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LOCALIZATION AND COMPLETION AT PRIMES GENERATED BY NORMALIZING SEQUENCES IN RIGHT NOETHERIAN RINGS

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1. Introduction. If P is a right localizable prime ideal in a right Noetherian ring R, it is known that the ring R_P is right Noetherian, that its Jacobson radical is the only maximal ideal, and that $R_P/J(R_P)$ is simple Artinian: in short it has several properties of the commutative local rings.

In the present work we examine the properties of R_P under the additional assumption that P is generated by, or is a minimal prime above, a normalizing sequence. It is shown that in such cases $J(R_P)$ satisfies the AR-property (i.e., P is classical) and that the rank of P coincides with the Krull dimension of R_P. The length of the normalizing sequence is shown to be an upper bound for the rank of P, and if P is generated by a normalizing sequence $x_1, x_2, \ldots, x_n$ then the rank of P equals n if and only if the P-closures of the ideals I_j generated by $x_1, x_2, \ldots, x_j$ $(j = 0, 1, \ldots, n)$, are all distinct primes. In such cases R_P is a prime ring of right global dimension n, and the images of $x_1, x_2, \ldots, x_n$ in R_P behave like the members of an R-sequence.

In § 6 we study the $J(R_P)$-adic completion of R_P. More generally, if I is an ideal generated by a normalizing sequence in a right Noetherian ring S, it is shown that the I-adic completion $\hat{S}$ of S is right Noetherian provided I has the right AR-property and satisfies a certain technical condition.

This condition is satisfied if I is generated by a centralizing sequence of generators, and is equivalent to I having the left AR-property if S is also left Noetherian. The condition is also satisfied when $S = R_P$, where P is a localizable prime minimal over an ideal generated by a normalizing sequence, in a right Noetherian ring R, so in this case $\hat{R}_P$ is right Noetherian.

In the final section we look at the question of whether $\hat{R}_P$ has a Morita duality. The key result (Theorem 7.4) asserts that if S is a right Noetherian semilocal ring whose Jacobson radical is generated by a normalizing sequence then the injective hulls of simple right S-modules are Artinian. If in addition J satisfies the technical condition referred to above (so that $\hat{S}$ is right Noetherian), then the injective hull U_S of S/J induces a Morita duality between $\hat{S}$ and $T = \mathrm{End}\ (U_S)$. Also T is left Noetherian and ${}_TU$ is Artinian.

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In the case where $S = R_P$, where P is a right localizable prime generated by a normalizing sequence, we conclude that the injective hull E of $(R/P)_+$ is Arthinian as a module over R_P, over $\hat{R}_P$, and over $T = \text{End}(E_R)$, and that E induces a duality between $\hat{R}_P$ and T.

2. Background. All rings discussed will have a unity and will be right (but not necessarily left) Noetherian, and all modules will be unital right modules, unless otherwise specified. If P is a prime ideal of R, one can form the hereditary torsion theory τ_P for which the torsion-free modules are those embeddable in a product of copies of the R-module injective hull of R/P ([**6**], [**10**], [**11**]).

If R satisfies the right Ore condition with respect to $\mathscr{C}(P)$, we say that P is *right localizable*. In this case, we can write each element q of the right of quotients R_P in the form $q = \bar{a}\bar{c}^{-1}$ where a is in R, c is in $\mathscr{C}(P)$, and $^-$ denotes coset modulo the torsion submodule $\tau_P(R)$. In such cases the homomorphism $\phi: R \rightarrow R_P$ can be decomposed as $R \twoheadrightarrow \bar{R} \rightarrowtail R_P$, where the first is injective, the second surjective. If I is a right ideal of R, $\bar{I}R_P$ can be identified as $\{\bar{u}\bar{c}^{-1}: u \in I \text{ and } c \in \mathscr{C}(P)\}$ and is a right ideal of R_P. If I is a two-sided ideal of R, then $\bar{I}R_P$ is known to be a two-sided ideal of R_P. In particular, $\bar{P}R_P$ is a maximal ideal in R_P: it is the Jacobson radical of R_P, and $R_P/\bar{P}R_P$ is Artinian.

For any right ideal I of R, the *closure* of I, denoted $\text{cl}_P(I)$, is $\{r \in R: rc \in I \text{ for some } c \text{ in } \mathscr{C}(P)\}$. It is identifiable either as $\phi^{-1}(\bar{I}R_P)$ or as the inverse image of $\tau_P(R/I)$ in the natural homomorphism $R \rightarrow R/I$ of R-modules. There is a one-to-one correspondence between the right ideals of R_P and the P-closed right ideals of R, and under this correspondence two-sided primes in R_P are in one-to-one correspondence with prime ideals of R contained in P. (Any prime ideal of R contained in P is necessarily $P-$closed.) Hence, $\text{rank}(\bar{P}R_P) = \text{rank}(P)$.

An ideal I of a ring R is said to have the *right* AR-*property* if for any right ideal E of R there is a natural number n such that $E \cap I^n \subseteq EI$. If P is a localizable prime of R and, in addition, $J(R_P)$ has the right AR-property, P is said to be a *classical* prime. (It is a consequence of this that $\bigcap_1^\infty J(R_P)^n$ must necessarily be zero if R is Noetherian.) The concepts of localizable and classical prime (or semiprime) ideals were discussed, for example, in [**11**].

An element x in a ring R is said to be *normal* (resp. *central*) if $xR = Rx$ (resp. $xa = ax$ for all a in R), and a sequence $x_1, x_2, \ldots, x_n$ is *normalizing* (resp. *centralizing*) if x_1 is normal (resp. central) in R and the coset $x_{j+1} + I_j$ is normal (resp. central) in R/I_j for $j = 0, 1, \ldots, n-1$, where $I_0 = \{0\}$ and $I_j = \sum_{i=1}^j x_iR$. Ideals which are generated by sequences which are normal, normalizing, and centralizing were discussed in [**13**] and [**21**].

In what follows P will always be a localizable prime in a right Noetherian ring unless otherwise specified. For I a right ideal of R and u in R, $u^{-1}I$ will denote $\{r \in R: ur \in I\}$, and if T is a left ideal, Tu^{-1} is $\{r \in R: ru \in T\}$. $J(R)$ always denotes the Jacobson radical of R, and $\kappa(R)$ is the Krull dimension of R, as discussed in [**4**]. The notation $I \triangleleft R$ means I is a two-sided ideal of R, and $\mathscr{C}(P)$ denotes $\{c \in R: ca \in P \Rightarrow a \in P\}$. For an element a in R, $\bar{a}$ will denote the coset $a + \tau_P(R)$, either as an element of $R/\tau_P(R)$ or of R_P. As is standard, $r(a)$ and $l(a)$ denote the right and left annihilators of a respectively.

3. Passing from R to R_P. Throughout this section we assume R is a right Noetherian ring and P is a right-localizable prime of R. We show that if P is minimal among the primes containing a (fixed) ideal generated by a normalizing sequence, then P is classical, and that normalizing sequences in R pass to normalizing sequences in R_P.

The first lemma reduces in the case $I = 0$ to the observation [**11**] that a right-Ore set in a right Noetherian ring is right reversible. Since the proof in the more general setting is almost identical, it will be omitted.

LEMMA 3.1. *Let $I \triangleleft R$ and $cr \in I$ where $c \in \mathscr{C}(P)$. Then $rd \in I$ for some $d \in \mathscr{C}(P)$.*

Note that this result can be stated as $r^{-1}I \cap \mathscr{C}(P) = \emptyset$ implies $Ir^{-1} \cap \mathscr{C}(P) = \emptyset$.

In [**7**] Proposition 2.11 it was shown that if x is a normal element in a prime right Noetherian ring R, then $xP = Px$ (assuming P is localizable). The next result generalizes that observation. Most of the lemma could be deduced from § 4 of [**2**], and also from the results of [**23**].

LEMMA 3.2. *Suppose I is an ideal of R contained in P, and x an element of R which is normal modulo I. If $x^{-1}I \subseteq P$ then* i) *$Px + I = xP + I$, and* ii) *if $xc - dx \in I$ for elements c and d of R, then d is in $\mathscr{C}(P)$ if and only if c is in $\mathscr{C}(P)$.*

Proof. Since R/P is prime and right Noetherian, any two-sided ideal of R is either contained in P or meets $\mathscr{C}(P)$. In particular, $x^{-1}I$ and Ix^{-1} are two-sided ideals. The assumption $x^{-1}I \subseteq P$ guarantees by Lemma 3.1 that $Ix^{-1} \subseteq P$ also.

i) Set $T = (xP + I)x^{-1}$, an ideal of R. If T were not contained in P, T would meet $\mathscr{C}(P)$ so by Lemma 3.1, $x^{-1}(xP + I)$ would meet $\mathscr{C}(P)$, say at d. Then $xd = xp + u$, where $p \in P$ and $u \in I$, so $d - p \in x^{-1}I \cap \mathscr{C}(P)$, contradicting $x^{-1}I \subseteq P$. Therefore $T \subseteq P$ and so $Tx + I \subseteq Px + I$. However x being normal modulo I implies $Tx + I = xP + I$, hence $xP + I \subseteq Px + I$.

Define $K_0 = P$ and $K_{i+1} = x^{-1}(K_ix + I)$ for $i = 0, 1, 2 \ldots$. Each $K_i \triangleleft R$, and $xK_{i+1} + I = K_ix + I$. Since $xP \subseteq Px + I$ we have

$K_0 \subseteq K_1$ and, by induction, $K_i \subseteq K_{i+1}$ for each i. R is Noetherian so we eventually get $K_{n+1} = K_n$. Then

$$K_{n-1}x + I = xK_n + I = xK_{n+1} + I = K_nx + I$$

so

$$K_n \subseteq K_{n-1} + Ix^{-1} \subseteq K_{n-1} + K_0 = K_{n-1}.$$

We conclude $K_n = K_{n-1}$, and by induction that $K_1 = K_0 = P$, so $xp + I = Px + I$.

ii) Suppose $xc - dx \in I$. We show that $c \in \mathscr{C}(P)$ implies $d \in \mathscr{C}(P)$: the proof of the converse is similar and will be omitted. Suppose then that $c \in \mathscr{C}(P)$, and that $da \in P$ for some a in R. We can write $ax = xr + u$, where $u \in I$ and $r \in R$. Then

$$xcr + I = dxr + I = dax + I \subseteq Px + I = xP + I,$$

so cr is in $P + x^{-1}I = P$ and hence $r \in P$. Then $ax \in Px + I$ and so a is in $P + Ix^{-1} = P$, as desired.

PROPOSITION 3.3. *If I is a two-sided ideal of R, and x in R is norm a modulo I, then $\bar{x}$ in R_P is normal modulo $\bar{I}R_P$.*

Proof. If I is not contained in P, then $I \cap \mathscr{C}(P) \neq \emptyset$, and (since elements of $\mathscr{C}(P)$ become units in R_P), $\bar{I}R_P = R_P$ so there is nothing to prove. Thus we may assume $I \subseteq P$. Furthermore, if $x^{-1}I \not\subseteq P$, then $x^{-1}I \cap \mathscr{C}(P)$ contains an element c. Then $\bar{x} = \bar{x}\bar{c}\,\bar{c}^{-1}$ is in $\bar{I}R_P$, and the result is trivial.

Thus we may assume $I \subseteq P$ and $x^{-1}I \subseteq P$. First of all, $Rx + I = xR + I$ is a two-sided ideal of R. Now

$$R_P\overline{(xR + I)} \subseteq \overline{(xR + I)}R_P = \bar{x}R_P + \bar{I}R_P,$$

so

$$R_P\bar{x} + \bar{I}R_P \subseteq \bar{x}R_P + \bar{I}R_P.$$

To show $\bar{x}R_P \subseteq R_P\bar{x} + \bar{I}R_P$, consider an element $\bar{x}q$ where q is in R_P. We may write $q = \bar{a}\,\bar{c}^{-1}$ for some a in R and c in $\mathscr{C}(P)$, and since x is normal modulo I we have $xa = rx + u$ and $xc = dx + v$ for some r, d in R and some u, v in I. By Lemma 3.2 (ii), d is in $\mathscr{C}(P)$. Now, (dropping the $^-$ for ease of notation)

$$\begin{aligned} xq &= xac^{-1} = (rx + u)c^{-1} = rd^{-1}(dx)c^{-1} + uc^{-1} \\ &= rd^{-1}(xc - v)c^{-1} + uc^{-1} = rd^{-1}x + (\text{elements of } R_PIR_P). \end{aligned}$$

Therefore $\bar{x}R_P \subseteq R_P\bar{x} + \bar{I}R_P$, as desired.

COROLLARY 3.4. *If I is an ideal of R generated by the normalizing sequence $x_1, x_2, \ldots, x_n$, then $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ is a normalizing sequence in R_P which generates $\bar{I}R_P$.*

Proof. Apply the obvious induction argument.

PROPOSITION 3.5. *Let $x_1, x_2, \ldots, x_n$ be a normalizing sequence in a right Noetherian ring S, and assume that* i) *$S/J(S)$ is Artinian, and* ii) *$J(S)$ is the only prime which contains $I_n = x_1S + x_2S + \ldots + x_nS$. Then $J(S)$ has the right* AR-*property.*

Proof. According to Lemma 8 of [**13**] it is sufficient to show that if N is a Noetherian S-module with an essential submodule M such that $MJ^s = 0$ for some s, then $NJ^t = 0$ for some t.

Since x_1 is normal, $\operatorname{Ann}_N(x_1^i)$ is a submodule of N and since N is Noetherian, there is a natural number t such that $\operatorname{Ann}_N(x_1^t) = \operatorname{Ann}_N(x_1^{t+j})$ for all $j \geqq 0$. It follows easily that Nx_1^t (which is a submodule) satisfies

$$Nx_1^t \cap \operatorname{Ann}_N(x_1^t) = 0.$$

However $x_1 \in J$ and $MJ^s = 0$ so

$$M \subseteq \operatorname{Ann}_N(x_1^s) \subseteq \operatorname{Ann}_N(x_1^{t+s}) = \operatorname{Ann}_N(x_1^t).$$

As M is essential in N, Nx_1^t is forced to be zero.

The proof now proceeds by induction on n. If $n = 1$ then $J = J(S)$ is the only prime over x_1R. As S is Noetherian, $J^k \subseteq x_1S$ for some k so $NJ^{kt} \subseteq Nx_1^tS = 0$, as desired.

The proof of the induction step follows exactly as in the proof of Lemma B of [**13**] provided we can show that $xJ = Jx$ for any normal element x. Now $xJ = Tx$ for some ideal T (namely $(xJ)x^{-1}$). If $T \not\subseteq J$ then, since J is the only maximal ideal, $T = S$ and $xJ = Sx = xS$. By Nakayma's Lemma this would force $x = 0$, in which case $xJ = Jx = 0$. If $x \neq 0$ then $T \subseteq J$ so $xJ = Tx \subseteq Jx$. Similarly $Jx = xT'$ for some ideal T' and if $T' \not\subseteq J$ then $Jx = xS = Sx$, again forcing $x = 0$. Thus either $x = 0$ or $xJ \subseteq Jx \subseteq xJ$: in either case $xJ = Jx$.

THEOREM 3.6. *Let $x_1, x_2, \ldots, x_n$ be a normalizing sequence in a right Noetherian ring R, and suppose P is a prime, minimal over $I = x_1R + x_2R + \ldots + x_nR$. If P is localizable then P is classical.*

Proof. By Corollary 3.4, $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ is a normalizing sequence in R_P. In view of the one-to-one order-preserving correspondence between primes in R contained in P and primes of R_P, the Jacobson radical $J(R_P)$ is minimal over $\bar{I}R_P$. Proposition 3.5 is applicable to R_P so $J(R_P)$ has the AR-property.

Remark. It is necessary to assume *a priori* that P is localizable in Theorem 3.6. For example, as is pointed out in [**17**], if we take R to be the enveloping algebra of the two-dimensional Lie Algebra (over a field of characteristic zero) with basis x, y where $[x, y] = x$, then the prime P

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generated by the normalizing sequence x, y is not localizable. However, if P is generated by a normalizing sequence and $P^2 = 0$ then P is localizable ([**2**], Theorem 4.5) and hence classical by Theorem 3.6. Moreover, if R is right Noetherian and P is generated by a centralizing, or by a normal, sequence, then P is classical; for in these cases P has the AR-property [**13**] and from this it follows that P is localizable and therefore classical. (See Proposition 2.1 of [**21**] or Theorem 4.6 of [**2**].)

4. Properties of R_P. We are interested in studying the properties of R_P in the case where P is a localizable prime minimal over an ideal generated by a normalizing sequence. We know now that R_P is right Noetherian, that $J(R_P) = \bar{P}R_P$ is minimal over a normalizing sequence in R_P, and that $J(R_P)$ has the AR-property.

One aspect we wish to study is the (right) Krull dimension $\kappa(R_P)$. Since P is localizable, the submodule lattice of R_P is isomorphic to the lattice of submodules of R which are closed relative to the torsion theory determined by P, so $\kappa(R_P)$ is the same as $\kappa_P(R)$ in the notation of [**7**]. From [**4**] we have that $\operatorname{rank}(P) = \operatorname{rank}(\bar{P}R_P) \leqq \kappa(R_P)$ and it follows from [**7**], Corollary 3.3, that $\kappa(R_P)$ is finite. In our present situation, we will show that $\operatorname{rank}(P)$ and $\kappa(R_P)$ are equal.

The assumptions on R and P are the same as in the last section. S will denote a right Noetherian ring: generally we will think of S as being the ring R_P.

The first result is essentially due to Walker [**23**].

PROPOSITION 4.1. *Let $x_1, x_2, \ldots, x_n$ be a normalizing sequence of elements of $J(S)$. Then $\kappa(S) \leqq n + \kappa(S/I)$ where $I = x_1S + x_2S + \ldots + x_nS$.*

Proof. Once the result for $n = 1$ is shown, the result follows easily by induction. To prove the case $n = 1$, one proceeds exactly as in Application 2 of [**19**] where the result is proved in the case where x_1 is central instead of normal. The only difficulty arises in showing that each right ideal of S is closed in the topology defined by the filtration $S \supseteq x_1S \supseteq x_1^2S \supseteq \ldots$ and this is done in the first part of the proof of Theorem 1.8 of [**23**].

THEOREM 4.2. *Let S be a right Noetherian ring for which $J = J(S)$ satisfies* i) *S/J is simple Artinian, and* ii) *J is a minimal prime above a finite normalizing sequence $x_1, x_2, \ldots, x_n$. Then $\kappa(S) = \operatorname{rank}(J)$, and this is bounded above by n.*

Proof. Let $I = x_1S + \ldots + x_nS$, an ideal of S. Since J is the only prime over I, $J^t \subseteq I$ for some t, whence S/I is right Artinian and $\kappa(S/I) = 0$. By Proposition 4.1, $\kappa(S) \leqq n$, so $k = \kappa(S)$ is finite. We show that $\operatorname{rank}(J)$ also equals k by induction on k. The case $k = 0$ (S is Artinian) is trivial, so assume the result is true for rings S' for which $k' = \kappa(S')$ is less than k. By [**4**] there is a prime ideal Q of S such that $\kappa(S) = \kappa(S/Q)$, and without loss of generality we may pass to S/Q and thus assume S is a prime ring. Then J contains a non-zero normal element x which must have zero right annihilator. As is well-known, this forces $\kappa(S/xS) < \kappa(S)$ which, together with 4.1, gives $(S/xS) = k - 1$. By our induction assumption, rank $(J/xS) = k - 1$ and (putting the zero ideal below a longest possible chain of primes between xS and J) we have $k \leqq \text{rank}(J)$. But $\text{rank}(J) \leqq \kappa(S)$ (see [**4**]), and we are done.

COROLLARY 4.3. *Let R be right Noetherian and P a localizable prime minimal among the primes containing a normalizing sequence of length n. Then $\kappa(R_P) = \text{rank}(P) \leqq n$.*

Proof. Apply the theorem and recall that $\text{rank}(P) = \text{rank}(\bar{P}R_P)$.

Suppose S is as in Theorem 4.2 and that J is minimal over the normalizing sequence $x_1, x_2, \ldots, x_n$. The next result shows that there is a maximal chain of primes related to the ideals $I_0, I_1, \ldots, I_n$ where $I_0 = 0$ and $I_j = \sum_{i=1}^{j} x_i S$ for $j = 1, 2, \ldots, n$.

PROPOSITION 4.3. *Let S be right Noetherian and suppose $J = J(S)$ satisfies* i) *S/J is simple Artinian, and* ii) *J is a minimal prime over the normalizing sequence $x_1, x_2, \ldots, x_n$. If* $\text{rank}(J) = k$ *then there exists a chain of distinct primes $Q_0 \subset Q_1 \subset \ldots Q_k = J$ and a sequence of integers $0 \leqq n_0 < n_1 < \ldots < n_k = n$ such for each j, $I_{n_j} \subseteq Q_j$ but $x_{n_j+1} \notin Q_j$.*

Proof. We proceed by induction on k. If $k = 0$, take then $n_0 = n$ and $Q_0 = J$. Assume then that $k > 0$, and that the result is known whenever the Krull dimension is less than k. Since $\kappa(S) = \kappa(S/I_0) = k$ and $\kappa(S/I_n) = \kappa(S/J) = 0$, there must be an integer n_0 such that $\kappa(S/I_{n_0}) = k$ but $\kappa(S/I_{n_0+1}) < k$. From [**4**] we know there is a prime Q_0 containing I_{n_0} such that $\kappa(S/Q_0) = k$. Clearly $x_{n_0+1} \notin Q_0$. Then $x' = x_{n_0+1} + Q_0$ is a normal element in the prime ring $S' = S/Q_0$, so (as in the previous theorem)

$$\kappa(S') = 1 + \kappa(S'/x'S'),$$

whence $\kappa(S'/x'S') = k - 1$. The Jacobson radical of $S'/x'S'$ is minimal above a normalizing sequence of length $n - n_0 - 1$ (the images of the x_i's for $i \geqq n_0 + 2$). By induction there is a chain of primes $Q_0'' \subset Q_1'' \subset \ldots Q_{k-1}''$ in $S'/x'S'$, each Q_i'' containing the appropriate members from the image in $S'/x'S'$ of this "tail" of the x-sequence. We pull each Q_i'' back to Q_{i+1} in S to obtain primes $Q_1 \subset Q_2 \subset \ldots Q_k$, all containing Q_0. It is routine to verify that these Q_i's contain the required members of the x-sequence.

Remark. The Q_0 that was constructed above is necessarily minimal over I_{n_0}, and Q_1 can be seen to be minimal over I_{n_1}. Also Q_n $(=J)$ is minimal over I_{n_k} $(=I_n)$. However, it is not clear whether the other Q_i's are necessarily minimal over the I_{n_i}'s. As the proof of the next result shows, this would follow easily if we knew that $\kappa(S/I_{n_j}) = k - j = 0, 1, \ldots, k$.

PROPOSITION 4.4. *Suppose that S is a right Noetherian ring such that S/J is simple Artinian and that J is a minimal prime over a normalizing sequence $x_1, x_2, \ldots, x_n$ of length n, and suppose further that* $\operatorname{rank}(J) = n$. *Then there is a chain of primes $Q_0 \subset Q_1 \subset \ldots Q_n = J$ such that each Q_i is minimal over I_i and $x_{i+1} \notin Q_i$.*

Proof. Since $\operatorname{rank}(J) = n$, the integers n_j of the previous theorem must be $n_0 = 0$, $n_1 = 1$, $n_2 = 2$, etc. Now, by repeated application of Proposition 4.1 we obtain

$$n = \kappa(S) \leqq 1 + \kappa(S/I_1) \leqq \ldots \leqq j + \kappa(S/I_j) \leqq \ldots$$
$$\leqq n + \kappa(S/I_n) = n,$$

so $\kappa(S/I_j) = n - j$ for $j = 0, 1, \ldots, n$. Thus $\operatorname{rank}(J/I_j) = n - j$, so the chain

$$Q_j/I_j \subset Q_{j+1}/I_j \subset \ldots \subset Q_n/I_j$$

cannot have any additional primes added to it. In particular, Q_j is minimal over I_j.

In many situations it will be the case that $\operatorname{rank}(J) = n$ and J is generated by a normalizing sequence of length n. This is the case studied in [**23**], and much more can be said there. This situation is described in the following theorem.

THEOREM 4.5. *Let S be a right Noetherian ring for which $S/J(S)$ is simple Artinian and $J(S)$ is generated by a normalizing sequence $x_1, x_2, \ldots, x_n$ of length n. Then the following are equivalent:*

1) $\kappa(S) = \operatorname{rank}(J) = n$;
2) *the coset $x_{j+1} + I_j$ is not a zero-divisor in S/I_j, for $j = 0, 1, \ldots, n$;*
3) *$I_0, I_1, \ldots, I_n$ are distinct prime ideals.*

If these conditions are satisfied, then the right global dimension of S equals n also, and S is a domain if S/J is a division ring.

Proof. The assertion that 2) implies all the other statements is precisely the content of [**23**], Theorem 2.7. Also, it is evident that 3) implies 2) since a normal element in a prime ring cannot be a zero-divisor. It remains to show that 1) implies 3). We know that I_n $(=J)$ is prime, so it suffices to show that if I_k is prime then so is I_{k-1}. From 4.1 we have

$$n = \kappa(S) \leqq 1 + \kappa(S/I_1) \leqq 2 + \kappa(S/I_2) \leqq \ldots$$
$$\leqq n + \kappa(S/I_n) = n,$$

so for each k we have

$$\kappa(S/I_{k-1}) = 1 + \kappa(S/I_k).$$

Passing to the ring $S' = S/I_k$ and denoting the image of x_k in S' by x', we have the following situation: x' is a normal element in $J(S')$ generating a prime ideal I', and we wish to conclude that S' is a prime ring. This will follow from Lemma 2.6 of [**23**] provided we can show that x' is not a zero-divisor. Thus, to complete the proof it remains to show that the conditions

i) $I' = x'S' = S'x'$ is a prime ideal, and

ii) $\kappa(S') = 1 + \kappa(S'/x'S')$

together guarantee that x' is not a zero-divisor in S'.

Since S' is Noetherian, the ascending chain $r(x') \subseteq r(x'^2) \subseteq \ldots$ is eventually stationary: suppose $r(x'^k) = r(x'^{k+1})$, and denote each $r(x'^i)$ by A_i. We show first that A_k is contained in I'.

From [**4**], Corollary 7.5, there is a prime Q' in S' for which $\kappa(S') = \kappa(S'/Q')$. Clearly Q' can't contain I' yet $I'^k A_k = 0$, hence $A_k \subseteq Q'$ and $\kappa(S') = \kappa(S'/A_k)$. Since $A_k = A_{k+1}$ the image $\bar{x}$ of x' in the ring $\bar{S} = S'/A_k$ has zero right annihilator. Also $\bar{x}$ is normal in $\bar{S}$ and is in $J(\bar{S})$. From 4.1 and the fact that $r(\bar{x}) = 0$ we have

$$\kappa(S') = \kappa(\bar{S}) = 1 + \kappa(\bar{S}/\bar{x}\bar{S}) = 1 + \kappa(S'/I' + A_k)$$

or

$$\kappa(S'/I' + A_k) = \kappa(S') - 1.$$

If it were the case that $A_k \not\subseteq I'$, then $(A_k + I')/I'$ would be essential in the prime ring S'/I', and we would have ([**4**], Proposition 6.1)

$$\kappa(S'/I' + A_k) < \kappa(S'/I') = \kappa(S') - 1,$$

which is false. Therefore $A_k \subseteq I'$.

Now we proceed inductively: suppose we have established that $A_k \subseteq x'^j S'$. Denoting $\{s \in S' : x'^j s \in A_k\}$ by T_j we have $A_k = x'^j T_j$. Then $0 = x'^k A_k = x'^{k+j} T_j$ so $T_j \subseteq A_{k+j} = A_k \subseteq x'S'$ and hence $A_k \subseteq x'^{j+1}S$. Therefore

$$A_k \subseteq \bigcap_1^\infty x'^j S'.$$

Now x' is normal and in $J(S')$, and S' is right Noetherian, so $I' = x'S'$ has the right AR-property, and this implies $\bigcap_1^\infty x'^j S' = 0$. Thus $A_k = 0$ and so $A_1 = r(x') = 0$. It follows [**12**] that $l(x') = 0$, so x' is not a zero-divisor. The proof that 1) implies 3) is complete.

If we start with a right localizable prime P in a right Noetherian ring R, the map $X \to \bar{X}R_P$ of right ideals of R to right ideals of R_P gives a one-to-one correspondence between all right R_P-ideals and right ideals of R which are closed in the torsion theory determined by P. Under this

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correspondence, primes of R_P correspond to primes of R contained in P, and vice-versa. The application of Theorem 4.5 (with $S = R_P$) to the original ring R is the following.

COROLLARY 4.6. *Let P be a right localizable prime in a right Noetherian ring R, and suppose P is generated by a normalizing sequence $x_1, x_2, \ldots, x_n$ of length n. Then the following are equivalent.*

1) $\text{rank}(P) = n$,

2) $\text{cl}_P(x_1R + \ldots + x_jR)$ *is a prime not containing* x_{j+1}, *for* $j = 0, 1, \ldots, n-1$,

3) x_{j+1} *is not a zero-divisor modulo* $\text{cl}_P(x_1R + \ldots x_jR)$, *for* $j = 0, 1, \ldots, n-1$.

5. Examples. 1) Let $\mathfrak{g}$ be a finite-dimensional solvable Lie algebra over a field of characteristic zero, and let R be the universal enveloping algebra of $\mathfrak{g}$. For $\mathfrak{h}$ an ideal of $\mathfrak{g}$, let P be the prime ideal of R generated by $\mathfrak{h}$. If $\mathfrak{h}$ is nilpotent, then R_P satisfies the conditions of Theorem 4.5, and $\kappa(R_P) = \dim(h)$ ([**23**], Theorem 3.7).

2) If R is either

a) the universal enveloping algebra of a finite-dimensional nilpotent Lie algebra over any field, or

b) the group ring FG of a finitely-generated group G with a finite normal subgroup H such that G/H is torsion-free nilpotent and $o(H)$ is a unit in a field F,

then for any prime P of R, R_P satisfies the conditions of 4.5, as is shown in [**22**]. Moreover, in these cases, each ideal of R is generated by a centralizing sequence. It follows easily that any ideal of R_P is then generated by a centralizing sequence.

3) If G is a polycyclic-by-finite group and F is a field for which either

a) $\text{char}(F) = 0$ and G is finite-by-nilpotent, or

b) $\text{char}(F) = p \neq 0$ and G is finite p'-by-(nilpotent-by-finite p),

then ([**18**], Corollary 3.12) every ideal of the group ring FG has a centralizing sequence of generators. Therefore 4.2, 4.3, and 4.4 are applicable to R_P for each prime P.

4) A commutative Noetherian local ring of finite global dimension is regular, and hence has its prime ideals generated by R-sequences. Since (2) of 4.5 attempts to extend the concept of R-sequence to noncommutative rings, it is reasonable to wonder whether, in the case where S is a right Noetherian ring for which $J(S)$ is generated by a normalizing sequence and $S/J(S)$ is simple Artinian, knowing gl. dim. (S) is finite might ensure that all the conditions of 4.5 hold. This is not the case, as the following example (pointed out to the author by K. A. Brown) demonstrates.

Let F be the field $GF(2)$ and G the group generated by x, y, z where

$[x, y] = z^4$, $z^{-1}xz = x^{-1}$ and $z^{-1}yz = y^{-1}$. G has a chain of normal subgroups

$$1 \subseteq \langle z^2 \rangle \subseteq \langle x, z^2 \rangle \subseteq \langle x, y, z^2 \rangle \subseteq G$$

where the top factor is of order 2 and the other factors are infinite cyclic. As in 3) above, every ideal of $R = FG$ has a centralizing sequence of generators, and R is left and right Noetherian. The FC subgroup of G is $\langle z^2 \rangle$, which is torsion-free, so R is prime ([**18**], Theorem 2.10). Also, gl. dim. (R) = Hirsch number $(G) = 3$. If we let P be the augmentation ideal then

$$\text{gl. dim. } (R_P) \leqq \text{gl. dim. } (R),$$

so R_P has finite global dimension.

If we let I_1 be the ideal of R generated by $1 - z^2$ then R/I_1 is the group ring over F of the group

$$\langle x, y, z\colon [x, y] = z^2 = 1, z^{-1}xz = x^{-1}, z^{-1}yz = y^{-1} \rangle,$$

and this is a prime ring. Therefore I_1 is a prime ideal. Similarly, the ideal I_2 generated by $1 - z^2$ and $1 - x$ is prime. Therefore $\operatorname{rank}(P) \geqq 3$. As noted above, P has a centralizing sequence of generators. Such a sequence is $u_1, u_2, \ldots, u_6$ where $u_1 = 1 - z^2$, $u_2 = x + x^{-1} = x^{-1}(1 - x^2)$, $u_3 = y + y^{-1} = y^{-1}(1 - y^2)$, $u_4 = 1 - x$, $u_5 = 1 - y$, $u_6 = 1 - z$. Note however that if T is the ideal generated by u_1, u_2, and u_3 then R/T is the group ring over F of the group $G/\langle x^2, y^2, z^2 \rangle$ (which is abelian) so R/T is commutative. Any prime ideal containing T also contains $1 - x^2 = (1 - x)^2$, and so contains u_4 (and, similarly, u_5 and u_6). Therefore P is a minimal prime above T. From 4.3 we have $\kappa(R_P) = \operatorname{rank}(P) \leqq 3$, so $\operatorname{rank}(P) = 3$.

If the conditions of 4.5 held in R_P, then there would be a sequence w_1, w_2, w_3 generating $\bar{P}R_P$ such that w_1 is regular (i.e., not a zero-divisor) in R_P, w_2 is regular modulo $w_1R_P = Q_1$, and w_3 is regular modulo $w_1R_P + w_2R_P = Q_2$, and Q_1 and Q_2 would be primes. In fact, $R_P/\bar{P}R_P$ is a field so R_P/Q_1 would be an integral domain. Now the ideal $Q_1' = Q_1 \cap R$ of R would contain a non-zero central element of R, i.e., an element of R whose support in G is in the centre $\langle z^2 \rangle$ of G. Multiplying by some power of z^2, Q_1' would contain a polynomial over F in z^2. Suppose $p(z^2) = a_0 + a_1z^2 + \ldots a_kz^{2k}$ is of smallest possible degree k in z^2 among such elements of Q_1'. Then $p(1) = 0$ since $p(z^2)$ is in the augmentation ideal P of R. Therefore

$$p(z^2) = p(z^2) - p(1) = (z^2 - 1)h(z^2)$$

where the degree of $h(z^2)$ in z^2 is $k - 1$. Since R_P/Q_1 and R/Q_1' are domains, $z^2 - 1$ or $h(z^2)$ is in Q_1'. Since k is minimal, we conclude $z^2 -$

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$1 = (z-1)^2$, and thus $z - 1$, is in Q_1'. But then

$$\begin{aligned} x^2 - 1 &= x(z^{-1}x^{-1}z) - xz^{-1}x^{-1} + xz^{-1}x^{-1} - 1 \\ &= xz^{-1}x^{-1}(z-1) + xz^{-1}(1-z)x^{-1} \end{aligned}$$

is in Q_1', and so Q_1' contains $1 - x$ and (by the same reasoning) $1 - y$. But then Q_1' contains P, which is not possible. Therefore, in this example, the conditions of Theorem 4.5 cannot hold, although R_P is of finite global dimension.

6. Completion of R_P. If P is a classical prime of a right Noetherian ring, then $J = J(R_P)$ has the right AR-property, which ensures that $\bigcap_1^\infty J^n = 0$. If we endow R_P with the J-adic topology, then the completion $\hat{R}_P$ of R_P is a ring, and can be identified as the bicommutator of $E(R_P/J)$, the R_P-injective hull of R_P/J [**8**]. The next section arises from the questions posed in [**17**], where it was asked when (if) $\hat{R}_P$ is Noetherian.

For any ideal I in any ring S, one can define the I-adic topology on S, and the completion $\hat{S}$ of S with respect to this topology is a ring. Questions relating to whether $\hat{S}$ is Noetherian were considered in [**14**] and more recently in [**15**]. In the latter, it was shown that if S has a centralizing set of generators and if S/I is right Artinian, or if S is right Noetherian, then $\hat{S}$ is right Noetherian. An example was cited of a left and right Noetherian ring S with ideal I generated by a normalizing sequence for which $\hat{S}$ is not left or right Noetherian, so one cannot replace "centralizing" by "normalizing" in the theorem. It is significant that in the example, I does not have the right or left AR property, for we will show that if S is right and left Noetherian, and I is generated by a normalizing sequence and has the AR properties, then $\hat{S}$ is right and left Noetherian. We also will obtain a "one-sided" result, but a technical assumption on I is needed.

As in the case of commutative rings, the passage from S to $\hat{S}$ is via the graded ring $G_I(S)$. As an additive group, $G_I(S) = S/I \oplus I/I^2 \oplus \ldots$, and multiplication proceeds by extending the obvious multiplication for which

$$(I^n/I^{n+1}) \cdot (I^m/I^{m+1}) \subseteq I^{n+m}/I^{n+m+1}.$$

We begin by citing the following results from [**14**].

LEMMA 6.1. a) $G_{\hat{I}}(S) \cong G_{\hat{I}}(\hat{S})$.

b) *If $G_I(\hat{S})$ is right Noetherian, so is $\hat{S}$.*

c) *If S/I is right Artinian and I finitely generated as a right ideal, then a right ideal F of $\hat{S}$ is finitely generated if and only if for each $k \geqq 0$ there is an $n \geqq 0$ such that $F \cap \hat{I}^n \subseteq F\hat{I}^k$. Hence $\hat{S}$ is right Noetherian if and only if $\hat{I}$ has the right* AR-*property.*

Proof. See page 145, Lemma 2.2, and page 149 of [**14**], respectively.

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LEMMA 6.2. ([**15**], Proposition). *Let x be a normal element of a ring S, and set $I = xS$. If* i) $r(x^n) = r(x^{n+1})$ *for some n,* ii) $l(x^m) = l(x^{m+1})$ *for some m, and* iii) *S/I is right Noetherian, then the I-adic completion $\hat{S}$ of S is right Noetherian.*

Proof. From Lemma 6.1, it suffices to show that the graded ring $G_I(S)$ is right Noetherian.

Case I. Suppose $r(x) = 0$. Then ii) guarantees $l(x) = 0$ as well, and the map $\rho:S \to S$ defined by $xs = s^\rho x$ is an automorphism of S. Since $I = I^\rho$, ρ induces an automorphism σ of $\bar{S} = S/I$ and, as is pointed out in [**23**], p. 32, $G_I(S)$ is isomorphic to the twisted polynomial ring $S[Z;\sigma]$ whose elements are polynomials $\bar{s}_nZ^n + \bar{s}_{n-1}Z^{n-1} + \ldots \bar{s}_0$ in Z with coefficients in $\bar{S}$, and multiplication defined by $Z\bar{s} = \bar{s}^\sigma Z$. The usual proof of the Hilbert Basis Theorem works in $\bar{S}[Z;\sigma]$ so $\bar{S}[Z;\sigma]$, and hence $G_I(S)$, is right Noetherian.

Case II. General case. By replacing n and m in i) and ii) by $k = n + m$ we may assume that $r(x^k) = r(x^{k+1})$ and $l(x^k) = l(x^{k+1})$. Therefore in the ring $S' = S/r(x^k)$ the element $x' = x + r(x^k)$ has zero right annihilator. Also S'/I' is right Noetherian, so by Case (i), $G_{I'}(S')$ is right Noetherian. In the ring $G_I(S)$, the ideal

$$W_k = \bigoplus_k^\infty x^iS/x^{i+1}S$$

can be regarded as a $G(S')$-module, where we define the multiplication via

$$\frac{x^mS}{x^{m+1}S} \times \frac{x'^iS'}{x'^{i+1}S'} \to \frac{x^{m+i}S}{x^{m+i+1}S}$$

where

$$(x^ma + x^{m+1}S, x'^is' + x'^{i+1}S') \mapsto (x^max^is + x^{m+i+1}S).$$

W_k is cyclic, generated by $x^k + x^{k+1}S$; the annihilator in $G_{I'}(S')$ of this is zero, and the $G_I(S)$ and $G_{I'}(S')$ submodules of W_k coincide. Therefore W_k is Noetherian as a $G_I(S)$ module, and $G_I(S)/W_k$ is finitely generated, and thus Noetherian, as an S/I-module, and as $G_I(S)$-module. Hence $G_I(S)$ is right Noetherian.

LEMMA 6.3. *Let $I \lhd S$, and $\hat{S}$ the I-adic completion. Suppose that I has the right* AR-*property, and let x be any element of S. Then:*

i) *If $Sx \subseteq xS$ then $\hat{S}x \subseteq x\hat{S}$;*

ii) *If $r_S(x^n) = r_S(x^{n+1})$, then $r_{\hat{S}}(x^n) = r_{\hat{S}}(x^{n+1})$.*

iii) *If $Sx = xS$ and $xI^k \subseteq Ix$ for some k, then $x\hat{S} = \hat{S}x$. If in addition we have $l_S(x^m) = l_S(x^{m+1})$, then*

$$l_{\hat{S}}(x^m) = l_{\hat{S}}(x^{m+1}).$$

Proof. First note that since I has the right AR-property, for any k and

t there is an integer $n = n(k, t)$ (depending on k and t) such that $x^t S \cap I^n \subseteq x^t I^k$.

i) Suppose $Sx \subseteq xS$ and consider $\sigma x \in \hat{S}x$. The element σ is the limit of a Cauchy sequence $(s_n)_1^\infty$, of elements in S, and for each n we can write $s_n x = x a_n$ where $a_n \in S$. Since the sequence $(s_n)_1^\infty$ is Cauchy, for each k there is an integer M_k such that $s_i - s_j \in I^{n(k,1)}$ for $i, j \geqq M_k$, and we may assume $M_1 < M_2 < M_3 \ldots$. Then, for $j \geqq M_k$,

$$x(a_j - a_{M_k}) = (s_j - s_{M_k})x \in xS \cap I^{n(k,1)} \subseteq xI^k$$

so

$$a_{M_{k+1}} - a_{M_k} \in I^k + r(x);$$

say $a_{M_{k+1}} - a_{M_k} = w_k + r_k$ where $w_k \in I^k$ and $xr_k = 0$. The sequence $(t_n)_1^\infty$ defined by $t_n = a_{M_n} - \sum_1^{n-1} r_i$, is Cauchy and so converges to an element τ of $\hat{S}$. Then

$$x\tau = \lim(xt_i) = \lim(xa_{M_i}) = \lim(s_{M_i}x) = \sigma x$$

and $\sigma x \in x\hat{S}$.

ii) Suppose $r_S(x^n) = r_S(x^{n+1})$ and let $\sigma \in r_{\hat{S}}(x^{n+1})$, where σ is the limit of the Cauchy sequence $(s_n)_1^\infty$ of elements of S. The sequence $(x^{n+1}s_j)_{j=1}^\infty$ converges to 0 in S, so for each k there is an M_k such that $x^{n+1}s_j \in I^{n(k,n+1)}$ for $j \geqq M_k$. But then $x^{n+1}s_j$ is in $x^{n+1}S \cap I^{n(k,n+1)} \subseteq x^{n+1}I^k$ so

$$s_j \in I^k + r(x^{n+1}) = I^k + r(x^n),$$

and so

$$x^n s_j \in x^n I^k \subseteq I^k \quad \text{for} \quad j \geqq M_k.$$

Therefore the sequence $(n^n s_j)_{j=1}^\infty$ converges to 0 and hence $x^n\sigma = 0$. Therefore $r_{\hat{S}}(x^{n+1}) \subseteq r_{\hat{S}}(x^n)$, as desired.

iii) Suppose $xS = Sx$ and $xI^k \subseteq Ix$. From (i) we know $\hat{S}x \subseteq x\hat{S}$, so we wish to show $x\hat{S} \subseteq \hat{S}x$. Let σ be in $\hat{S}$, suppose $\sigma = \lim_{i\to\infty}(s_i)$ where $s_i \in S$, and suppose $xs_i = t_i x$. We may assume $s_{n+1} - s_n \in I^n$ for each n, so

$$(t_{k(n+1)} - t_{kn})x = x(s_{k(n+1)} - s_{kn}) \in xI^{kn} \subseteq I^n x$$

and

$$t_{k(n+1)} - t_{kn} \in I^n + l(x).$$

If we write $t_{k(n+1)} - t_{kn} = v_n + c_n$ where $v_n \in I^n$ and $c_n x = 0$, then the sequence $(b_j)_1^\infty$ defined by

$$b_j = t_{kj} - \sum_{i=1}^{j-1} c_i$$

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is Cauchy and so converges to (say) β in $\hat{S}$. Then

$$\beta x = \lim_{j\to\infty} (b_j x) = \lim_{j\to\infty} [t_{kj} x) = \lim_{j\to\infty} (x s_{kj}) = x\sigma$$

and $x\sigma \in \hat{S}x$.

Just as this proof is a modification of the proof of i), so the proof of the last part of iii) is a modification of the proof of ii). The details are omitted.

If M is an S module, and $I \lhd S$, then one can define a topology on M by taking as neighbourhoods of 0 the sub-modules MI^n for $n = 1, 2, \ldots$. Denote by $\hat{M}$ the completion of M in this topology: then $\hat{M}$ can be regarded as a right $\hat{S}$-module. The following lemma and its proof (exactly as in the commutative case) are well known.

LEMMA 6.4. *Let S be right Noetherian and I an ideal of S with the right* AR-*property. If $0 \to X_1 \to X_2 \to X_3 \to 0$ is an exact sequence of finitely generated right S-modules, then $0 \to \hat{X}_1 \to \hat{X}_2 \to \hat{X}_3 \to 0$ is an exact sequence of $\hat{S}$-modules, and each $\hat{X}_i$ is generated as an $\hat{S}$ module by any finite generating set for X_i. Furthermore, $\hat{X}_1$ is a closed submodule of $\hat{X}_2$ in the $\hat{I}$-adic topology.*

In particular, we note that if S is right Noetherian, then for any right ideal E of S, $\hat{E} = E\hat{S}$ is a closed right ideal of $\hat{S}$. This means $\hat{E} = \bigcap_1^\infty (\hat{E} + \hat{I}^n)$.

THEOREM 6.5. *Let S be a right Noetherian ring and I an ideal of S for which:*

i) *I is generated by a normalizing sequence $x_1, \ldots, x_n$,*

ii) *I has the right* AR-*property, and*

iii) *there is an integer k such that, for $1 \leqq j \leqq n$,*

$$x_j I^k \subseteq I x_j + \sum_{i=1}^{j-1} x_i S.$$

Then the I-adic completion $\hat{S}$ of S is right Noetherian.

Proof. The proof of the theorem is by induction on n (c.f. [**15**]). When $n = 1$, Lemma 6.2 gives the result. Assume the result is true for ideals generated by normalizing sequences of length $n - 1$ or less. By the induction hypothesis, the completion of $\bar{S} = S/x_1S$ in the $\bar{I}$-adic topology is right Noetherian. However, the $\bar{I}$-adic topology on $\bar{S}$ is the same as the I-adic topology, and so we have that the I-adic completion $\bar{S}\hat{}$ of $\bar{S}$ is Noetherian as a right $\hat{S}$-module. From Lemma 6.4, $0 \to x_1\hat{S} \to \hat{S} \to \bar{S}\hat{} \to 0$ is exact, and from Lemma 6.3 we see that x_1 is a normal element of $\hat{S}$, that $l_{\hat{S}}(x^m) = l_{\hat{S}}(x^{m+1})$ and $r_{\hat{S}}(x^n) = r_{\hat{S}}(x^{n+1})$ for some integers m and n. Therefore by Lemma 6.2, the $x_1\hat{S}$-adic completion of $\hat{S}$ is right Noetherian.

It remains to show that $\hat{S}$ is itself complete in the $x_1\hat{S}$-adic topology. Let $(\rho_j)_1^\infty$ be a Cauchy sequence in the $x_1\hat{S}$-adic topology: we may assume $\rho_{n+1} - \rho_n \in (x_1\hat{S})^n = x_1^n\hat{S}$. This is then Cauchy in the $\hat{I}$-adic topology and since $\hat{S}$ is $\hat{I}$-complete, $(\rho_j)_1^\infty$ converges to (say) ρ in the $\hat{I}$-adic topology. We may assume $\rho - \rho_j \in \hat{I}^j$ by passing to a subsequence if necessary. Then for any n and any r,

$$\rho - \rho_n = \rho - \rho_{n+r} + \rho_{n+r} - \rho_n \in \hat{I}^{n+r} + x_1^n\hat{S},$$

so

$$\rho - \rho_n \in \bigcap_{r=1}^{\infty} (x_1^n\hat{S} + \hat{I}^r) = x_1^n\hat{S}$$

(from Lemma 6.4). Therefore $(\rho_j)_1^\infty$ converges to ρ in the $x_1\hat{S}$-adic topology, as desired.

Remark. If I has the left AR-property, then for each j we have

$$I^k \cap \left(\sum_1^j Sx_i\right) \subseteq I\left(\sum_1^j Sx_i\right) \subseteq Ix_j + \sum_1^{j-1} x_iS$$

for some k, from which iii) easily follows. Conversely, if S is also left Noetherian, the proof of Lemma B of [**12**], Corrigenda and Addenda can easily be adapted to show that iii) implies I has the left AR-property. (Caveat: Lemma B is incorrect as stated: the expressions x_1I and Ix_1 should be interchanged, as examination of the proof will reveal.)

COROLLARY 6.6. *If P is a right localizable prime of a right Noetherian ring R, and if P is a minimal prime for an ideal I generated by a normalizing sequence $x_1, \ldots, x_n$, then the $J(R_P)$-adic completion $\hat{R}_P$ of R_P is right Noetherian.*

Proof. From 3.4 and 3.6, the sequence $\bar{x}_1, \ldots, \bar{x}_n$ is a normalizing sequence in R_P, $J = J(R_P)$ is the only prime containing the ideal $\bar{I} = \sum_1^n \bar{x}_i R_P$ (so $J^k \subseteq I$ for some k), and J has the right AR-property. Since $J^k \subseteq I \subseteq J$, it follows that J_k and then I have the right AR-property, and that the I-adic and J-adic topologies on R_P are the same. Furthermore, J is the only maximal ideal of R_P, so (as in the proof of 3.5) in any homomorphic image R_P' of R_P we have $xJ' = J'x$ for any normal element x of R_P'. In any such image we thus have

$$x(I')^k \subseteq xJ'^k = J'^kx \subseteq I'x.$$

By 6.5, the I-adic ($=J$-adic) completion of R_P is right Noetherian.

7. Duality for $\hat{R}_P$. Let S be a right Noetherian ring whose Jacobson radical J has the right AR property, and assume further that S is *semi-local;* i.e., that S/J is Artinian. It was shown in [**8**] that under these assumptions, the J-adic completion $\hat{S}$ of S is canonically isomorphic to the bicommutator of U_S, the injective hull in Mod-S of S/J. Furthermore, U can be regarded in a natural way as right $\hat{S}$-module and the conditions i) $\hat{S}$ is right Noetherian, ii) $U_{\hat{S}}$ is Σ-injective, and iii) $\hat{J}$ has the right AR-property were shown to be equivalent.

In studying the question of whether $\hat{S}$ is right Noetherian, several authors have found that this question is closely related to the questions of whether U_S is Artinian, and whether U induces a Morita duality between $\hat{S}$ and $T = \mathrm{End}(U_S) = \mathrm{End}(U_{\hat{S}})$. (To say that a bimodule ${}_TU_S$ induces a *Morita duality* between S and T means U_S and ${}_TU$ are injective cogenerators, that $S = \mathrm{End}({}_TU)$ and that $T = \mathrm{End}(U_S)$.) For example, if S is a (left and right) Noetherian semilocal P.I. ring, then Vámos [**24**] showed essentially that U induces a duality between $\hat{S}$ and T, that $\hat{S}$ and T are Noetherian, and that U_S and ${}_TU$ are Artinian. Also, Deshpande [**3**] showed that if S is an HNP ring with non-zero Jacobson radical, (S is automatically semilocal) the same conclusions may be drawn. (Note that some authors take U to be the injective hull of S/J, others take U to be the injective hull of the direct sum of simple modules, one from each isomorphism class: the proofs for one case are easily translated into proofs for the other.)

The approach taken in establishing results such as those just cited is to show first that U_S is Artinian. Once this is known, many other things follow, as the next result shows.

PROPOSITION 7.1. *Let S be a right Noetherian semilocal ring whose Jacobson radical J satisfies the right* AR-*property, and let U_S be the injective hull in* Mod-S *of S/J, and $T = \mathrm{End}(U_S)$. Then the following are equivalent:*

1) *U_S is Artinian.*
2) *U_S is linearly compact.*
3) *T is left Noetherian.*
4) *${}_TU$ is injective.*

Furthermore, when these conditions are satisfied, T is a left Noetherian ring whose Jacobson radical H satisfies the left AR *property.*

Proof. For (2) $\Leftrightarrow$ (4) see [**16**], and (1) $\Rightarrow$ (3) is contained in Theorem B of [**24**]. In that proof, it is shown *inter alia* that for any finite set $t_1, \ldots, t_n$ in T, we have

$$\mathrm{Ann}_T\left(\bigcap_1^m \mathrm{Ker}\, t_i\right) = \sum_1^m Tt_i:$$

from this one concludes $\mathrm{Ann}_T(\mathrm{Ann}\quad UL) = L$ for each finitely generated left ideal L of T. On the other hand, ([**5**], Proposition 23.13) it is known that

$$\mathrm{Ann}_U(\mathrm{Ann}_T\ V) = V$$

for each submodule V of U_S. Therefore if T is left Noetherian, $\mathrm{Ann}_U(\ )$

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and $\mathrm{Ann}_T(\ )$ are inverse anti-isomorphisms between the submodule lattices of ${}_TT$ and U_S, so U_S is Artinian: hence (3) $\Rightarrow$ (1). It is immediate from the definition of linearly compact (see [**15**]) that (1) $\Rightarrow$ (2). To show that (2) implies (1) recall ([**5**] Corollary 19.16B) that a module is Artinian if every factor module has finitely generated essential socle. In our situation, the fact that J has the AR-property implies

$$U_S = \bigcup_1^\infty \mathrm{Ann}_U(J^n),$$

whence every factor module of U_S has an essential socle. Now if U_S is linearly compact, so is any submodule of any factor module. In particular, the socle of any factor module is linearly compact. An infinite direct sum of nonzero modules cannot be linearly compact, so the socle of each factor module of U must be finitely generated as well. Hence (2) $\Rightarrow$ (1).

Suppose these conditions are satisfied. Then (using the fact that U_S is a cogenerator), one can show that, for each n,

$$\mathrm{Soc}_n({}_TU) = \mathrm{Ann}_U(H^n) = \mathrm{Ann}_U(J^n) = \mathrm{Soc}_n(U_S),$$

and that for each n $\mathrm{Soc}_n(U_S)$ induces a duality between S/J^n and T/H^n. Also T is semiperfect so T/H is semisimple: thus T is left Noetherian and semilocal. Since ${}_{\hat{T}}U$ is an injective cogenerator and equals $\bigcup_1^\infty \mathrm{Ann}_U(H^n)$, it follows easily that if ${}_TX$ is a finitely generated module with an essential submodule ${}_TY$ such that $HY = 0$, then $H^nX = 0$ for some n. This is equivalent to saying H has the left AR-property [**13**].

COROLLARY 7.2. *Suppose that S is right Noetherian and semi-local, and that J has the right AR-property. Then U induces a Morita duality between $\hat{S}$ and T if and only if U_S is Artinian and one/all of the following hold:*

i) *$\hat{S}$ is right Noetherian.*

ii) *$U_{\hat{S}}$ is injective.*

iii) *${}_TU$ is Artinian.*

Proof. If U induces a duality between T and $\hat{S}$ then U_S and ${}_TU$ are injective. By 7.1, U_S is Artinian.

Conversely, suppose U_S is Artinian. We can use 7.1 to conclude that ${}_TU$ is injective, that T is left Noetherian and semilocal and that H has the left AR-property. Now we can apply the left-hand version of 7.1 to the module ${}_TU$ and conclude that i), ii) and iii) are equivalent. Thus if U_S is Artinian and any/all of (i)-(iii) hold, then $U_{\hat{S}}$ and ${}_TU$ are injective cogenerators: hence we have a duality.

The next lemma is due to Jategaonkar. The proof given here is more direct than the one given in [**9**].

LEMMA 7.3. *Let S be right Noetherian and I an ideal of S generated by a*

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normalizing sequence $x_1, \ldots, x_n$. If M is a simple right S-module such that $MI = 0$, then $E_S(M)$ is Artinian if and only if $E_{S/I}(M)$ is Artinian.

Proof. Set $E = E_S(M)$. Also, let $I_k = x_1S + \ldots + x_kS$ and $A_k = \mathrm{Ann}_E(I_k)$, for $k = 0, 1, \ldots, n$. Then $A_k = E_{S/I_k}(M)$, as is well-known. The "only if" is therefore clear, and the "if" part follows easily by induction once we have established the result for $n = 1$. We may assume then that $I = xS = Sx$.

The ideal I is generated by a normal element, and hence has the right AR-property, from which we obtain [**13**] $E = \bigcup_1^\infty E_n$, where $E_n = \mathrm{Ann}_E(x^n)$. By assumption, E_1 is Artinian, and it now follows by induction that each E_i is Artinian. For suppose E_n is Artinian. For any submodule W of E_{n+1}, Wx is a submodule of E_n, and $W \mapsto Wx$ induces a one-to-one order-preserving correspondence between submodules of E_{n+1} containing E_1 and submodules of E_n. Hence E_{n+1}/E_1 is Artinian, whence E_{n+1} is, too.

If E were not Artinian, then some factor E/X would fail to have a finitely-generated and essential socle, and by Zorn's Lemma there is an X which is minimal with this property; call it W. The minimality of W guarantees that W is an Artinian module.

Since E is the union of the (Artinian) E_i's, the socle of E/W is essential, and is contained in $\mathrm{Ann}_{E/W}(I)$. Suppose this annihilator is A/W, where $A = \{e \in E : eI \subseteq W\}$. For any submodule Z satisfying $E_1 + W \subseteq Z \subseteq A$, Zx is a submodule of W, and $Z \mapsto Zx$ induces a one-to-one order-preserving correspondence from the submodule lattice of $A/(E_1 + W)$ to that of W/Wx: hence $A/(E_1 + W)$ is Artinian. But $(E_1 + W)/W$ is also Artinian, and thus A/W is Artinian. Therefore the socle of E/W is Artinian (and essential in E/W), a contradiction. Therefore E is Artinian.

THEOREM 7.4. (a) *Let S be a right Noetherian semilocal ring whose Jacobson radical J is generated by a normalizing sequence $x_1, \ldots, x_n$. Then J has the right* AR-*property and U_S is Artinian.*

(b) *If in addition there is an integer k such that $x_jJ^k \subseteq Jx_j + x_1S + \ldots x_{j-1}S$ for $1 \leqq j \leqq n$, then U_S and ${}_TU$ are Artinian, ${}_TT$ and $\hat{S}_{\hat{S}}$ are Noetherian, and U induces a duality between $\hat{S}$ and T.*

Proof. (a) S is semilocal, so each simple right module M is injective as an S/J module. By Lemma 7.3, each $E_S(M)$ is Artinian so U_S, a finite direct sum of $E_S(M)$'s, is Artinian as well. For any u in U, the sequence $uJ, uJ^2, \ldots$ must eventually become stationary, so by Nakayama's Lemma $uJ^t = 0$ for some t. Hence

$$U = \bigcup_1^\infty \mathrm{Ann}_U(J^i).$$

To show that J has the AR-property, it suffices ([**13**], Lemma 8) to show that if Y is a Noetherian right S-module with an essential submodule X such that $XJ = 0$, then $YJ^m = 0$ for some m. For such an X and Y, X would be a Noetherian semisimple module, embeddable in a direct sum of (say t) copies of S/J. By injectivity we get an embedding of Y into U^t. Thus Y is Artinian and Noetherian, so Y has a composition series, and the result follows.

(b) $\hat{S}$ is right Noetherian by Theorem 6.5, and the rest of the assertions follow from 7.1 and 7.2.

THEOREM 7.5. *Let R be a right Noetherian ring, P a right localizable prime generated by a normalizing sequence. Let E be the injective hull in* Mod-R *of* R/P, $T = \mathrm{End}(E_R)$ *and* $R' = \mathrm{End}({}_TE)$. *Then T is left Noetherian R' is right Noetherian; ${}_TE$ and $E_{R'}$ are Artinian; E induces a duality between R' and T; and R' is canonically isomorphic to $\hat{R}_P$.*

Proof. Since P is right localizable, we have the commutative diagram in Mod-R:

$$\begin{tikzcd} & & & 0 \arrow[d] & \\ 0 \arrow[r] & P \arrow[r] \arrow[d, "\eta_p"] & R \arrow[r] \arrow[d, "\eta_R"] & R/P \arrow[r] \arrow[d, "\eta_{R/P}"] & 0 \\ 0 \arrow[r] & PR_P \arrow[r] & R_P \arrow[r] & \dfrac{R_P}{PR_P} \arrow[r] & 0 \end{tikzcd}$$

where the second row can be regarded as the application of the localization functor Q to the first row (c.f. [**6**], § 3). The map $\eta_{R/P}$ may be regarded both as a homomorphism of R-modules and as the embedding of the prime right Noetherian ring R/P into its classical right quotient ring. As an R-module homomorphism, $\eta_{R/P}$ is one-to-one, and $\mathrm{Im}(\eta_{R/P})$ is essential as an R-submodule of R_P/PR_P. We identify R/P with its image under $\eta_{R/P}$, and we can embed R_P/PR_P, as an R module, in E. But then E may be regarded as an R_P-module in a canonical fashion, as is well known. (Briefly: for c in $\mathscr{C}(P)$ and e in E, the map $cr \mapsto er$ of cR to E is well defined and lifts to θ: $R \to E$. Then $\theta(1)c = e$, so $E = Ec$ for any c in $\mathscr{C}(P)$. Since P is localizable, the elements of $\mathscr{C}(P)$ are not zerodivisors on E. Therefore for e in E, ec^{-1} is defined as the (unique) solution f of $fc = e$.)

As an R_P module, E is essential over R_P/PR_P and E is in fact the Mod-R_P injective hull of R_P/PR_P. (Show that the embedding of E into this hull must split in Mod-R, and then that the other summand must be zero.) Furthermore (again as is well-known),

$$\mathrm{End}(E_R) = \mathrm{End}(E_{R_P}).$$

Now, $J(R_P) = PR_P$ is generated by a normalizing sequence (Corollary 3.4) and has the right AR-property (Theorem 3.6). As the proof of Corollary 6.6 shows, the condition of 7.4(b) is satisfied so R_P (which can be identified with $\operatorname{End}({}_TE)$) has a duality with T by Theorem 7.4.

Remark. In 3.6 and 6.6, it was sufficient to have the localizable prime P a minimal prime for an ideal I generated by a normalizing sequence, in order that $J(R_P)$ have the AR-property and that $\hat{R}_P$ be Noetherian. However, in order to show that $\hat{R}_P$ has a duality, it appears necessary to assume in 7.5 that P is generated by a normalizing sequence (or rather, that PR_P is so generated).

An example was given in [**20**] of a left and right Artinian ring S with $J^2 = 0$ such that (in the context of 7.3) U_S is not Artinian. The example is constructed by considering division rings $T \subseteq D$ such that $\dim({}_TD)$ is finite but $\dim(D_T)$ is not. ([**1**], Chapter VI). Let $S = \begin{pmatrix} D & 0 \\ D & T \end{pmatrix}$, a subring of the 2×2 matrices over D. It is easy to verify that $(D, 0)$ is simple, and is an essential submodule of the non-Artinian module $(D, D) = \{(e, f) : e, f \in D\}$, so the injective hulls of simple right S-modules need not be Artinian.

In this example, J is not a prime ideal. However, as is pointed out in [**20**], suppose there exists a division ring D and an endomorphism ϕ of D such that $\dim({}_{\phi(D)}D)$ is finite but $\dim(D_{\phi(D)})$ is not. Then the ring

$$S = \left\{ \begin{pmatrix} d & 0 \\ d' & \phi(d) \end{pmatrix} : d, d' \in D \right\}$$

would be left and right Artinian, yet U_S would not be Artinian. (Again consider $(D, 0)$ and (D, D).) In this case J would be a minimal prime above the ideal 0.

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Note added in press. The additional assumption in (b) of Theorem 7.4 is superfluous, for under the assumptions of (a) it follows that $\hat{S}$ is right and left Noetherian. (See Theorem 4.3 of J. C. McConnell, Israel J. Math. *32* (1979), 305-310.)

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