance under translations of $\mathbb{R}^d$ for the distribution of the process $(\tilde{z}_{(i)}^o, \ i \in \mathbb{Z})$ defined in (4).

Furthermore, the translation invariance of G guarantees that the life time distributions of the cells remain translation invariant too. Finally, because Λ is invariant under translations, it follows that $\Lambda([z + x])^{-1}\Lambda(A + x \cap [z + x]) = \Lambda([z])^{-1}\Lambda(A \cap [z])$ for all $z \in \mathcal{P}_d$, $x \in \mathbb{R}^d$, and Borel sets $A \subset [z]$. Hence, the division law is translation equivariant. $\qquad\square$

4.3. Construction of the cut-out appearing in a bounded window *W*

For simulations of the model it is of interest to construct the restriction $(T_t \wedge W, \ t > 0)$ of $(T_t, \ t > 0)$ to a bounded window $W \in \mathcal{P}_d$, where $T_t \wedge W := \{z \cap W : z \in T_t, \ \mathrm{int}(z \cap W) \neq \emptyset\}$. It was shown in [21] that, for STIT tessellations, the (L-Λ) (D-Λ) cell division process can be launched in W without regarding boundary effects, because the STIT model is spatially consistent. And in [19] it was shown that all the other models considered in the present paper are not spatially consistent, which means that for the construction of $(T_t \wedge W, \ t > 0)$ information outside of W is also needed.

The construction described in the proof of Theorem 1 also shows how $(T_t \wedge W, \ t > 0)$ can be realized. For a given window W construct a zero-cell $\tilde{z}_{(i_W)}^o$ from the process $(\tilde{z}_{(i)}^o, \ i \in \mathbb{Z})$ defined in (4) such that $W \subset \tilde{z}_{(i_W)}^o$. This can be done as follows. Let $R > 0$ be such that $W \subset \mathsf{B}_R$. Simulate the restricted Poisson point process $\hat{X}^* \cap ([\mathsf{B}_{nR}] \times (0, 1/n))$, starting with $n = 2$. If the generated zero-cell contains W, then choose it as $\tilde{z}_{(i_W)}^o$; otherwise update $n := n + 1$ until an appropriate zero-cell is simulated.

Then launch the (L-G) (D-Λ) cell division process inside $\tilde{z}_{(i_W)}^o$. Its restriction to W has exactly the distribution of $(T_{t + \tau_{i_W}} \wedge W, \ t > 0)$. Here, the jump time τ_{i_W} of the zero-cell process is unknown because we do not know the value of the series in (9).

5. The Mondrian directional distribution

Now we study a particular class of directional distributions φ and functionals G where the sufficient condition of Theorem 1 is satisfied.

Denote by $e_1, \ldots, e_d$ the orthonormal base of $\mathbb{R}^d$ with $e_k := (0, \ldots, 0, 1, 0 \ldots 0)$ where the 1 is at the kth position, $k \in \{1, \ldots, d\}$. Let δ_{e_k} denote the Dirac probability measure defined on S^{d-1} and with mass 1 on e_k.
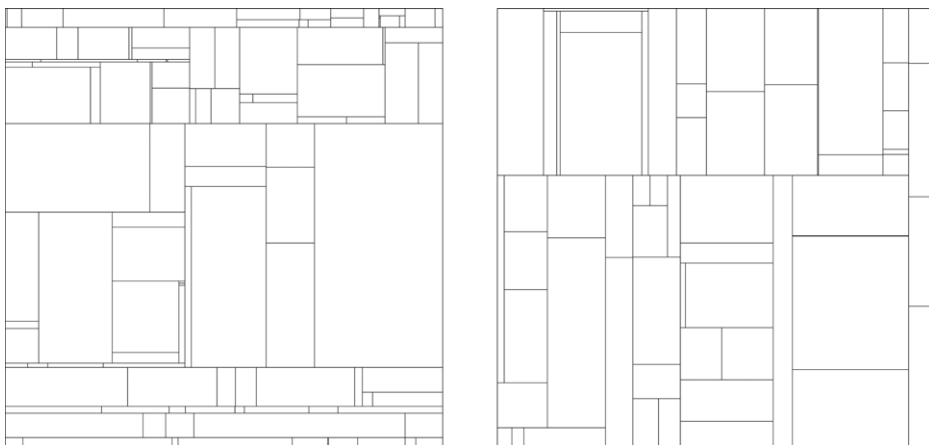
FIGURE 1. Simulations of cell division tessellations in a quadratic window with (L-Λ) (left panel) and (L-V_2) (right panel), both with (D-Λ). The directional distribution is $\varphi = \frac{1}{2}\delta_{e_1} + \frac{1}{2}\delta_{e_2}$. (Generated with the software from [10].)

In this section we consider spherical directional distributions of the form

$$\varphi = \frac{1}{2}\sum_{k=1}^{d} p_k(\delta_{e_k} + \delta_{-e_k}), \qquad p_k > 0, \ \sum_{k=1}^{d} p_k = 1. \tag{11}$$

Referring to recent papers like [24], we will call them Mondrian directional distributions.

The random tessellations generated by Mondrian directional distributions consist of cuboids (orthogonal parallelepipeds). As mentioned above, the volume $G = V_d$ is a functional of interest for the life time distribution, and so are the other intrinsic volumes $G = V_n$, $n \in \{1, \ldots, d-1\}$, including the surface content and the mean width of a cell. For cuboids these intrinsic volumes depend on the side lengths only, see (15). Thus, the sufficient condition of Theorem 1 can be checked by proving that these side lengths are 'long enough'. See Figure 1 for an example.

Theorems 3 and 4 provide the first concrete models of cell-division processes in $\mathbb{R}^d$ that differ from the STIT model.

5.1. Existence results

For a d-dimensional cuboid z with all $(d-1)$-dimensional faces parallel or orthogonal to the axes in $\mathbb{R}^d$, denote by $l^{(1)}, \ldots, l^{(d)}$ the side lengths of the cuboid, where $l^{(k)}$ is the length of the sides that are parallel to e_k, $k \in \{1, \ldots, d\}$. Write $S(z) := \sum_{k=1}^{d} l^{(k)}$ if z is a cuboid and $S(z) = 1$ elsewhere.

Theorem 3. *Let Λ be a translation-invariant measure on the space of hyperplanes in $\mathbb{R}^d$ satisfying* (1), *where φ is a Mondrian directional distribution* (11). *Then there exists a Markov tessellation-valued process $(T_t, \ t > 0)$ that is an (L-S) (D-Λ) cell division process according to Definition* 2.

Proof. Consider the auxiliary Poisson process $\hat{X}^*$ of hyperplanes that are marked with birth times, i.e. the Poisson point process on $\mathcal{H} \times (0, \infty)$ with the intensity measure $\Lambda \otimes \lambda_+$, see Section 2, and Λ defined in (1) with φ as in (11). We show that the sufficient condition of

Theorem 1 is satisfied for the functional $G = S$, which is monotone and translation invariant on the class of cuboids.

The Poisson point process $\hat{X}^*$ can be described as a superposition of d i.i.d. Poisson point processes $\hat{X}^{(k)*}$, i.e. $\hat{X}^* \overset{D}{=} \bigcup_{k=1}^d \hat{X}^{(k)*}$ where $\hat{X}^{(k)*}$ is a Poisson point process on $\mathcal{H} \times (0, \infty)$ with the intensity measure $p_k \Lambda^{(k)} \otimes \lambda_+$, and $\Lambda^{(k)}$ is as defined in (1) with $\varphi^{(k)} = \frac{1}{2}(\delta_{e_k} + \delta_{-e_k})$, $k \in \{1, \ldots, d\}$ (for which condition (2) is obviously not satisfied).

For $k \in \{1, \ldots, d\}$ we define the Poisson point processes Φ^{k+} and Φ^{k-} on $(0, \infty) \times (0, \infty)$ as the birth-time-marked processes of intersection points of the hyperplanes of $\hat{X}^{(k)*}$ with the positive axis and the negative axis in direction e_k and $-e_k$, respectively. Formally,

$$\Phi^{k+} := \{(x, t) \in (0, \infty) \times (0, \infty) : (h(e_k, x), t) \in \hat{X}^{(k)*}\},$$
$$\Phi^{k-} := \{(x, t) \in (0, \infty) \times (0, \infty) : (h(-e_k, x), t) \in \hat{X}^{(k)*}\}.$$

Both point processes have the intensity measure $p_k \lambda_+ \otimes \lambda_+$. Now we define the Markov chains $M^{k+} = ((x_n^{k+}, t_n^{k+}), n \in \mathbb{N}_0)$ with $(x_n^{k+}, t_n^{k+}) \in \Phi^{k+}$ and $x_0^{k+} := \min\{x : (x, t) \in \Phi^{k+}, t \leq 1\}$, $x_{n+1}^{k+} := \min\{x : (x, t) \in \Phi^{k+}, t \leq t_n^{k+}\}$, $n \in \mathbb{N}_0$.

The times t_n^{k+} are a.s. uniquely defined by the condition $(x_n^{k+}, t_n^{k+}) \in \Phi^{k+}$. Correspondingly, M^{k-} is defined by replacing $k+$ by $k-$.

The properties of the Poisson process $\hat{X}^*$ yield that, for fixed $k \in \{1, \ldots, d\}$, the Markov chains M^{k+} and M^{k-} are i.i.d.

Let us consider some properties of M^{k+}. The random variable x_0^{k+} is exponentially distributed with parameter p_k, and t_0^{k+} is uniformly distributed in the interval $(0,1)$. Then, $x_{n+1}^{k+} - x_n^{k+}$ is exponentially distributed with the parameter $p_k t_n^{k+}$ and is independent of $x_0^{k+}, \ldots, x_n^{k+}$ and, given t_n^{k+}, conditionally independent of $t_0^{k+}, \ldots, t_{n-1}^{k+}$. Furthermore, t_{n+1}^{k+} is uniformly distributed in the interval $(0, t_n^{k+})$ and, given t_n^{k+}, conditionally independent of $x_0^{k+}, \ldots, x_n^{k+}$ and of $t_0^{k+}, \ldots, t_{n-1}^{k+}$.

Now consider the Markov chain $((S_i, t_i), i \in \mathbb{Z}_-)$, where $(t_i, i \in \mathbb{Z})$ is the ordered sequence of jump times of $(\check{z}_t^o, t > 0)$ with $t_0 < 1 < t_1$, as defined in Section 2, and $S_i := S(\check{z}_{(i)}^o)$. Thus we have $t_0 = \max\{t_0^{k+}, t_0^{k-}, k \in \{1, \ldots, d\}\}$, and $S_0 = \sum_{k=1}^d x_0^{k+} + x_0^{k-}$. By recursion, $t_{i-1} = \max\{t_n^{k+} < t_i, t_n^{k-} < t_i, k \in \{1, \ldots, d\}, n \in \mathbb{Z}_-\}$, and

$$S_{i-1} = \begin{cases} S_i + x_{m+1}^{k+} - x_m^{k+} & \text{if } t_i = t_m^{k+} \text{ for some } m \in \mathbb{Z}_-, \\ S_i + x_{m+1}^{k-} - x_m^{k-} & \text{if } t_i = t_m^{k-} \text{ for some } m \in \mathbb{Z}_-. \end{cases} \tag{12}$$

Let $(R_i, i \in \mathbb{Z})$ be a sequence of i.i.d. random variables with the probability density $f(r) = 2dr^{2d-1}\mathbf{1}\{0 < r < 1\}$, i.e. R_i has the distribution of the maximum of $2d$ i.i.d. random variables that are uniformly distributed in the interval $(0,1)$.

Thus, the law of the time component t_i in the Markov chain $((S_i, t_i), i \in \mathbb{Z}_-)$ can be described by $t_0 \overset{D}{=} R_0$, and, by recursion, $t_{i-1} \overset{D}{=} t_i R_{i-1}, i \in \mathbb{Z}_-$. Thus,

$$(t_i, i \in \mathbb{Z}_-) \overset{D}{=} \left(\prod_{j=i}^0 R_j, i \in \mathbb{Z}_-\right).$$

The law of $S_{i-1} - S_i$ depends on both t_i and the index k^+ or k^-. It is the exponential distribution with parameter $p_k t_i$, not depending on the sign of the index k.

Let $(\kappa_i,\ i \in \mathbb{Z})$ be a sequence of i.i.d. random variables, exponentially distributed with parameter 1. This sequence is assumed to be independent of all the other random variables we have considered so far. Then the conditional distribution is

$$S_{i-1} - S_i \overset{\mathrm{D}}{=} \left(p_k \textstyle\prod_{j=i}^0 R_j\right)^{-1} \kappa_i \le p_{\min}^{-1} \left(\textstyle\prod_{j=i}^0 R_j\right)^{-1} \kappa_i, \tag{13}$$

under the condition that $t_i = t_m^{k+}$ or $t_i = t_m^{k-}$ for some $m \in \mathbb{N}_0$, with $p_{\min} := \min\{p_1, \ldots, p_k\}$.

Thus, a sufficient condition for

$$\textstyle\sum_{i \le 0} S_i^{-1} = S_0^{-1} + \sum_{i \le 0} \left(S_0 + \sum_{\ell=i}^0 (S_{\ell-1} - S_\ell)\right)^{-1} < \infty \quad \text{a.s.}$$

is

$$\textstyle\sum_{i \le 0} \left(\sum_{\ell=i}^0 \left(\prod_{j=\ell}^0 R_j\right)^{-1} \kappa_\ell\right)^{-1} < \infty \quad \text{a.s.} \tag{14}$$

Note that in (14) the item S_0^{-1} is neglected in order to simplify the technicalities. To verify (14) we provide an upper bound for the expectation of the random variable on the left-hand side:

$$\mathbb{E} \textstyle\sum_{i \le 0} \left(\sum_{\ell=i}^0 \left(\prod_{j=\ell}^0 R_j\right)^{-1} \kappa_\ell\right)^{-1} \le \mathbb{E} \textstyle\sum_{i \le 0} \left(\sum_{\ell=i}^{i+1} \left(\prod_{j=\ell}^0 R_j\right)^{-1} \kappa_\ell\right)^{-1}$$

$$= \mathbb{E} \sum_{i \le 0} \frac{\prod_{j=i+1}^0 R_j}{R_i^{-1} \kappa_i + \kappa_{i+1}}$$

$$\le \mathbb{E} \sum_{i \le 0} \frac{\prod_{j=i+1}^0 R_j}{\kappa_i + \kappa_{i+1}}$$

$$= \mathbb{E} \frac{1}{\kappa_0 + \kappa_1} \sum_{i \le 0} \left(\prod_{j=i+1}^0 \mathbb{E} R_j\right)$$

$$= \sum_{i \le 0} \left(\prod_{j=i+1}^0 \frac{2d}{2d+1}\right) < \infty.$$

In the last equation we used that the sum of two i.i.d. exponentially distributed random variables with parameter 1 has a gamma distribution with parameter $(2,1)$, also referred to as an Erlang distribution, and hence $\mathbb{E}(\kappa_0 + \kappa_1)^{-1} = 1$. We have thus shown that (14) is satisfied. $\square$

In order to generalize the result for Mondrian directional distributions, we consider the individual side lengths of the zero-cells. Denote by $l_i^{(1)}, \ldots, l_i^{(d)}$ the side lengths of the cuboid $\tilde{z}_{(i)}^o$, where $l_i^{(k)}$ is the length of the sides that are parallel to e_k, $k \in \{1, \ldots, d\}$, $i \in \mathbb{Z}$. The following technical lemma prepares us for the proof of Theorem 4.

Lemma 2. *With the assumptions of Theorem 3,*

$$\mathbb{P}(\text{there exists } i_0 \in \mathbb{Z}: \text{ for all } i \le i_0 \text{ and all } k \in \{1, \ldots, d\}, l_i^{(k)} > d) = 1.$$

Proof. Fix some $k \in \{1, \ldots, d\}$ and, using the notation in the proof of Theorem 3, consider (13), which expresses the growth of the length $l_i^{(k)}$ if $t_i = t_m^{k+}$ or $t_i = t_m^{k-}$ for some $m \in \mathbb{N}_0$. The index k is chosen with probability p_k. Obviously, $\left(p_k \prod_{j=i}^0 R_j\right)^{-1} > 1$.

Hence, the events $B_i := \{\kappa_i > d, \ t_i = t_m^{k+} \text{ or } t_i = t_m^{k-} \text{ for some } m \in \mathbb{N}_0\}$, $i \in \mathbb{Z}$, which are independent, have a strictly positive probability that does not depend on i. Thus, the Borel–Cantelli lemma yields $\mathbb{P}\big(\bigcap_{n \leq 0} \bigcup_{i \leq n} B_i\big) = 1$, i.e. there are infinitely many of the B_i a.s. This, together with the monotonicity of the sequence $(l_i^{(k)}, \ i \in \mathbb{Z})$ yields that, for all $k \in \{1, \ldots, d\}$, $\mathbb{P}(\text{there exists } i_0(k) \in \mathbb{Z}: \text{ for all } i \leq i_0(k), l_i^{(k)} > d) = 1$, and hence $\mathbb{P}(\text{for all } k \in \{1, \ldots, d\} \text{ there exists } i_0(k) \in \mathbb{Z}: \text{ for all } i \leq i_0(k), l_i^{(k)} > d) = 1$. Because there are only finitely many k, the assertion of the lemma follows. $\square$

Lemma 2 can be used to check the sufficient condition (8) when the functional G is chosen as a function of the side lengths of the zero-cell. Here we consider the important cases of the intrinsic volumes V_n, $n \in \{1, \ldots, d\}$. The functional V_d is the volume, V_1 the mean width (or breadth) up to a constant factor. For $n = 3$ the intrinsic volume V_2 is, up to a constant factor, the surface area. The intrinsic volume V_0 is the Euler characteristic which has the value 1 for all nonempty convex sets, and hence it is clear that for $G = V_0$ the sufficient condition (8) is not satisfied.

A definition of the intrinsic volumes for convex bodies can be found in [25]. Here we make use of a formula for cuboids which is a particular case of the formula for polytopes, see [8, (4.9)] or [25, (14.35)]. For $n \geq 1$,

$$V_n(\tilde{z}_{(i)}^o) = \sum_{1 \leq k_1 < \cdots < k_n \leq d} \prod_{j=1}^{n} l_i^{(k_j)}. \tag{15}$$

Theorem 4. *Let Λ be a translation-invariant measure on the space of hyperplanes in $\mathbb{R}^d$ satisfying* (1), *where φ is a Mondrian directional distribution* (11). *Then, for all $n \in \{1, \ldots, d\}$, there exists a Markov tessellation-valued process $(T_t, \ t > 0)$ which is an (L-V_n) (D-Λ) cell division process according to Definition* 2.

Proof. It is sufficient to show that (8) is satisfied for $G = V_n$ for all $n \in \{1, \ldots, d\}$. For $n = 1$ it follows immediately from Theorem 3. For $n \in \{2, \ldots, k\}$ and $i \in \mathbb{Z}$,

$$\left(\sum_{k_1 < \cdots < k_n} \prod_{j=1}^{n} l_i^{(k_j)} \right) - \sum_{k=1}^{d} l_i^{(k)} = \frac{1}{\binom{d-1}{n-1}} \sum_{k=1}^{d} l_i^{(k)} \left(\sum_{k_1 < \cdots < k_{n-1}, \, k_j \neq k} \left(\frac{\binom{d}{n}}{d} \prod_{j=1}^{n-1} l_i^{(k_j)} - 1 \right) \right).$$

Note that $\binom{d}{d}/d = 1/d$ and $\binom{d}{n}/d \geq 1$ for $n \in \{1, \ldots, d-1\}$. By Lemma 2 there exists a.s. an $i_0 \in \mathbb{Z}$ such that, for all $i \leq i_0$ and all $k \in \{1, \ldots, d\}$, the side lengths $l_i^{(k)} > d$, which implies that the difference on the left-hand side is positive. Recall that $S_i = \sum_{k=1}^{d} l_i^{(k)}$ for all $i \in \mathbb{Z}$, and that in the proof of Theorem 3 it was shown that $\sum_{i \leq 0} S_i^{-1} < \infty$ a.s. Hence, for $n \in \{2, \ldots, k\}$, $\sum_{i \leq 0} \big(\sum_{k_1 < \cdots < k_n} \prod_{j=1}^{n} l_i^{(k_j)} \big)^{-1} < \infty$ a.s. also, and, by (15), $\sum_{i \leq 0} V_n\big(\tilde{z}_{(i)}^o\big)^{-1} < \infty$ a.s. $\square$

Remark 1. Theorem 4 can easily be generalized to (L-V_n^α) (D-Λ) cell division processes for exponents $\alpha \geq 1$. The larger α is, the longer will be the life time of cells z with $V_n(z) < 1$ and the shorter the life time if $V_n(z) > 1$.

Note that for the Mondrian distribution defined in (11) with $p_1 = \ldots = p_d = 1/d$, the (L-V_1) (D-Λ) cell division process is, up to a scaling factor, the same as the STIT tessellation process driven by Λ, which is a (L-Λ) (D-Λ) cell division process.

5.2. Distribution of cell volumes

Now we consider the particular cell division process in $\mathbb{R}^d$ with volume-weighted life time distribution (L-V_d) and (D-Λ), with a Mondrian directional distribution (11). We make use of an idea, used in [3], developed for so-called 'geometry-independent apportionment of volume' (GIA), which means a division rule such that the ratio of the volume of a daughter cell and the volume of the mother cell is uniformly distributed on the interval (0,1).

In contrast to this, in Section 5.3 we consider an (L-V_d) (D-Λ) process in a *bounded* window W and describe its relation to fragmentation.

For a random stationary tessellation of $\mathbb{R}^d$ the concept of the 'typical cell' is formally defined using the Palm measure; see [25, p. 450], for example. Intuitively, it can be imagined as a cell, chosen at random, where all cells have the same chance to be chosen. Let T be a random spatially stationary tessellation in $\mathbb{R}^d$ with law $\mathbb{P}_T$. Furthermore, let $c \colon \mathcal{P}_d \to \mathbb{R}^d$ be a center function, which means that c is measurable and translation covariant, i.e. $c(z + x) = c(z) + x$ for all $z \in \mathcal{P}_d$, $x \in \mathbb{R}^d$. For example, $c(z)$ can be chosen as the circumcenter of z. The intensity of the center points of cells of T is the mean number of those points per unit volume, $\gamma_T := \mathbb{E} \sum_{z \in T} \mathbf{1}\{c(z) \in [0, 1]^d\}$, where $\mathbf{1}\{\ldots\}$ denotes the indicator function with value 1 if the condition in brackets is satisfied and 0 otherwise. Then the distribution $\mathbb{Q}_T$ of the typical cell of T is defined by

$$\mathbb{Q}_T(A) := \gamma_T^{-1} \mathbb{E} \sum_{z \in T} \mathbf{1}\{c(z) \in [0, 1]^d\} \mathbf{1}\{z - c(z) \in A\}$$

for all measurable sets $A \subseteq \mathcal{P}_d$. A random polytope with law $\mathbb{Q}_T$ is called the typical cell of T.

Lemma 3. *Let Λ be a translation-invariant measure on the space of hyperplanes in $\mathbb{R}^d$ satisfying (1), where φ is a Mondrian directional distribution (11), and let $(T_t, \ t > 0)$ be the (L-V_d) (D-Λ) cell division process as described in the proof of Theorem 1. Then, for all $t > 0$, the law of the volume $V_d(z)$ of the typical cell of T_t is the exponential distribution with parameter t, and the law of the volume $V_d(z_t^o)$ of the zero-cell of T_t is the gamma distribution with parameter (2,t), which is also referred to as an Erlang distribution.*

Proof. The volumes of cells of a tessellation T_t, $t > 0$, are represented by the lengths of intervals on $\mathbb{R}$. The idea of the proof is the following: The tessellation T_t is mapped to a point process $(y^{(j)}, \ j \in \mathbb{Z})$ on $\mathbb{R}$ such that the lengths of the intervals between adjacent points correspond to the volumes of the cells of T_t.

In order to formalize this idea, it is an involved technical issue to bring these intervals into an appropriate linear order. As before, we start with the process of zero cells as defined in (10), and we use a method analogous to the one in the proof of Theorem 3. But here we consider the differences of volumes when cells are split.

If, for some $\varepsilon > 0$, the tessellation T_ε is mapped to a point process $(y_\varepsilon^{(j)}, \ j \in \mathbb{Z})$ on $\mathbb{R}$ such that the lengths of the intervals between adjacent points correspond to the volumes of the cells of T_ε, then the division of the cell volumes in the continuation of the cell division process corresponds to a *Poisson* point process that is superposed on $(y_\varepsilon^{(j)}, \ j \in \mathbb{Z})$. Because this can be done for any $\varepsilon > 0$, we can conclude that the cell volumes of T_t are represented by the lengths of the interval between two adjacent points of a Poisson point process. Summarizing the idea, we will show this for a Poisson point process on $\mathbb{R}^d \times (0, \infty)$ with intensity measure $\lambda \otimes \lambda_+$.

Fix an $\varepsilon > 0$ and $i \in \mathbb{Z}$ such that $\tau_i \leq \varepsilon < \tau_{i+1}$, where τ_i is defined in (9) with $G = V_d$. Let $\hat{y}_{\varepsilon}^{(-1)} < 0 < \hat{y}_{\varepsilon}^{(0)}$ be real valued such that $\hat{y}_{\varepsilon}^{(0)} - \hat{y}_{\varepsilon}^{(-1)} = V_d(z_{\tau_i})$, and 0 is uniformly distributed in the interval $(\hat{y}_{\varepsilon}^{(-1)}, \hat{y}_{\varepsilon}^{(0)})$.

Initialize counters $r := 0$ and $\ell := 1$. If, in (12), $t_i = t_m^{k+}$ for some k and m, then put $\hat{y}_{\varepsilon}^{(1)} := \hat{y}_{\varepsilon}^{(0)} + V_d(z_{\tau_{i-1}} \setminus z_{\tau_i})$, $t^{(0)} := \tau_i$, and update $r := 1$. If $t_i = t_m^{k-}$ for some k and m, then define $\hat{y}_{\varepsilon}^{(-2)} := \hat{y}_{\varepsilon}^{(-1)} - V_d(z_{\tau_{i-1}} \setminus z_{\tau_i})$, $t^{(-1)} := \tau_i$, and update $\ell := 2$.

Having defined $(\hat{y}_{\varepsilon}^{(-\ell)}, t^{(-\ell)}), \ldots, (\hat{y}_{\varepsilon}^{(r)}, t^{(r)})$ for some $\ell \in \mathbb{N}_0$ and $r \in \mathbb{N}_0$, the next item of the sequence is that if, in (12), $t_{i-r-\ell-1} = t_m^{k+}$ for some k and m, then $\hat{y}_{\varepsilon}^{(r+1)} := \hat{y}_{\varepsilon}^{(r)} + V_d(z_{\tau_{i-r-\ell-2}} \setminus z_{\tau_{i-r-\ell-1}})$, $t^{(r+1)} := \tau_{i-r-\ell-1}$, and update $r := r + 1$. If $t_{i-r-\ell-1} = t_m^{k-}$ for some k and m, then $\hat{y}_{\varepsilon}^{(-\ell-1)} := \hat{y}_{\varepsilon}^{(-\ell)} - V_d(z_{\tau_{i-r-\ell-2}} \setminus z_{\tau_{i-r-\ell-1}})$, $t^{(-\ell-1)} := \tau_{i-r-\ell-1}$, and update $\ell := \ell + 1$.

This yields the sequence $((\hat{y}_{\varepsilon}^{(j)}, t^{(j)}), j \in \mathbb{Z})$ pertaining to the process of zero-cells. It has to be complemented by the points which correspond to the divisions of the cells $\mathrm{cl}(z_{\tau_{i-r-\ell-2}} \setminus z_{\tau_{i-r-\ell-1}})$ in the time interval $(\tau_{i-r-\ell-1}, t)$.

This can be described as follows. The volume of the cell $\mathrm{cl}(z_{\tau_{i-r-\ell-2}} \setminus z_{\tau_{i-r-\ell-1}})$ equals the length of the interval $(\hat{y}_{\varepsilon}^{(r)}, \hat{y}_{\varepsilon}^{(r+1)})$ or $(\hat{y}_{\varepsilon}^{(-\ell-1)}, \hat{y}_{\varepsilon}^{(-\ell)})$, respectively. For the sake of simplicity, in the following we consider only the part of $((\hat{y}_{\varepsilon}^{(j)}, t^{(j)}), j \in \mathbb{Z})$ that belongs to $(0, \infty) \times (0, \infty)$. The other part on $(-\infty, 0) \times (0, \infty)$ can be dealt with quite analogously. When the cell $\mathrm{cl}(z_{\tau_{i-r-\ell-2}} \setminus z_{\tau_{i-r-\ell-1}})$ is divided by a hyperplane, the respective interval is divided by a point such that the two new interval lengths equal the two volumes of the daughter cells. We define the new interval that is closer to $0 \in \mathbb{R}$ to pertain to the daughter cell which is contained in the half-space of the dividing hyperplane that contains the origin $o \in \mathbb{R}^d$. The point dividing the interval is marked with the birth time of the hyperplane dividing the cell. Thus, the interval $(\hat{y}_{\varepsilon}^{(r)}, \hat{y}_{\varepsilon}^{(r+1)})$ is divided after a life time that is exponentially distributed with the parameter $\hat{y}_{\varepsilon}^{(r+1)} - \hat{y}_{\varepsilon}^{(r)}$, and the dividing point is (due to the translation invariance of Λ) uniformly distributed in this interval. The new daughter intervals are subsequently divided following analogous rules. Thus, the birth-time-marked point process appearing in the interval $(\hat{y}_{\varepsilon}^{(r)}, \hat{y}_{\varepsilon}^{(r+1)})$ has the same distribution as the Poisson point process on $(\hat{y}_{\varepsilon}^{(r)}, \hat{y}_{\varepsilon}^{(r+1)}) \times (\tau_{i-r-\ell-1}, t)$, with the two-dimensional Lebesgue measure (restricted to this set) as its intensity measure.

Taking all the marked points together, we obtain a point process $Y_{\varepsilon} := ((y_{\varepsilon}^{(j)}, t^{(j)}), j \in \mathbb{Z})$. Given this point process for some time $\varepsilon > 0$, and regarding the independence assumptions in Definition 2, the generation of the point process $Y_t = ((y_t^{(j)}, t^{(j)}), j \in \mathbb{Z})$ for $t > \varepsilon$ can be described using an auxiliary Poisson point process Φ^* on $\mathbb{R} \times (0, \infty)$ with intensity measure $\lambda \otimes \lambda_+$.

For all $0 < \varepsilon < t$, let $\Phi_{(\varepsilon,t)}^* := \{(x, t') \in \Phi^* \colon \varepsilon < t' < t\}$. We obtain $Y_t \overset{\mathrm{D}}{=} Y_{\varepsilon} \cup \Phi_{(\varepsilon,t)}^*$, which is the superposition of two independent point processes.

The sequence $((\hat{y}_{\varepsilon}^{(j)}, t^{(j)}), j \in \mathbb{Z})$ pertaining to the process of zero cells is monotone in ε in the following sense. For $\varepsilon_2 < \varepsilon_1$ the sequence $((\hat{y}_{\varepsilon_2}^{(j)}, t^{(j)}), j \in \mathbb{Z})$ emerges from $((\hat{y}_{\varepsilon_1}^{(i)}, t^{(i)}), i \in \mathbb{Z})$ by deleting the points with $t^{(i+1)} > \varepsilon_2$ for $i \geq 0$ or $t^{(i-1)} > \varepsilon_2$ for $i < 0$, followed by a respective rearrangement of the indexes $i \in \mathbb{Z}$. Accordingly, $Y_{\varepsilon_2} = ((y_{\varepsilon_2}^{(j)}, t^{(j)}), j \in \mathbb{Z})$ can, up to a rearrangement of the indexes, be considered as a subsequence of $Y_{\varepsilon_1} = ((y_{\varepsilon_1}^{(j)}, t^{(j)}), j \in \mathbb{Z})$.

Denoting by $Y_{(\varepsilon_2, \varepsilon_1)}$ the difference between Y_{ε_1} and Y_{ε_2}, we can write

$$Y_{\varepsilon_2} \cup \Phi_{(\varepsilon_2,t)}^* \overset{\mathrm{D}}{=} Y_{\varepsilon_2} \cup \Phi_{(\varepsilon_2,\varepsilon_1)}^* \cup \Phi_{(\varepsilon_1,t)}^* \overset{\mathrm{D}}{=} Y_{\varepsilon_2} \cup Y_{(\varepsilon_2,\varepsilon_1)} \cup \Phi_{(\varepsilon_1,t)}^*,$$

where the superposed point processes are independent. This yields that $Y_{(\varepsilon_2,\varepsilon_1)} \overset{\mathrm{D}}{=} \Phi^*_{(\varepsilon_2,\varepsilon_1)}$. Intuitively, this means that the sequence $((\hat{y}^{(i)}_{\varepsilon_1}, t^{(i)}), \; i \in \mathbb{Z})$ pertaining to the process of zero-cells until time ε_1 is consistent with the Poisson point process properties, which are characteristic for the consecutive division of cell volumes.

Because this holds for all $\varepsilon_2 > \varepsilon_1 > 0$, we have shown that $Y_t = ((y^{(j)}_t, t^{(j)}), \; j \in \mathbb{Z})$ is a Poisson point process on $\mathbb{R} \times (0, t)$ with intensity measure $\lambda \otimes \lambda_{(0,t)}$, where $\lambda_{(0,t)}$ denotes the restriction of the Lebesgue measure to the interval $(0,t)$. Hence, the point process $(y^{(j)}_t, \; j \in \mathbb{Z})$ of points projected onto $\mathbb{R}$ is a homogeneous Poisson point process with intensity measure $t\lambda$, and therefore the length of the typical interval between two points is exponentially distributed with parameter t, while the length of the interval containing the origin o has a gamma distribution with parameter $(2, t)$. $\qquad\square$

Cell division processes are potential models for real crack or fracture patterns appearing in materials science, geology, soft matter, or nanotechnology. The examples considered in [12, 20] illustrate that a useful and simple initial goodness-of-fit criterion for such models is the coefficient of variation (CV) of the volume of the typical cell, and it became obvious that the CV of STIT is too large compared to the CV for data of real crack patterns.

Corollary 2. *With the assumptions of Lemma* 3, *for all* $t > 0$,

$$\mathrm{CV}(V_d(z)) := \frac{\sqrt{\mathrm{Var}(V_d(z))}}{\mathbb{E}V_d(z)} = 1,$$

where z is a random cuboid with the distribution of the typical cell of T_t, and Var *denotes the variance of a random variable.*

For a cell division process with (L-Λ) (D-Λ), which is the STIT tessellation process, at any time $t > 0$ the typical cell has the same distribution as the typical cell of Poisson hyperplane tessellation for the same Λ at the same time [21]. Hence, for the translation-invariant measure Λ satisfying (1) where φ is a Mondrian directional distribution (11), the CV of the volume of the typical cell equals $\mathrm{CV}\big(\prod_{j=1}^{d} Z_j\big)$, where $Z_1, \ldots, Z_d$ are independent exponentially distributed random variables with the parameters $p_1 t, \ldots, p_d t$, respectively. Thus $\mathrm{CV}(V_d(z^{\mathrm{STIT}})) = 2^d - 1$, where z^{STIT} is a random cuboid with the distribution of the typical cell of the STIT tessellation.

Note that these CVs depend neither on the time t nor on the probabilities $p_1, \ldots, p_d$. Obviously, for $d \geq 2$ the variability of the volume of the typical cell of the (L-V_d) (D-Λ) cell division process is considerably smaller than that of STIT.

We remark that the method of the proof of Lemma 3 is crucially based on the above-mentioned GIA principle, and in our context this can only be applied to the volume of cells. Therefore, it can neither be used to deal with other intrinsic volumes of the typical cell of the (L-V_d) (D-Λ) cell division process, nor on the typical cell of any (L-V_n) (D-Λ) cell division process with $n < d$. In those cases the representation of the functional values by intervals on the real line does not yield a Poisson point process.

5.3. A cell division process in a bounded window W and its relation to fragmentation

Now consider a fixed window $W \in \mathcal{P}_d$ which is a cuboid with its sides parallel or orthogonal to all axes of $\mathbb{R}^d$. An (L-V_d) (D-Λ) cell division process $(T_{W,t}, \; t \geq 0)$ *within* W is launched with $T_{W,0} := \{W\}$. Note that its distribution is different from $(T_t \wedge W, \; t > 0)$, which is the restriction to W of the (L-V_d) (D-Λ) cell division process $(T_t, \; t > 0)$ in the whole of $\mathbb{R}^d$.

The process $(T_{W,t}, \ t \geq 0)$ can be related to a particular case of a conservative fragmentation process as described in [1].

Let $G \colon \mathcal{P}_d \to [0, \infty)$ be a measurable functional defined on the set of polytopes, and assume that $G(W) = 1$. For $t > 0$ and $T_{W,t} = \{z_1, \ldots, z_n\}$, define $\mathbf{G}(T_{W,t}) := (G(z_{i_1}), \ldots, G(z_{i_n}), 0, 0, \ldots)$ such that $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$ and $G(z_{i_1}) \geq \cdots \geq G(z_{i_n})$.

If we choose $G(z) = V_d(z)$ for $z \in \mathcal{P}_d$, then the (L-V_d) (D-Λ) cell division process $(T_{W,t}, \ t \geq 0)$ induces a pure jump Markov process $(\mathbf{G}(T_{W,t}), \ t \geq 0)$ on $\mathfrak{S} := (0, \infty)^{\mathbb{N}}$ with the following properties:

(i) $\sum_{z \in T_{W,t}} G(z) = 1$ for all $t \geq 0$.

(ii) The holding time of a state $(G(z_{i_1}), \ldots, G(z_{i_n}), 0, 0, \ldots)$ is exponentially distributed with parameter 1.

(iii) At the nth jump time τ_n, $n \in \mathbb{N}$, the process jumps from a state $(G(z_{i_1}), \ldots, G(z_{i_n}), 0, 0, \ldots)$ to $(G(z_{j_1}), \ldots, G(z_{j_{n+1}}), 0, 0, \ldots)$ by the following rule:
The index i_k, $k \in \{1, \ldots, n\}$, is chosen at random with probability $G(z_{i_k}) = G(z_{i_k}) / \sum_{\ell \in \{1, \ldots, n\}} G(z_{i_\ell})$. Then the cell z_{i_k} is divided into two new fragments, z'_{i_k} and z''_{i_k} say, and

$$(G(z'_{i_k}), G(z''_{i_k})) = (U G(z_{i_k}), (1 - U) G(z_{i_k})), \tag{16}$$

where U is a random variable, uniformly distributed on (0,1) and independent of all the other random variables considered here.

(iv) Putting the G-values of the $n + 1$ cells into descending order yields the new state $(G(z_{j_1}), \ldots, G(z_{j_{n+1}}), 0, 0, \ldots)$.

This shows that, according to [1, Definition 1.1], the process $(\mathbf{G}(T_{W,t}), \ t \geq 0)$ is a self-similar fragmentation chain with index of self-similarity $\alpha = 1$ and the dislocation measure ν defined by (16), i.e. ν is the law of $(\xi, 1 - \xi, 0, 0, \ldots)$, where ξ is a random variable which is uniformly distributed on the interval $\left[\frac{1}{2}, 1\right]$.

Note that if G is chosen as another intrinsic volume V_n, $n < d$, then $\sum_{z \in T_{W,t}} G(z) \geq 1$ with '>' after the first division. Then this is no longer the *conservative* scenario in the sense of [1].

6. Discussion and open problems

The 'constructive proof' of the existence result of Theorem 1 of a cell division process does not imply a uniqueness result. We conjecture that the distribution of the process is uniquely determined by G and Λ, but it is an open problem whether for (L-G) and (D-Λ) other distributions of cell division processes also exist. Up to now, only the uniqueness of the (L-Λ) (D-Λ) cell division process has been shown, which is the STIT tessellation process driven by Λ.

Somehow related to the uniqueness problem is the question of whether the sufficient condition in (8) for the existence of an (L-G) (D-Λ) cell division process is also a necessary one.

In a cell division process $(T_t, \ t > 0)$ the cells z are divided subsequently as long as $G(z) > 0$. Therefore, in general there will not be a limiting random tessellation for $t \to \infty$. Thus the problem arises of whether there is a scaling of the process which stabilizes the one-dimensional

distributions of the process or even yields a time-stationary process. For a tessellation $T \in \mathbb{T}$ the scaling by a factor $c > 0$ means $cT := \{cz; z \in T\}$, with $cz := \{cx \colon x \in z\}$.

For the STIT tessellation process $(Y_t, \ t > 0)$ driven by an arbitrary Λ satisfying (1) and (2), it was shown in [21] that $tY_t \overset{\mathrm{D}}{=} Y_1$ for all $t > 0$. Furthermore, in [13] it was proven that for $a > 1$ the process $a^t Y_{a^t}$ is stationary in time.

For (L-V_d) (D-Λ) and the Mondrian directional distribution, by Lemma 3, for all $t > 0$ and for the zero-cell z_t^0 of T_t, we have $tV_d(z_t^0) \overset{\mathrm{D}}{=} V_d(z_1^0)$ and hence $V_d(t^{1/d} z_t^0) \overset{\mathrm{D}}{=} V_d(z_1^0)$. This does not yet show a scaling property for the whole random tessellation T_t but we can see that, if there is a scaling factor, it can only be $t^{1/d}$.

Regarding tail triviality and other ergodic properties of the tessellations generated by cell division, the so-called method of encapsulation used in [14, 15] cannot be applied here directly for models which are not spatially consistent. But the method used in the proof of Lemma 1 for zero-cells can be developed further.

Real crack structures, some of them mentioned in the introduction, show the tendency that the crack of a cell divides it close to the center. This is not yet incorporated in the models with a (D-Λ) division rule where Λ is a translation-invariant measure. Maybe this can be partially compensated in some cases using the life time rule (L-G^α) instead of (L-G), with a large value of the exponent α. The impact of a large α is that the life time of small cells is considerably increased while the life time of large cells is decreased compared to the setting with $\alpha = 1$, see also Remark 1. Thus, if the division of a mother cell generates a very small daughter cell and a relatively large one, then the larger one will be more frequently divided than the smaller one. This can result in a tessellation which is more 'homogeneous' in the sense that the sizes, measured by the function G, of the cells do not differ too much. And this can emulate a partition rule where the cells are divided by hyperplanes close to their center.

Nevertheless, it is desirable to develop models for cell division processes with alternative division rules as well as life time distributions. Here, a combination of the results of [4, 29] with the present approach could be promising. Simulation studies for tessellations in a bounded window of the plane $\mathbb{R}^2$ with several life time rules L and also a variety of division rules, aimed to adapt models to real structures, are presented in [11, 12]. Code for simulation of those tessellations is available in [10].

Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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