BOOK REVIEWS - COMPTES RENDUS CRITIQUES

Lectures on Calculus, EDITED BY K. O. MAY. Holden-Day, San Francisco (1967). vi + 180 pp.

The Lectures are selected from the programme of visiting lecturers sponsored by the Mathematical Association of America. Stated briefly the purpose of these Lectures is to encourage contact between mathematicians who are active in scholarship and research, who are, for the most part, in universities which emphasize graduate work, and staff members and students of the smaller universities and colleges. While not specifically stated, an important purpose of the programme should be, and without doubt is, that of influencing instructors in all universities and colleges to take a second and hard look at the content of their courses and their methods of instruction, especially in first and second level courses.

The first Lecture is *Formal logic as a tool for instruction* by Arthur H. Copeland Sr. The postulates and theorems are those of Boolian algebra. The notations of logic are adequately explained and used to describe continuity and uniform continuity of a function of a single real variable. The discussion is helpful to those of us, of whom there are many, who have no formal training in the notations and methods of logic. It would be even more helpful had a formal proof that a function continuous on a closed interval is uniformly continuous on the interval been carried through to completion.

Elementary functions by Julian D. Mancill treats in an interesting way the exponential and logarithmic functions. It is assumed that there exists a function $E(x)$ for which Postulate 1: $E(x).E(y)=E(x+y)$ and Postulate 2: $\lim_{x\to 0+} \times [E(x)-1]/x=1$ are true. It is then shown that the function $E(x)$ is unique and $E(x)=1+x+(x^2/2!)+(x^3/3!)+\cdots$ which identifies $E(x)$ with e^x. For $a>0$ it is shown that $a=e^k$ has an unique solution k and a^x is defined to be $a^x=e^{kx}$, and all the properties of the exponential function $y=a^x$ follow from this definition, including

$$\frac{dy}{dx} = ke^{kx} = ka^x = ky.$$

Then

$$dx = \frac{1}{k}\frac{dy}{y}, \qquad x = \frac{1}{k}\int_1^y \frac{dy}{y}.$$

This leads to the definitions

$$\log_a y = \frac{1}{k}\int_1^y \frac{dy}{y}, \quad \ln y = \int_1^y \frac{dy}{y}.$$

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All the properties of the logarithmic function then follow.

This approach to the exponential and logarithmic functions could well replace the usual approaches found in calculus texts. The standard formulas for sin $(x-y)$ and cos $(x-y)$ together with sin $x/x \to 1$ as $x \to 0+$ are taken as postulates and properties of the trigonometric functions follow. The lecture is an interesting and concise treatment of the topics considered, and would fit neatly into first or second year Calculus courses.

The lecture *Area and volumes without limit processes* by Donald E. Richmond is novel and interesting. Let $f(x)$ be strictly increasing and nonnegative. Let $F(x)$ be a function which is such that $F(a)=0$ and for x, x' on $[a, b]$ with $x<x'$

$$(1) \qquad (x'-x)f(x) < F(x')-F(x) < (x'-x)F(x').$$

It is proved that if such a function exists it is unique. Then $F(x)$ is defined to be the area under the graph of $y=f(x)$ and over the interval $[a, x]$. If f is strictly decreasing a similar treatment leads to (1) with the inequality signs reversed.

It is shown that for the parabola over the interval $[0, a]$ the function $x^3/3$ qualifies as a function $F(x)$ which satisfies (1). Also the existence of functions $F(x)$ which satisfy (1) are established for polynomials, for sin x on $(0, \pi/2)$ and for $1/x$ on the interval $[1, x]$, and this without recourse to limit processes. There seems to be no general proof that a function $F(x)$ satisfying (1) exists for every strictly increasing function $f(x)$ with $f(x)\geq 0$. Can the existence of such a function be established without recourse to limiting processes?

Area and integration by Hans Sagen starts with the function $f(x)$ or $[0, 1]$ defined as follows:

$$f(x) = \left.\begin{matrix}0\\1\end{matrix}\right\} \text{ for } \frac{1}{n} < x < \frac{1}{n-1} \text{ if } \left.\begin{matrix}\\\end{matrix}\right\} \begin{matrix}n \text{ is odd}\\ n \text{ is even}\end{matrix}.$$

It is shown that the Riemann integral of $f(x)$ exists, and that the discontinuities form a denumerable set. At points of discontinuity the jump is unity. The function $g(x)$ is then defined on $[0, 1]$ as follows:

$$g(x) = \left\{\begin{matrix}1 \text{ for } x \text{ irrational}\\ 0 \text{ for } x \text{ rational}\end{matrix}\right\} \quad 0 \leq x \leq 1.$$

Since at each rational number $g(x)$ jumps from zero to unity and back to zero it appears that $g(x)$ is discontinuous only at the set of rationals which is denumerable. The fact is, however, that $g(x)$ is discontinuous at all x on $[0, 1]$, which is not a denumerable set. It is then shown that the Riemann integral of $g(x)$ over $[0, 1]$ does not exist.

The next step is that of lining up the rationals on $[0, 1]$ in an ordered set $\{r_1, r_2, \ldots\}$, and defining $g_n(x)$ on $[0, 1]$ as follows:

$$g_n(n) = \left\{\begin{matrix}0 \text{ on the set } \{r_1, r_2, \ldots, r_n\}\\ 1 \text{ on the set } [0, 1]-\{r_1, r_2, \ldots, r_n\}\end{matrix}\right\}$$

Then

$$\lim_{n\to\infty} g_n(x) = g(x), \qquad \int_0^1 g_n(x)dx = 1$$

and the integral is a Riemann integral. Now the integral of $g(x)$ over [0, 1] is defined by the relation

$$\int_0^1 g(x)\,dx = \lim_{n\to\infty} \int_0^1 g_n(x)dx.$$

It is proved in the lecture that the integral on the left does not exist as a Riemann integral. It is, however, the Lebesque integral of $g(x)$. It is pointed out that this is a special case of the general definition of the Lebesque integral as the limit of a sequence of Riemann integrals of step functions.

In the reviewer's opinion this lecture exhibits examples which should become a part of an introductory course in real analysis.

Geometrical applications of integral calculus by Henrie W. Cuggenheimer illustrates the use of the calculus in the study of smooth, closed convex curves (ovals). The results of this lecture are interesting, and surprising too, to one who is not familiar with them. By the use of the integral Calculus such results as the following are obtained.

The radius of curvature of an oval has at least two relative maxima and two relative minima on the oval.

An oval has at least four circles of curvature whose perimeters equal the perimeter of the curve.

There are many other similar results. It is probably too special a topic to become a part of any usual undergraduate course. There is a fairly extensive bibliography. This together with the interesting results, twenty-five in all, and the methods of obtaining them, makes the lecture a promising starting point for an honours thesis or an M.A. thesis.

To the reviewer the most interesting lecture of the series is *Additive functions* by Albert Wilanski. A function f with domain and range in the set of reals is additive if $f(x+y)=f(x)+f(y)$. The function $f(x)=kx$ is additive. It is also linear. Are there nonlinear functions which are additive? Not if they are from the class of continuous functions. Theorem 1 says that if f is continuous and additive it is linear. Hence if f is nonlinear and additive it must be discontinuous. How badly can it be discontinuous? Theorems 4 and 14 say:

If f is nonlinear and additive then the graph of $f(x)$, the set $S=[x, f(x)]$, is dense in the plane; in every circle no matter how near zero is its radius, there are points of the set S. If f is additive and bounded on a set of positive measure it is linear. A measurable function is linear.

A full appreciation of the content of this lecture requires considerable maturity in real analysis. This probably excludes it from serious consideration in any undergraduate course. Nevertheless, its content, together with its extensive bibliography, makes it, like the preceding lecture, a useful starting point for honours and master's theses.

The lecture, *Manipulations with differentials made respectable* by M. Evans Munroe is an important contribution to what has been, and still is, a controversial topic of the calculus relative to how and where it should be taught. It is the reviewer's opinion that this lecture by Professor Munroe should be given more consideration than the space allotted for a review of *Lectures on calculus* allows! A separate review is being prepared, which will also include *Exterior differential calculus and Maxwell's equations* by Oswald Wyler.

There remains for consideration *Continuous square roots of mappings* by M. K. Fort, Jr.

C_0 is the set of all non-zero complex numbers. If f and φ are complex valued functions which have the same domain D then φ is a continuous square root of f if φ is a mapping (i.e. is continuous) and $\varphi^2=f$, $z \in D$.

THEOREM 1. *If f is a mapping whose domain is a closed disk D having center at* 0, *the origin, and whose range is contained in C_0, then f has a continuous square root.*

THEOREM 2. *If f is a mapping whose domain is a circle S with center* 0, *the origin whose range is contained in C_0, and which satisfies $f(-z)=-f(z)$ for all $z \in S$, then f does not have a continuous square root.*

THEOREM 3. *Let D be a closed disk with center at* 0 *and let S be the boundary of D. If f is a mapping of D into the complex plane such that* (A) *$f(-z)=-f(z)$ for all $z \in S$, or* (B) *there is an integer $n \neq 0$ such that $f(z)=z^n$ for all $z \in S$, then there exists $q \in D$ such that $f(q)=0$.*

These theorems lead up to a proof of the Fundamental Theorem of Algebra and to the Brower Fixed Point theorem for a disk. If f is a mapping whose domain is a closed disk D and whose range is contained in D, then f has a fixed point in D.

The proofs of these theorems are neat, and have no serious subtleties. They will be included in the course in complex variable theory given by the reviewer. It is not going too far to say they should be given consideration in any first course in complex variable theory.

Finally, a few general remarks about *Lectures on calculus*. The editor is to be congratulated on the selection of topics and the excellence of format. These lectures will be studied by many instructors and students in many universities and colleges. The lectures provide solid backing for a continuation of the programme of visiting lecturers. We all look forward to the appearance of further issues.

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