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CONVOLUTION OF L^p FUNCTIONS ON NON-UNIMODULAR GROUPS

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In this note we prove the following

THEOREM. *If G is a nonunimodular locally compact group and $1<p<\infty$, then there is an open set, U, in G and there are functions, f simultaneously in every $L^r(G)$, $p\le r\le\infty$, and g simultaneously in every $L^q(G)$, $1\le q\le\infty$, such that the convolution, $f*g(y)$, is not defined for any y in U.*

REMARK 1. Rickert proved a theorem similar to this in [1]. He proved that, if $1<p<\infty$, $1<q<\infty$, $1/p+1/q<1$ and G is an arbitrary noncompact locally compact group, then there is an open set, U, in G and there are functions, $f\in L^p$ and $g\in L^q$, such that $f*g(y)$ is not defined for any $y\in U$.

REMARK 2. This theorem provides another proof that $L^p(G)$ is not an algebra under convolution for any $p>1$, if G is not unimodular [2, Lemma 1].

Proof of Theorem. Let V be a symmetric neighbourhood of the identity, e, of G such that $\frac{3}{4}\le\Delta(x)\le\frac{4}{3}$ $\forall x\in V$ and $0<\mu(V)<\infty$, where Δ is the modular function and μ is left Haar measure on G. $\exists$ an open set, $U\subset V$, such that $\mu(yV\cap V)>0$, $\forall y\in U$.

Choose $t\in G$ such that $\Delta(t)=a\ge 4$; then $Vt^n\cap V^2t^m=\varnothing$ if $m\ne n$. For $1\le n<\infty$, let f_n (g_n) be the characteristic function of Vt^n $(t^{-n}V)$. Note that $g_n(x)=f_n(x^{-1})$ $\forall x\in G$ and $\forall n$, $f_n(x)f_m(y^{-1}x)=0$ $\forall x\in G$ if $y\in U$ and $m\ne n$, and

$$\int f_n(x)f_n(y^{-1}x)\,d\mu(x) = \mu((yV\cap V)t^n) = a^n\mu(yV\cap V).$$

Put $f=\sum f_n/(a^{n/p}n^2)$, $g=\sum g_m/m^2$. (All sums are taken from 1 to ∞.) $f\in L^r$, $p\le r\le\infty$, and $g\in L^q$, $1\le q\le\infty$, as required. If $y\in U$,

$$\begin{aligned} f*g(y) &= \sum_{m,n} 1/(a^{n/p}n^2m^2)\int f_n(x)g_m(x^{-1}y)\,d\mu(x)\\ &= \sum_n 1/(a^{n/p}n^4)\int f_n(x)f_n(y^{-1}x)\,d\mu(x)\\ &= \mu(yV\cap V)\sum_n 1/n^4a^{n(1-1/p)},\end{aligned}$$

which does not converge.

One might think that, in this setting, the map, $(f,g)\to f*\check{g}$, where $\check{g}(x)=g(x^{-1})$, would take $L^p\times L^q$ into L^s for some choices of p, q and s. Using the same technique

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as in the theorem, but different g, it is easy to show that this is the case only if $1/p+1/q=1$, and then $s=\infty$.

References

1. N. W. Rickert, *Convolution of L^p functions*, Proc. Amer. Math. Soc. **18** (1967), 762–763.
2. W. Zelazko, *A note on L^p-algebras*, Colloq. Math. **10** (1963), 53–56.

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