THE ELEMENTARY SOLUTION OF DUAL INTEGRAL EQUATIONS†

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(Received 13 April, 1959)

1. When the theory of Hankel transforms is applied to the solution of certain mixed boundary value problems in mathematical physics, the problems are reduced to the solution of dual integral equations of the type

$$\int_0^\infty \xi^\alpha \psi(\xi) J_\nu(\xi\rho)\, d\xi = f(\rho) \quad (0 < \rho < 1),$$

$$\int_0^\infty \psi(\xi) J_\nu(\xi\rho)\, d\xi = 0 \qquad (\rho > 1),$$

where α and ν are prescribed constants and $f(\rho)$ is a prescribed function of ρ [1]. The formal solution of these equations was first derived by Titchmarsh [2]. The method employed by Titchmarsh in deriving the solution in the general case is difficult, involving the theory of Mellin transforms and what is essentially a Wiener-Hopf procedure. In lecturing to students on this subject one often feels the need for an elementary solution of these equations, especially in the cases $\alpha = \pm 1, \nu = 0$. That such an elementary solution exists is suggested by Copson's solution [3] of the problem of the electrified disc which corresponds to the case $\alpha = -1, \nu = 0$. A systematic use of a procedure similar to Copson's has in fact been made by Noble [4] to find the solution of a pair of general dual integral equations, but again the analysis is involved and long. The object of the present note is to give a simple solution of the pairs of equations which arise most frequently in physical applications. The method of solution was suggested by a procedure used by Lebedev and Uflyand [5] in the solution of a much more general problem.

2. To solve the pair of dual integral equations

$$\int_0^\infty \xi^{-1} \psi(\xi) J_0(\xi\rho)\, d\xi = f(\rho) \quad (0 < \rho < 1), \qquad \text{.........................(1)}$$

$$\int_0^\infty \psi(\xi) J_0(\xi\rho)\, d\xi = 0 \quad (\rho > 1), \qquad \text{.........................(2)}$$

which occur in the discussion of the flow of a jet of a perfect fluid through a circular aperture in a plane rigid screen, of the field due to an electrified circular disc, and of the punch problem in elasticity, we set

$$\psi(\xi) = \xi \int_0^1 \phi(t) \cos(\xi t)\, dt. \qquad \text{.........................(3)}$$

For this form of the function $\psi(\xi)$ we find that

† This work, which was done in the Department of Mathematics, Duke University, North Carolina, has been supported by the Air Force Office of Scientific Research of the Air Research and Development Command under Contract No. AF 18(600)1341.

$$\int_0^\infty \psi(\xi)J_0(\rho\xi)\,d\xi = \int_0^\infty J_0(\xi\rho)\,d\xi \int_0^1 \phi(t)\xi\cos(\xi t)\,dt$$
$$= \phi(1)\int_0^\infty J_0(\xi\rho)\sin\xi\,d\xi - \int_0^\infty J_0(\xi\rho)\,d\xi\int_0^1 \phi'(t)\sin(\xi t)\,dt.$$

If we invert the order of integration in the double integral on the right and make use of the result [**6**]

$$\int_0^\infty J_0(\xi\rho)\sin(\xi t)\,d\xi = 0 \quad (t < \rho), \qquad \text{(4)}$$

we see that the form (3) satisfies equation (2).

Substituting from equation (3) into equation (1) we find that $\phi(t)$ must satisfy the equation

$$\int_0^\infty J_0(\rho\xi)\,d\xi\int_0^1 \phi(t)\cos(\xi t)\,dt = f(\rho) \quad (0<\rho<1).$$

If we interchange the order of integration and make use of a well-known integral [**6**]

$$\int_0^\infty J_0(\rho\xi)\cos(\xi t)\,d\xi = \begin{cases} 0 & (0<\rho<t), \\ (\rho^2-t^2)^{-\frac{1}{2}} & (\rho>t), \end{cases} \qquad \text{(5)}$$

we see that $\phi(t)$ must satisfy the integral equation

$$\int_0^\rho \frac{\phi(t)\,dt}{\sqrt{(\rho^2-t^2)}} = f(\rho) \quad (0<\rho<1).$$

The simple transformation $t = \rho\sin\theta$ turns this integral equation into Schlömilch's integral equation, the solution of which is elementary [7]. We find that, under conditions which are sufficient for our present purpose, the solution of this equation is

$$\phi(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{\rho f(\rho)\,d\rho}{\sqrt{(t^2-\rho^2)}} \qquad \text{(6)}$$

(Compare Lemma 2 of [**3**]).

Substituting from (6) into (3) and integrating by parts we find that

$$\psi(\xi) = \frac{2\xi}{\pi}\left\{\cos\xi\int_0^1 \frac{\rho f(\rho)\,d\rho}{\sqrt{(1-\rho^2)}} + \xi\int_0^1 \sin(\xi t)\,dt\int_0^t \frac{\rho f(\rho)\,d\rho}{\sqrt{(t^2-\rho^2)}}\right\},$$

which is easily transformed to the familiar form (cf. [**1**, p. 456])

$$\psi(\xi) = \frac{2\xi}{\pi}\left\{\cos\xi\int_0^1 \frac{\rho f(\rho)\,d\rho}{\sqrt{(1-\rho^2)}} + \xi\int_0^1 \frac{u\,du}{\sqrt{(1-u^2)}}\int_0^1 vf(uv)\sin(\xi v)\,dv\right\}. \qquad \text{(7)}$$

3. We can proceed in a similar way to solve the pair of dual integral equations

$$\int_0^\infty \xi\psi(\xi)J_0(\xi\rho)\,d\xi = f(\rho) \quad (0<\rho<1), \qquad \text{(8)}$$

$$\int_0^\infty \psi(\xi)J_0(\xi\rho)\,d\xi = 0 \quad (\rho>1), \qquad \text{(9)}$$

which arise in crack problems in the theory of elasticity. In this case we put

$$\psi(\xi) = \int_0^1 \chi(t)\sin(\xi t)\,dt, \quad \chi(0) = 0. \qquad \text{(10)}$$

It is obvious from equation (4) that the form (10) satisfies the equation (9). If we substitute from (10) into (8) we find that

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$$\int_0^\infty \xi\psi(\xi)J_0(\rho\xi)\,d\xi = \int_0^\infty J_0(\rho\xi)\,d\xi\left\{-\chi(1)\cos\xi + \int_0^1 \chi'(t)\cos(\xi t)\,dt\right\}$$
$$= \int_0^1 \chi'(t)\,dt\int_0^\infty J_0(\rho\xi)\cos(\xi t)\,d\xi - \chi(1)\int_0^\infty J_0(\rho\xi)\cos\xi\,d\xi.$$

Making use of the result expressed by equations (5) we see that the integral equation determining $\chi(t)$ is

$$\int_0^\rho \frac{\chi'(t)\,dt}{\surd(\rho^2-t^2)} = f(\rho) \qquad (0 < \rho < 1).$$

This is exactly the same equation as for $\phi(t)$ in §2, and so has solution

$$\chi'(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{\rho f(\rho)\,d\rho}{\surd(t^2-\rho^2)}.$$

If we integrate with respect to t and use the fact that $\chi(0) = 0$, we have

$$\chi(t) = \frac{2}{\pi}\int_0^t \frac{\rho f(\rho)\,d\rho}{\surd(t^2-\rho^2)}.$$

Inserting this expression in equation (10) we get the solution

$$\psi(\xi) = \frac{2}{\pi}\int_0^1 \sin(\xi t)\,dt\int_0^t \frac{\rho f(\rho)\,d\rho}{\surd(t^2-\rho^2)}. \qquad \text{.............................(11)}$$

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https://doi.org/10.1017/S2040618500034006 Published online by Cambridge University Press