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LOCALLY GRADED GROUPS WITH A NILPOTENCY CONDITION ON INFINITE SUBSETS

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Dedicated to Professor B. H. Neumann on the occasion of his 90th birthday

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Abstract

A group G is locally graded if every finitely generated nontrivial subgroup of G has a nontrivial finite image. Let $N(2, k)^*$ denote the class of groups in which every infinite subset contains a pair of elements that generate a nilpotent subgroup of class at most k. We show that if G is a finitely generated locally graded $N(2, k)^*$-group, then there is a positive integer c depending only on k such that $G/Z_c(G)$ is finite.

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Let k be a fixed positive integer. We denote by $N(2, k)$ the class of groups in which every 2-generator subgroup is nilpotent of class at most k, and by $N(2, k)^*$ the class of groups in which every infinite subset contains a pair of elements that generate a nilpotent subgroup of class at most k. The main result of [3] states that a finitely generated residually finite group G belongs to $N(2, 2)^*$ if and only if $G/Z_2(G)$ is finite. In [1] it was proved that if G is finitely generated and soluble, then $G \in N(2, k)^*$ if and only if $G \in \mathfrak{F}N(2, k)$, where $\mathfrak{F}$ denotes the class of finite groups. It is remarked in [1] that if $G/Z_k(G)$ is finite, then $G \in \mathfrak{F}N(2, k)$ but that the converse is false for $k \geq 3$, even if G is finitely generated and soluble of derived length three. The examples cited, which are due to Newman [8], are torsion-free nilpotent and hence residually finite. For $k = 2$ there is the result from [2] that a finitely generated soluble group G belongs to $N(2, 2)^*$ if and only if $G/Z_2(G)$ is finite. Using some deep results of Zel'manov and Lubotzky and Mann, we are able to establish a theorem that provides

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what is essentially a generalization of each of the results mentioned above. First we recall that a group G is *locally graded* if every finitely generated nontrivial subgroup of G has a nontrivial finite image.

THEOREM 1. *Let G be a finitely generated locally graded group and suppose that $G \in N(2,k)^*$. Then there is a positive integer c depending only on k such that $G/Z_c(G)$ is finite.*

Let us consider for the moment the special case where the group G is finite-by-nilpotent and satisfies the hypotheses of the theorem. Then G has a finite normal subgroup N with G/N torsion-free nilpotent and, by [1, Lemma 2], $G/N \in N(2,k)$. A result of Zel'manov [11] now tells us that the class t of G/N is bounded in terms of k only, and it follows that $\gamma_{t+1}(G)$ is finite and hence that $G/Z_t(G)$ is finite [5]. Thus in order to establish the theorem we need to show that the group in question is finite-by-nilpotent. In fact, it is the residually finite case that we need to deal with, and we shall see that the result follows quite easily once we have proved the following (where $\mathfrak{N}$ denotes the class of nilpotent groups).

PROPOSITION 1. *Let G be a finitely generated residually finite group in the class $N(2,k)^*$. Then $G \in \mathfrak{F}\mathfrak{N}$.*

In order to establish Proposition 1 we require a number of preliminary results. But let us note at this point that there is a gap at the beginning of the proof of Theorem 1 in [3], in that the deduction that G is hypercentre-by-finite relies on the assumption that G is residually finite modulo its hypercentre, a fact that is not at all clear. It is possible to salvage the argument by means of techniques similar to those adopted in [3] (where Zel'manov's results are certainly not required), but we omit the details and proceed directly to the matter in hand.

In what follows $\mathfrak{N}_k$ denotes the class of nilpotent groups of class at most k.

LEMMA 1. *Let $G \in N(2,k)^*$ and let $x, y \in G$. Then $\langle x\rangle^{\langle y\rangle}$ is finitely generated.*

PROOF. If y is of finite order, then the result is clear, so assume y has infinite order and consider the set $\{x, xy, xy^2, \dots\}$. For some i, j with $0 \le i < j$ we have $\langle xy^i, xy^j\rangle \in \mathfrak{N}_k$ and hence $\langle xy^i, y^m\rangle \in \mathfrak{N}_k$, where $m = j - i$, so that $1 = [xy^i, {}_k y^m] = [x, {}_k y^m]$. For $g \in G$ and $r \in \mathbb{N}$, an easy induction shows that $\langle [x, {}_i g]; 0 \le i \le r\rangle = \langle x^{g^i}; 0 \le i \le r\rangle$, (read $[x, {}_0 g] = x$), and it follows easily that $\langle x\rangle^{\langle y\rangle} \le \langle x^{y^i}; |i| \le km\rangle$, thus giving the result. □

With the notation of [6], a group in the class $N(2,k)^*$ is therefore restrained, and repeated use of Corollary 4 of that paper gives the following (where $G^{(i)}$ denotes the i-th term of the derived series of G).

COROLLARY 1. *Let $G \in N(2,k)^*$ and suppose that G is finitely generated. Then $G^{(i)}$ is finitely generated for each positive integer i.*

Next, suppose that G is a finitely generated group in $N(2,k)^*$ and let K be an arbitrary soluble image of G. Then K is finite-by-nilpotent by [7, Theorem A], and hence finite-by-(nilpotent of k-bounded class), as we saw earlier. Thus the Hirsch length of K is bounded by some integer that depends only on k and the number of generators of G, and it follows that there is a positive integer $s = s(G)$ such that $G^{(j)}/G^{(j+1)}$ is finite for all $j \geq s$.

LEMMA 2. *Let G be a finitely generated group in the class $N(2,k)^*$ and let H be an arbitrary subgroup of finite index in $G^{(s)}$, where $s = s(G)$. Then H/H' is finite.*

PROOF. By Corollary 1 we know that $G^{(s)}$ is finitely generated, and it follows that $G^{(s)}$ is finite modulo the core C of H in G. If C/C' is finite, then so is H/H', so we may assume that H is normal in G. Suppose for a contradiction that H/H' is infinite, and let K/H' denote its torsion subgroup. Choose $h \in H\backslash K$, $x \in G^{(s)}$ and consider the set $\{x, xh, xh^2, \dots\}$. By hypothesis there are distinct integers i, j such that $\langle xh^i, xh^j\rangle \in \mathfrak{N}_k$, and then $[h^{j-i}, {}_k x] = 1$, so that x has nontrivial centralizer C_1/K in H/K. If this is not of finite index in H/K then we may repeat the argument to obtain a series $K < C_1 < C_2 \leq H$, where $[C_2, x] \leq C_1$. Since $G^{(s)}/H$ is finite there is a positive integer n such that $x^n \in H$, and H/K is of course abelian. Thus modulo K we have $1 = [C_2, x^n] = [C_2, x]^n$; but H/K is torsion-free and we have the contradiction $[C_2, x] \leq K$. We deduce that H/C_1 is finite and hence that $H = C_1$, that is, $[H, x] \leq K$. Since x was arbitrary we now have $[H, G^{(s)}] \leq K$ and hence $G^{(s)}/K$ centre-by-finite, and it follows that $G^{(s+1)}K/K$ is finite. But $G^{(s)}/G^{(s+1)}$ is also finite, giving the contradiction that H/K is finite. The lemma is therefore proved. □

COROLLARY 2. *Let G and s be as in Lemma 2 and suppose that G is residually finite. If $H \in N(2,k)$ for some H of finite index in $G^{(s)}$ then $G \in \mathfrak{F}\mathfrak{N}$.*

PROOF. As before, H is finitely generated. It is also residually finite and, by hypothesis, k-Engel. By [10, Theorem 2], therefore, H is nilpotent. But H/H' is finite, by Lemma 2, and we deduce that H is finite and hence that G is finite-by-soluble and therefore finite-by-nilpotent [7]. □

Our next result, in conjunction with those above, will allow us to restrict our attention to groups G that have a very special structure.

LEMMA 3. *Let L be a finitely generated residually finite group belonging to $N(2,k)^*$ and suppose that $H \notin N(2,k)$ for every subgroup H of finite index in L. Then there*

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exists a normal subgroup G of finite index in L such that $G = N\langle t\rangle$ for some normal subgroup N of G and element t satisfying $\langle at, bt\rangle \in \mathfrak{N}_k$ for all $a, b \in N$.

PROOF. Assume for a contradiction that L has no such subgroup G. Then there exist elements x_0, y_0 of L such that $\langle x_0, y_0\rangle \notin \mathfrak{N}_k$, and we may choose a normal subgroup G_1 of finite index in L such that $\langle x_0, y_0\rangle G_1/G_1 \notin \mathfrak{N}_k$. Put $w_0 = y_0$.

If $\langle w_0x, w_0y\rangle \in \mathfrak{N}_k$ for all $x, y \in G_1$, then we may take $N = G_1$ and $w_0 = t$ for a contradiction. Thus $\langle w_0x, w_0y\rangle \notin \mathfrak{N}_k$ for some $x_1, y_1 \in G_1$, and there is a normal subgroup G_2 of finite index in L and contained in G_1 such that $\langle w_0x_1, w_0y_1\rangle G_2/G_2 \notin \mathfrak{N}_k$. Put $w_1 = w_0y_1$ and repeat the argument with G_1 replaced by G_2 and w_0 by w_1.

Continuing, we obtain a sequence of elements $z_i = w_{i-1}x_i$, where $w_i = w_{i-1}y_i$ for all $i \geq 1$, and a chain of subgroups $L > G_1 > G_2 \ldots$ with $\langle z_i, w_i\rangle G_{i+1}/G_{i+1} \notin \mathfrak{N}_k$ for each $i \geq 1$. Suppose $j > i \geq 1$. It is straightforward to show that $z_j \equiv w_i$ mod G_{i+1}, and thus we have $\langle z_i, z_j\rangle G_{i+1}/G_{i+1} \notin \mathfrak{N}_k$. It follows that $\langle z_i, z_j\rangle \notin \mathfrak{N}_k$ for all distinct i, j, contradicting the fact that $G \in N(2, k)^*$ and thus establishing the lemma. □

In view of Lemma 2, Corollary 2 and Lemma 3, we may assume for the proof of Proposition 1 that G has the following properties, say $(*)$:

G is finitely generated and residually finite, $G = N\langle t\rangle$ for some normal subgroup N and element t such that $\langle at, bt\rangle \in \mathfrak{N}_k$ for all $a, b \in N$, and H/H' is finite for every subgroup H of finite index in G.

LEMMA 4. *If G satisfies $(*)$, then every finite image of G is nilpotent.*

PROOF. For this we may assume that G is itself finite. Let $x, y \in N$ be of order p^r, q^s respectively, where p, q are distinct primes. We show that $[x, y] = 1$.

Write $t = t_1t_2$, where each of t_1 and t_2 is a power of t, t_1 is a p'-element and t_2 is a q'-element. For arbitrary $g \in G$ we have $\langle x^g, t\rangle = \langle x^g t, t\rangle \in \mathfrak{N}_k$. It follows that $[x^g, t_1] = 1$ for all $g \in G$ and hence that x centralizes $\langle t_1\rangle^G$. Thus $\langle x\rangle^{\langle t_2y\rangle} = \langle x\rangle^{\langle ty\rangle}$, which is a p-group since $\langle x, ty\rangle = \langle tyx, ty\rangle = \langle yxt, yt\rangle^{t^{-1}} \in \mathfrak{N}_k$. Since $\langle y, t\rangle \in \mathfrak{N}_k$ we have similarly that $[y, t_2] = 1$, so there is a q'-number c such that $(t_2y)^c = y^c$, and it follows that $y \in \langle t_2y\rangle$ and hence that $\langle x\rangle^{\langle y\rangle}$ is a p-group. Similarly $\langle y\rangle^{\langle x\rangle}$ is a q-group, and we have $[x, y] = 1$ as claimed.

It follows that N is nilpotent and so $\gamma_{s+1}(N) = 1$ for some integer s. Also $\langle x, t\rangle \in \mathfrak{N}_k$ for all $x \in N$, and so for each $i \geq 1$ we get $[\gamma_i(N), {}_kG] \leq [\gamma_i(N), {}_k\langle t\rangle]\gamma_{i+1}(N) \leq \gamma_{i+1}(N)$. Hence $[N, {}_{ks}G] = 1$ and the result follows. □

If G satisfies $(*)$, then of course G/G' is finite, and Lemma 4 then implies that G is residually a finite π-group for some finite set π of primes. Thus $\bigcap_{p\in\pi} R_p = 1$, where R_p is the finite p-residual of G. Accordingly, it is only the case where G is residually

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finite-p that needs to be considered. Our final prerequisite requires the notion of a powerful p-group (for definitions and essential properties the reader is referred to [4, Chapter 2].

LEMMA 5. *Let G be a finite p-group that satisfies $(*)$. Then there is an integer d depending only on k and p, such that $(N^d)'$ is a powerful subgroup of N.*

PROOF. Let A be a normal subgroup of G contained in N and let $a \in A, x \in N$ and write $z = xt$. Since $\langle a, z\rangle \in \mathfrak{N}_k$, $\langle a\rangle^{\langle z\rangle} = \langle a^{z^i}; 0 \le i \le k-1\rangle$ (much as in the proof of Lemma 1). Similarly $\langle a\rangle^{\langle t\rangle} = \langle a^{t^i}; 0 \le i \le k-1\rangle$. Also, since $\langle z, t\rangle \in \mathfrak{N}_k$ there is an integer $c = c(k)$ such that every element of $\langle z, t\rangle$ has the form $t^{u_1}z^{v_1}\cdots t^{u_c}z^{v_c}$, for integers u_i, v_i. Thus

$$\begin{aligned}\left\langle a^{\langle x\rangle}\right\rangle \le \left\langle a^{\langle z,x\rangle}\right\rangle &= \left\langle a^{t^{u_1}z^{v_1}\cdots t^{u_c}z^{v_c}}; u_i, v_i \in \mathbb{Z}\right\rangle \\ &= \left\langle (a^{t^{u_1}})^{z^{v_1}\cdots t^{u_c}z^{v_c}}; 0 \le u_1 < k; v_1, u_2, \ldots, u_c, v_c \in \mathbb{Z}\right\rangle \\ &= \left\langle (a^{t^{u_1}z^{v_1}})^{t^{u_2}z^{v_2}\cdots t^{u_c}z^{v_c}}; 0 \le u_1, v_1 < k; u_2, v_2, \ldots, u_c, v_c \in \mathbb{Z}\right\rangle.\end{aligned}$$

Inductively, we see that this is

$$= \left\langle a^{t^{u_1}z^{v_1}t^{u_2}z^{v_2}\cdots t^{u_c}z^{v_c}}; 0 \le u_1, v_1, u_2, v_2, \ldots, u_c, v_c < k\right\rangle.$$

Define M to be A^pA' if p is odd, A^4A' if $p = 2$. Then modulo M, $\langle a\rangle^{\langle z,x\rangle}$ has order bounded by some function of p and k. It follows that $[a, x^d] \in M$ for some $d = d(p, k)$. But x and a were arbitrary, and we deduce that $[A, N^d] \le M$.

With $J = N^d$ and $J' = A$ we obtain $\gamma_3(J) = [J', J] \le (J')^p\gamma_4(J)$ (respectively $(J')^4\gamma_4(J)$). Since J is nilpotent it follows that $[J', J] \le (J')^p$ (respectively $(J')^4$). Thus J' is powerfully embedded in J and is therefore powerful and the lemma is proved. □

PROOF OF PROPOSITION 1. As we have seen, we may assume that G satisfies $(*)$ and that G is residually a finite p-group for some prime p. Let d be as in Lemma 5, so that $(N^d)'$ is powerful in every finite quotient of G. Let J denote the intersection of all subgroups N^dK_λ, where K_λ is normal in N and N/K_λ is a finite p-group. By $(*)$, N has finite index in G and is therefore finitely generated. So N/J is a finitely generated residually finite group of exponent at most d and hence finite, by [12] and [13]. It follows that G/J' is finite.

Write $P_1 = J'$, and for each positive integer i let $P_{i+1} = (P_i)^p[P_i, J']$. By [4, Theorem 2.9] the rank of J'/P_i is precisely $d(J')$, the minimum number of elements required to generate J'. With the notation of [4, Definition 6.2], the system (P_i) therefore has finite rank, and [4, Theorem 6.3] then implies that G is a linear group. Certainly G does not contain a non-abelian free group, since such groups do not belong to $N(2, k)^*$, and so G is soluble-by-finite [9]. Therefore G is soluble by Lemma 4, and the result follows from [7, Theorem A]. □

PROOF OF THEOREM 1. Let G be as stated and let R denote the finite residual of G. By the Proposition there is a finite normal subgroup U/R of G/R with G/U nilpotent. By Lemma 1 above and repeated application of [6, Lemma 3], U is finitely generated and hence R is finitely generated. If $R = 1$ then of course we are done, so assume R is nontrivial, so that R has a proper normal subgroup S of finite index. Indeed, we may choose S to be normal in G, and then G/S is finite-by-nilpotent and therefore residually finite, contradicting the definition of R. The result follows. □

References

[1] A. Abdollahi and B. Taeri, 'A condition on finitely generated soluble groups', *Comm. Algebra*, to appear.

[2] C. Delizia, 'Finitely generated soluble groups with a condition on infinite subsets', *Istit. Lombardo Accad. Sci. Lett. Rend. A* **128** (1994), 201–208.

[3] ———, 'On certain residually finite groups', *Comm. Algebra* **24** (1996), 3531–3535.

[4] J. D. Dixon, M. P. F. du Sautoy, A. Mann and D. Segal, *Analytic pro-p groups*, London Math. Soc. Lecture Note Series 157 (Cambridge Univ. Press, Cambridge, 1991).

[5] P. Hall, 'Finite-by-nilpotent groups', *Proc. Cambridge Philos. Soc.* **52** (1956), 611–616.

[6] Y. K. Kim and A. H. Rhemtulla, 'Weak maximality condition and polycyclic groups', *Proc. Amer. Math. Soc.* **123** (1995), 711–714.

[7] J. C. Lennox and J. Wiegold, 'Extensions of a problem of Paul Erdös on groups', *J. Austral. Math. Soc.* **31** (1981), 459–463.

[8] M. F. Newman, 'Some varieties of groups', *J. Austral. Math. Soc.* **16** (1973), 481–494.

[9] J. Tits, 'Free subgroups in linear groups', *J. Algebra* **20** (1972), 250–270.

[10] J. S. Wilson, 'Two-generator conditions for residually finite groups', *Bull. London Math. Soc.* **23** (1991), 239–248.

[11] E. I. Zel'manov, 'On some problems of group theory and Lie algebras', *Mat. Sb.* **180** (1989), 159–167. English translation: *Math. USSR-Sb.* **66** (1990), 159–168.

[12] ———, 'Solution of the restricted Burnside problem for groups of odd exponent', *Izv. Akad. Nauk SSSR Ser. Mat.* **54** (1990), 42–59. English translation: *Math. USSR-Izv.* **36** (1991), 41–60.

[13] ———, 'Solution of the restricted Burnside problem for 2-groups', *Mat. Sb.* **182** (1991), 568–592.

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