ABSTRACT KERNELS AND COHOMOLOGY

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Let G, N be groups, let $A(N)$ be the automorphism group of N and let $I(N)$ be the subgroup of inner automorphisms. A homomorphism

$$\theta : G \to A(N)/I(N)$$

will be denoted by (G, N, θ) and called an *abstract kernel.* (G, N, θ) induces in an obvious manner a structure of a (left) G-module on the centre C of N. A well known construction of Eilenberg and MacLane [1, § 7–9] assigns to (G, N, θ) its obstruction Obs $(G, N, \theta) \in H^3(G, C)$. This assignment is such that if C is an arbitrary G-module then every element of $H^3(G, C)$ is of the form Obs (G, N, θ) for a suitable abstract kernel (G, N, θ).

We have discussed in [4, § 7, p. 302] abstract kernels

$$\theta : V \to A(N)/I(N)$$

where V is a local group. Generalizing the construction for groups, we have assigned to each (V, N, θ) its obstruction Obs $(V, N, \theta) \in H^3(V, C)$. (The V-module structure of C is induced by (V, N, θ) and the cohomology of V is the one defined in [4, § 4, p. 298] or, if V is contained in a group, the one defined in [2, § 5, p. 396]).

The purpose of this note is to show that the analogy with the group case does not go further, i.e. we shall prove the

THEOREM. *There exists a local group V and a V-module C such that a certain element of $H^3(V, C)$ is not of the form* Obs (V, N, θ) *for any abstract kernel* (V, N, θ).

Say that the local group V is *embedded* in a group G if V is a subset of G and the multiplication in V is taken from G (whenever performable in V). Say that this G is *V-monodrome* if V generates G and every morphism $V \to H$, where H is a group, can be extended to a morphism $G \to H$ [4, § 2, p. 294]. In this case there is a natural identification of G-modules and V-modules [5, § 2.3], so that $H^n(V, C)$ and $H^n(G, C)$ may be considered with the same C. Further, the inclusion $V \subset G$ induces the restriction

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morphism $H^n(G, C) \to H^n(V, C)$ (by restriction of cochains). The proof of our theorem follows from the two lemmas below.

LEMMA 1. *If V is embedded in a V-monodrome group G and C is a G-module such that every element of $H^3(V, C)$ is the obstruction for some abstract kernel, then the restriction morphism $H^3(G, C) \to H^3(V, C)$ is onto.*

PROOF. To obtain the obstruction of an abstract kernel (V, N, θ)

(i) select a map (not necessarily morphism) $\alpha : V \to A(N)$ such that θ is the composite of α and the quotient morphism $A(N) \to A(N)/I(N)$,

(ii) to every $v_1 v_2 \in V$ with $v_1 v_2$ defined assign an $h(v_1, v_2) \in N$ such that $\alpha(v_1)\alpha(v_2)(\alpha(v_1 v_2))^{-1}$ is the inner automorphism of N by $h(v_1, v_2)$,

(iii) for every $v_1, v_2, v_3 \in V$ with $v_1 v_2, v_2 v_3, v_1 v_2 v_3$ defined, denote

$$f_{\alpha,h}(v_1, v_2, v_3) = {}^{\alpha(v_1)}h(v_2, v_3)h(v_1, v_2 v_3)h^{-1}(v_1 v_2, v_3)h^{-1}(v_1, v_2).$$

Then $f_{\alpha,h}$ is a C-valued cocycle where $C = \text{centre } N$ is a V-module via (V, N, θ). The cohomology class $\{f_{\alpha,h}\} \in H^3(V, C)$ is, by definition [4, § 7, p. 303], the required Obs (V, N, θ).

Now suppose that V, C and G satisfy the assumptions of the lemma. Let $\gamma \in H^3(V, C)$ be arbitrary. Then $\gamma = \text{Obs}\,(V, N, \theta)$ for some abstract kernel. But as G is V-monodrome, (V, N, θ) can be uniquely extended to an abstract kernel $(G, N, \bar\theta)$, i.e. with $\bar\theta | V = \theta$. Let $\bar\alpha : G \to A(N)$, $\bar h : G \times G \to N$ satisfy the conditions obtained from (i), (ii) above by replacing V by G. Then $f_{\bar\alpha,\bar h}$ (defined by analogy with (iii)) is a cocycle in the class $\bar\gamma = \text{Obs}\,(G, N, \bar\theta) \in H^3(G, C)$. Now it is obvious that if we define α to be the restriction of $\bar\alpha$ to V and $h(v_1, v_2) = \bar h(v_1, v_2)$ whenever $v_1, v_2, v_1 v_2 \in V$, then $f_{\alpha,h}$ is the restriction of $f_{\bar\alpha,\bar h}$ and

$$\{f_{\alpha,h}\} = \text{Obs}\,(V, N, \theta) = \gamma.$$

Thus γ is the image of $\bar\gamma$ under $H^3(G, C) \to H^3(V, C)$.

LEMMA 2. There exists a local group V, embedded in a V-monodrome group G, and a G-module C such that $H^3(G, C) = 0$ and $H^3(V, C) \neq 0$.

PROOF. Let V be a local group embedded in a group G. Denote by Γ_G^V the simplicial scheme [3, p. 37] the set of whose vertices is G and such that $\{g_0, \cdots, g_n\} \subset G$ is a simplex iff $g_i^{-1} g_j \in V$ for all i, j. Let $H^n(\Gamma_G^V)$ be its cohomology with integral coefficients.

Define the (coinduced) G-module C to be $\text{Hom}_Z(ZG, Z)$ where ZG is the group ring and the action of $g \in G$ on $f : ZG \to Z$ is given by $(gf)(x) = f(xg)$. Clearly $H^3(G, C) = 0$. On the other hand, we have by [5, § 2.4, Thm 1] that

$$H^3(V, C) = H^3(\Gamma_G^V).$$

To complete the proof, we have to find an example where $H^3(\Gamma_G^V) \neq 0$ and G is V-monodrome.

Let G be the group of unit length quaternions. Then G is topologically a 3-sphere whence its singular cohomology $H^3_{\text{top}}(G)$ (integral coefficients) is Z. Denote by $\mathscr{V}$ the family of symmetric neighbourhoods of the identity in G. Then the set $\{H^3(\Gamma_G^V) | V \in \mathscr{V}\}$, together with the restriction morphisms

$$H^3(\Gamma_G^V) \to H^3(\Gamma_G^{\bar{V}}) \text{ for } \bar{V} \subset V,$$

forms a directed system. Since G is a connected Lie group, we have from a theorem of van Est [2, § 11.1, p. 410] that

$$\lim_{\rightarrow} H^3(\Gamma_G^V) = H^3_{\text{top}}(G) = Z.$$

Thus $H^3(\Gamma_G^V) \neq 0$ provided V is sufficiently small. If, moreover, V is connected, then G will be V-monodrome by the simple connectedness of G [4, § 11, p. 000]. This completes the proof.

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