DECIDING IF AN AUTOMORPHISM OF AN INFINITE SOLUBLE GROUP IS INNER

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1. Introduction. Let G be a group with a finite set of generators $x_1, x_2, \ldots, x_n$ and a recursive set of defining relators in the generators. Then an endomorphism η of G is completely determined by the images of the generators $x_i^\eta = w_i(\mathbf{x})$, and hence by the n-tuple of words in $\mathbf{x}$, $(w_1, \ldots, w_n)$. This allows the formulation of algorithmic problems about endomorphisms and automorphisms. For example, can one decide if a given n-tuple of words represents an endomorphism, and if so, an automorphism? Some results on these questions may be found in **[2]** and **[12]**. Here we shall be concerned with a similar problem: given that an n-tuple of words represents an automorphism of the group G, does there exist an algorithm which decides if the automorphism is inner?

As one would expect at this level of generality, the answer is negative. Indeed Baumslag, Gildenhuys and Strebel, as a by-product of their work on the insolubility of the word problem for finitely presented soluble groups **[6]**, exhibited a finitely presented soluble group of derived length 3 for which there is no algorithm to decide if an automorphism is inner.

On the other hand, there are classes of finitely generated soluble groups which are known to behave well algorithmically, in the sense that the word problem at least is soluble. Here we have two classes in mind. The first class consists of finitely generated soluble groups of finite (Prüfer) rank with a recursive presentation; such groups were shown in **[7]** to have soluble word problem; in fact they also have soluble generalized word problem and soluble conjugacy problem, although in a weaker sense **[11]**. The second class is that of all nilpotent-by-polycyclic groups satisfying max-n, the maximal condition on normal subgroups. As is well-known, groups of this type are finitely generated, and it was shown in **[3]** that they have soluble word problem. Also the subclasses of finitely generated abelian-by-nilpotent groups and finitely generated metabelian groups have respectively soluble generalized word problem and soluble conjugacy problem **[13, 9]**.

Our purpose here is to prove that for both of the above classes of groups algorithms exist to decide if automorphisms are inner.

THEOREM A. *Let G be a soluble-by-finite group of finite Prüfer rank which has a finitely generated recursive presentation. Then there is an algorithm which, when an automorphism of G is given by its effect on the generators of the presentation, decides if the automorphism is inner.*

Of course this result applies to finitely presented soluble groups of finite rank. It also applies to finitely generated soluble groups of finite rank that are residually finite, by a theorem of Baumslag and Bieri **[1]**. On the other hand, Theorem A does not hold for arbitrary finitely generated soluble groups of finite rank, as can be seen by taking one of the groups $G(\alpha)$ constructed in **[7]**, with α a non-computable p-adic integer, and forming the direct product $G(\alpha) \times \mathbb{Z}$.

THEOREM B. *Let G be a nilpotent-by-polycylic-by-finite group satisfying* max-n. *Then there is an algorithm which, when an automorphism of G is given by its effect on the generators of a finitely generated presentation, decides if the automorphism is inner.*

COROLLARY. *Let G be a soluble group which is finitely presented in some variety* $\mathbf{A}^m$ *and which either has finite rank or is nilpotent-by-polycyclic with* max-n. *Then* Out G, *the outer automorphism group, has a presentation for which the word problem is soluble.*

For Out G has a recursive presentation (see [**2**, Theorem 5.4]), and it follows from Theorems A and B that the word problem is soluble for Out G. The Corollary applies in particular to a finitely generated metabelian group G. Here one should keep in mind that Out G need not be finitely generated even if G is a finitely presented metabelian group of finite rank, by an example of Lewin [**8**].

We conclude with a question. Does there exist a finitely presented soluble group with soluble word problem for which there is no algorithm to decide if automorphisms are inner?

2. Soluble groups of finite rank. In this section we shall prove Theorem A. First of all we note a straightforward reduction to normal subgroups of finite index.

LEMMA. *Let G be a finitely generated recursively presented group of finite rank, and let N be a normal subgroup with finite index in G. If there is an algorithm to decide if an automorphism of N is inner, then there is an algorithm which can do the same for automorphisms of G.*

Proof. Notice that N is also a finitely generated recursively presented group. Let $\alpha \in \operatorname{Aut} G$ be given by its effect on the generators of G. One first decides if $N^\alpha = N$, which is possible because G/N is finite. Of course α will be outer if this fails to hold. Assume therefore that $N^\alpha = N$, and decide if α induces an inner automorphism in G/N. Again α will be outer if this is false. Thus it can be assumed, after modifying α by a suitable inner automorphism, that $[G, \alpha] \leq N$. If α is to be inner, it will have to be induced by an element of $C = C_G(G/N)$. Observe that $N \leq C$, and let $\{t_1, \ldots, t_k\}$ be a transversal to N in C. If α is inner, then some $\alpha(t_i^{-1})'$ induces an inner automorphism in N; here x' denotes conjugation by x. One can decide if this holds for some i. Hence it may be assumed, after modifying α by an inner automorphism, that $[N, \alpha] = 1$. Therefore the map $xN \mapsto [x, \alpha]$ is a derivation α^* from G/N to $A := Z(N)$. Write $E = C \cap C_G(N)$; then α is inner if and only if it is induced by an element of E. If $e \in E$, then $xN \mapsto [x, e]$ is also a derivation e^* from G/N to A. In addition $eZ(G) \mapsto e^*$ is an isomorphism from $E/Z(G)$ to a subgroup E^* of $D = \operatorname{Der}(G/N, A)$. Clearly α is inner precisely when $\alpha^* \in E^*$. Let $I = \operatorname{Inn}(G/N, A)$, the subgroup of inner derivations; then $I \leq E^*$ and it is easy to see that $D/I = H^1(G/N, A)$ is finite. It is assumed that there is given a finite number of derivations, expressed in terms of the generators of G, which form a transversal to I in D. Express $\alpha^* + I$ in terms of these derivations and decide if it belongs to E^*/I.

The form of this proof is fairly typical. One reduces to the case where the automorphism operates trivially first on a quotient group, and then on the associated normal subgroup. An argument with derivations then comes into play.

Recall that if G is a finitely generated soluble-by-finite group of finite rank, then G is

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a finite extension of a soluble minimax group. Thus there is a series of finite length in G whose infinite factors are either cyclic or quasicyclic. The number of infinite factors is easily seen to be an invariant $m(G)$, the minimax length. (For additional information about the structure of G see [**10**, Chapters 9 and 10]).

Proof of Theorem A. The proof is by induction on $m(G)$; of course, if $m(G)=0$, then G is finite and the result is obvious, so let $m(G)>0$. The group G is now infinite, and it is a simple matter to show that G has an infinite abelian normal subgroup A which is either torsion-free or else a divisible torsion group. The two cases are handled quite differently. In what follows α is an automorphism of G given by its effect on the generators of the presentation.

Case I: A torsion-free. (i) By an easy argument with torsion-free ranks it may be shown that there is a subgroup G_1, with finite index in G, which has a (non-trivial) torsion-free abelian normal subgroup A_1 that is rationally irreducible with respect to every subgroup of finite index in G_1. Now $G_1/C_{G_1}(A_1)$, being an irreducible soluble-by-finite linear group, possesses an abelian normal subgroup of finite index, say $G_2/C_{G_1}(A_1)$; then, for some $m>0$, one has $\bar{G}:=G^m\leq G_2$; also $\bar{A}:=A_1^m\lhd\bar{G}$ and $\bar{A}$ is a rationally irreducible $\mathbb{Z}\bar{G}$-module, while $[\bar{A},\bar{G}']=1$.

By the Lemma one can substitute $\bar{G}$ for G and $\bar{A}$ for A, i.e. one may assume that A is a rationally irreducible $\mathbb{Z}G$-module and $[A,G']=1$. In addition there is nothing to be lost in supposing A to be a cyclic $\mathbb{Z}G$-module; let a be a generator of A. Assuming a to be expressed in terms of the presentation of G, one observes that A is recursively enumerable, so that G/A has a finitely generated recursive presentation. Since $m(G/A)<m(G)$, the induction hypothesis implies the existence of an algorithm to decide if an automorphism of G/A is inner.

(ii) *One can assume that* $[G,\alpha]\leq A$. The first step is to decide if $A^\alpha=A$. By [**7**] the word problem is soluble for G/A, so one can decide if $a^\alpha\in A$. If this is not true, then α is certainly outer. Assume that $a^\alpha\in A$, so that $A^\alpha\leq A$. Next find $a^{\alpha^{-1}}$ by enumerating elements b of A and checking if $b^\alpha=a$, using solubility of the word problem in G. Then check if $a^{\alpha^{-1}}\in A$, i.e. $A^{\alpha^{-1}}\leq A$. Thus it may be assumed that $A=A^\alpha$.

Let α induce an automorphism α' in G/A. Naturally α' is described by its effect on the generators of the presentation of G/A; thus it is possible to decide if α' is inner. It may therefore be assumed that this is the case. After modifying α by an appropriate inner automorphism (which can be found by enumeration), one can further assume that α acts trivially on G/A.

(iii) *One can assume in addition that* $[A,\alpha]=1$. Let $\bar{G}=G/C_G(A)$, an abelian group. One can identify A with $\mathbb{Z}\bar{G}/I$ where the ideal I is the annihilator of a in $\mathbb{Z}\bar{G}$. Rational irreducibility of A implies that I is a prime ideal, so $R=\mathbb{Z}\bar{G}/I$ is a finitely generated domain, and by a theorem of Samuel [**14**] its group of units $U(R)$ is finitely generated. It will be assumed that an explicit finite presentation of $U(R)$ is known, the generators being expressed in terms of the generators in the presentation of G. Since $[G,\alpha]\leq A$, the automorphism $\bar{\alpha}$ of A induced by α is a $\mathbb{Z}\bar{G}$-automorphism, and hence $(r+I)\bar{\alpha}=ur+I$ where $u+I=(1+I)\bar{\alpha}$ is a unit of R. Note that u can be found from the equation $a^\alpha=a^u$.

Let $D=C_G(G/A)$. If α is inner, then $\bar{\alpha}$ is multiplication by an element of $\bar{D}=DC_G(A)/C_G(A)$, that is, $u+I\in\bar{D}+I/I$. But $\bar{D}+I/I$ is a subgroup of $U(R)$, so one can use the presentation of $U(R)$ to decide if $u + I \in \bar{D} + I/I$. Therefore one can assume this to be true and modify α by a suitable inner automorphism to reach the situation where α operates trivially on A, as well as G/A.

(iv) *Final step.* If α is to be inner, it will have to be induced by some element of the subgroup $K = C_G(A) \cap C_G(G/A)$. Observe that K is recursive; for given $g \in G$, one first decides if $g \in C_G(G/A)$, and if so, one tests to see if $[a, g] = 1$.

Note that $D = \mathrm{Der}(G/A, A)$ is isomorphic with a subgroup of the direct product B of n copies of A, where n is the number of generators in the presentation of G. The assignment $xA \mapsto [x, \alpha]$ is a derivation α^* from G/A to A. Likewise for k in K, the mapping $xA \mapsto [x, k]$ is a derivation k^*, and $k \mapsto k^*$ is a homomorphism from K to B with kernel $Z(G)$; the image K^* is evidently a recursively enumerable subgroup of B. Now B is a torsion-free abelian minimax group with a recursive presentation since A is of this type. Also α^* can be expressed in terms of the presentation of B by applying α to the generators of G. Since the word problem is soluble for B/K^* [**11**, Theorem 2.3*], it can be decided if $\alpha^* \in K^*$, i.e., if α is inner.

Case II. A a divisible torsion group. The following facts can be found in [**10**, §9.3 and §10.3]. There is a unique maximum divisible abelian torsion subgroup of G, say D, which has finite rank. (Of course $A \leq D$.) Also G/D is residually finite and possesses a torsion-free normal subgroup N/D with finite index. By the Lemma it is possible to replace G by N; thus G/D becomes torsion-free. The first point to establish is that D is recursively enumerable. To see this, write $\{d_1, \ldots, d_l\}$ for a maximal independent subset of D consisting of elements of prime order: the d_i are assumed to be known. Let d_i have order p_i. Find x_{i1} in G such that $x_{i1}^{p_i} = d_i$ by enumerating elements of the form $y^{-p_i}d_i$, $y \in G$, and waiting for the identity to appear. Then find successively elements $x_{i2}, x_{i3}, \ldots$ of G such that $x_{ir+1}^{p_i} = x_{ir}$. The elements x_{ij}, $i = 1, 2, \ldots, l$, $j = 1, 2, \ldots,$ belong to D since G/D is torsion-free; it is clear from the structure of D that they generate it.

Consequently the group G/D has a finitely generated recursive presentation; also $m(G/D) < m(G)$, so there is an algorithm to decide if an automorphism of G/D is inner. Now $D^\alpha = D$, so the usual reduction may be applied to get $[G, \alpha] \leq D$. Let $G = \langle x_1, \ldots, x_n \rangle$. Then $[x_i, \alpha] \in D$ and $[x_i, \alpha]^u = 1$, $i = 1, 2, \ldots, n$, for some integer u. Since D has finite rank, it follows that $[G, \alpha]$ is finite, whence $[D, \alpha] = 1$. Let $L = C_G(G/D)$: if α is to be inner, it will have to be induced by an element of L. Notice that L is recursively enumerable since G/D has soluble word problem. Also $[D, L] = 1$ by the argument that led to $[D, \alpha] = 1$.

Now consider $F = \mathrm{Der}(G/D, D)$; this may be identified with a subgroup of the direct sum B of n copies of D. In the usual way associate with α and with each l in L elements α^* and l^* of F and note that $L/Z(G) \simeq L^* \leq B$, where L^* is the image of $l \mapsto l^*$. The problem is to decide if $\alpha^* \in L^*$. This can be done since L^* is a recursively enumerable subgroup of B, so that a recursive presentation of B/L^* is available and the word problem for B/L^* can be solved [**11**].

3. Submodule computability. We pause to give a brief discussion of submodule computability, and to note an extension of a result of Baumslag, Cannonito and Miller that will be required in the proof of Theorem B.

Let Q be a finitely generated group with soluble word problem; assume also that $\mathbb{Z}Q$ is right noetherian, so that Q satisfies the maximal condition on subgroups. Let $\mathbf{S}$ be a set of subgroups of Q. Then $\mathbb{Z}Q$ will be called $\mathbf{S}$-*submodule computable* if there are uniform recursive procedures which, when given (a) a finite presentation of a right $\mathbb{Z}Q$-module M, (b) a finite subset $\{a, a_1, \ldots, a_r\}$ of M, and (c) a subgroup H in $\mathbf{S}$,

(i) find a finite presentation of the $\mathbb{Z}H$-submodule M_0 generated by $a_1, \ldots, a_r$, and

(ii) decide if a belongs to M_0.

In the case where $\mathbf{S} = \{Q\}$ this reduces to submodule computability of $\mathbb{Z}Q$ in the sense of Baumslag, Cannonito and Miller [**4**]. The result needed in the proof of Theorem B is

THEOREM 1. *Let Q be a polycyclic-by-finite group, and let $\mathbf{S}$ be the set of all subnormal subgroups of Q. Then $\mathbb{Z}Q$ is $\mathbf{S}$-submodule computable.*

The result is essentially implicit in [**4**], as we shall now indicate. Suppose that we are given M, $\{a, a_1, \ldots, a_r\}$ and H as in the definition. First find a finite presentation of H (see [**2**] or [**5**]). The proof of Theorem 2.14 of [**4**] yields a recursive procedure to accomplish (i) above. (Note that this proof is valid when K is a subnormal subgroup of a polycyclic-by-finite group G.) As for (ii), one uses (i) to find a finite presentation for the $\mathbb{Z}H$-module generated by $a, a_1, \ldots, a_r$; then one decides whether a belongs to the $\mathbb{Z}H$-module generated by $a_1, \ldots, a_r$, using the fact that $\mathbb{Z}H$ is submodule computable [**4**, Theorem 2.12].

Theorem 1 is useful for other purposes. For example, it can be used to give a proof of Romanovskiĭ's theorem on the solubility of the generalized word problem for finitely generated abelian-by-nilpotent groups (see [**13**] and [**12**]). It is an open question whether Theorem 1 is true when $\mathbf{S}$ is the set of all subgroups of Q; if so, it would follow that the generalized word problem is soluble for finitely generated abelian-by-polycyclic groups.

4. Nilpotent-by-polycyclic groups. Our aim in this final section is to prove Theorem B. A related module theoretic result is the main tool in the proof.

THEOREM 2. *Let Q be a polycyclic-by-finite group, H a subgroup of Q, and M a finitely generated right $\mathbb{Z}Q$-module. Then there is an algorithm to decide if a given $\mathbb{Z}Q$-automorphism of M is induced by an element of H.*

Proof. Observe that M is a finitely presented $\mathbb{Z}Q$-module since $\mathbb{Z}Q$ is right noetherian. It is clearly no loss to suppose that Q acts faithfully on M. Let α be a $\mathbb{Z}Q$-automorphism of M given by its effect on the module generators. If α is induced by an element of H, that element must belong to $H \cap Z(Q)$. Thus it can be assumed that $H \leq Z(Q)$.

(i) *Reduction to the case of a cyclic module.* Since M is a noetherian $\mathbb{Z}Q$-module, there is a series of submodules

$$0 = M_0 < M_1 < \ldots < M_l = M$$

with each M_{i+1}/M_i a cyclic module. Assume that the case of a cyclic module has been settled, and argue by induction on $l > 1$. Write $M_1 = a(\mathbb{Z}Q)$. The first step is to decide if

$M_1\alpha = M_1$, or equivalently if $a\alpha \in M_1$, because of the noetherian condition. This can be done by submodule computability of $\mathbb{Z}Q$, for example. One can suppose that $M_1\alpha = M_1$; otherwise α cannot be induced by any element of Q.

By induction hypothesis it may be decided if α induces in M/M_1 an automorphism arising from an element of H. Assuming that this is true, one can modify α so as to make it operate trivially on M/M_1. Then, using the algorithm for the case of a cyclic module, one decides if the restriction of α to M_1 is induced by an element of $C_H(M/M_1)$. Again one can assume this to be true and modify α to make it act trivially on M_1 as well as M/M_1.

Put $K = C_H(M/M_1) \cap C_H(M_1)$. The problem is now to decide if some element of K induces α. Let $k \in K$; then the mapping $a + M_1 \mapsto a(k-1)$ is a $\mathbb{Z}Q$-homomorphism k^* from M/M_1 to M_1 since $H \leq Z(Q)$. Moreover $kC_K(M) \mapsto k^*$ is a monomorphism from $K/C_K(M)$ to $V = \mathrm{Hom}_{\mathbb{Z}Q}(M/M_1, M_1)$. The image K^* is a finitely generated additive subgroup of the finitely presented $\mathbb{Z}Q$-module V. Now the mapping $a + M_1 \mapsto a(\alpha - 1)$ is an element α^* of V; also α^* can be described explicitly in terms of the generators of V. The problem is now to decide if $\alpha^* \in K^*$. By Theorem 1 this is decidable.

(ii) *The case of a cyclic module.* Let $M = a(\mathbb{Z}Q)$ and let R denote the annihilator of a in $\mathbb{Z}Q$. Thus R is a right ideal of $\mathbb{Z}Q$ and $M \simeq \mathbb{Z}Q/R$. Note that R is finitely generated. Now $(a)\alpha = au$ for some $u \in \mathbb{Z}Q$ and α is determined by u. The element u can be found by enumeration. Because H is contained in $Z(Q)$, the $\mathbb{Z}Q$-automorphism α is induced by an element of H if and only if $u \in H + R$. The first step is to decide if $u + R \in \mathbb{Z}H + R/R$. It is a consequence of Theorem 1 that this is possible. Thus one can assume that $u \in \mathbb{Z}H + R$, and find by enumeration u_0 in $\mathbb{Z}H$ such that $u \in u_0 + R$. Replacing u by u_0, one may suppose that $u \in \mathbb{Z}H$. Put $R_0 = R \cap \mathbb{Z}H$, and observe that $u \in H + R$ if and only if $u \in H + R_0$. Now replace Q by H and M by $a(\mathbb{Z}H) \simeq \mathbb{Z}H/R_0$. Thus Q can be assumed abelian.

(iii) *Conclusion.* The situation now is that $\mathbb{Z}Q$ is a finitely generated commutative noetherian ring and M is a noetherian $\mathbb{Z}Q$-module. A standard result in commutative algebra asserts that there is a series of submodules

$$0 = M_0 < M_1 < \ldots < M_l = M$$

such that $M_{i+1}/M_i \simeq \mathbb{Z}Q/P_i$ where P_i is a prime ideal of $\mathbb{Z}Q$. By the argument of (i) one can assume that $l = 1$ and $M = a(\mathbb{Z}Q) \simeq \mathbb{Z}Q/P$ with P a prime ideal.

Write $(a)\alpha = au$ with $u \in \mathbb{Z}Q$. The problem is to decide if $u \in Q + P$. Since $\mathbb{Z}Q/P$ is a finitely generated domain, its group of units is finitely generated, by Samuel's theorem. Thus one can decide if $u + P \in Q + P/P$, as required.

Proof of Theorem B. By hypothesis there is a normal nilpotent subgroup N such that G/N is polycyclic-by-finite. The proof is by induction on c, the nilpotent class of N. If $c = 0$, then G is polycyclic-by-finite and the result follows from Theorem A. Assume that $c > 0$. Write $M = Z(N)$ and $Q = G/N$, so that M is a $\mathbb{Z}Q$-module in the natural way.

Let there be given an automorphism α of G by its effect on the generators of a finitely generated recursive presentation of G. Since Q is finitely presented, N is generated as a G-operator group by a finite subset, say $\{y_1, \ldots, y_r\}$. Using solubility of the word problem in G/N, one can decide if all the y_i^α belong to N; a negative answer will mean that α is outer. If this is true, then $N^\alpha \leq N$, and therefore $N^\alpha = N$ because of max-n. Hence $M^\alpha = M$.

By induction one can suppose that α induces an inner automorphism in G/M. After modification by an inner automorphism of G, one obtains $[G, \alpha] \leq M$, so that α induces a $\mathbb{Z}Q$-automorphism in M. By Theorem 2 it is possible to decide if the restriction of α to M is induced by conjugation by an element of $C_G(G/M)$: this must be the case if α is to be inner. Assume this to be true and modify α in the usual way to reach the situation where α acts trivially on both M and G/M. Consequently $[N', \alpha] = 1$.

Let $V = \mathrm{Hom}_{\mathbb{Z}Q}(N_{ab}, M)$, a finitely generated $\mathbb{Z}Q$-module. The assignment $xN' \mapsto [x, \alpha]$ is a well-defined element $\bar{\alpha}$ of V; for if $g \in G$ and $x \in N$,

$$[x^g, \alpha] = [x, [g^{-1}, \alpha^{-1}]\alpha]^g = [x, \alpha]^g.$$

Next write $H = C_G(M) \cap C_G(G/M)$. By max-n $Z(G/M)$ is finitely generated, from which it follows that $H/C_H(N)$ is finitely generated. If $h \in H$, the assignment $xN' \mapsto [x, h]$ yields an element $\bar{h}$ of V; also $h \mapsto \bar{h}$ determines a monomorphism $H/C_H(N) \to V$, with image $\bar{H}$ say. Note that $\bar{H}$ is a finitely generated subgroup of V. Thus Theorem 2 allows a decision to be made as to whether $\bar{\alpha} \in \bar{H}$. This must be true if $\bar{\alpha}$ is inner, so modify α and assume that α acts trivially on N, as well as on G/M.

Now write $D = \mathrm{Der}(G/N, M)$, and let α^* denote the derivation $xN \mapsto [x, \alpha]$. If K denotes $C_G(N) \cap C_G(G/N)$, then for each $k \in K$ the mapping $xN \mapsto [x, k]$ is a derivation k^* in D, and $k \mapsto k^*$ induces a monomorphism $K/Z(G) \to D$, with image K^* say. Now $I \leq K^* \leq D$ where $I = \mathrm{Inn}(Q, M)$. Also $D/I \simeq H^1(Q, M)$, which is finitely generated since Q is polycyclic-by-finite and M is finitely generated $\mathbb{Z}Q$-module [**4**, Corollary 5.5]. Assuming a finite presentation is known for D/I, one can decide if $\alpha^* + I \in K^*/I$, and so if α is induced by an element of K.

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