GENERAL RADICALS THAT COINCIDE WITH THE CLASSICAL RADICAL ON RINGS WITH D.C.C.

N. DIVINSKY

General radical theories were obtained by Amitsur (**1**; **2**; **3**) and Kurosh (**6**). Following Kurosh we say that a property $\mathfrak{S}$ of rings is a radical property if:

(a) *Every homomorphic image of an $\mathfrak{S}$-ring is an $\mathfrak{S}$-ring;*

(b) *Every ring R contains an $\mathfrak{S}$-ideal S which contains every other $\mathfrak{S}$-ideal of R;*

(c) *The factor ring R/S is $\mathfrak{S}$-semi-simple (that is, has no non-zero $\mathfrak{S}$-ideals).*

The property $\mathfrak{N}$ of being nil is a radical property and for rings with D.C.C. (the descending chain condition on left ideals) this becomes the so-called Classical Radical. Nilpotency is not a radical property for the union of all the nilpotent ideals of a general ring need not be nilpotent. However, for rings with D.C.C. all nil radicals are nilpotent.

The question we are concerned with is which general radical properties coincide with $\mathfrak{N}$ on rings with D.C.C. If $\mathfrak{S}$ and $\mathfrak{T}$ are two radical properties we say $\mathfrak{S} \leqq \mathfrak{T}$ if every $\mathfrak{S}$-radical ring is also $\mathfrak{T}$-radical, and if we work with the class of all (associative) rings this is equivalent to the statement that for any ring R, the $\mathfrak{S}$-radical of R is contained in the $\mathfrak{T}$-radical of R. We say $\mathfrak{S} = \mathfrak{T}$ if a ring is $\mathfrak{S}$-radical if and only if it is $\mathfrak{T}$-radical, or if, for every ring R, its $\mathfrak{S}$-radical equals its $\mathfrak{T}$-radical. However, if we consider only rings with D.C.C. these statements are not equivalent for an ideal of a ring with D.C.C. may not have D.C.C. itself. Thus it is possible to have properties $\mathfrak{S}$ and $\mathfrak{T}$ such that a ring with D.C.C. is $\mathfrak{S}$-radical if and only if it is $\mathfrak{T}$-radical, but there exist rings R with D.C.C. whose $\mathfrak{S}$-radicals are smaller than their $\mathfrak{T}$-radicals. We seek general radical properties $\mathfrak{Q}$ which coincide with $\mathfrak{N}$ on rings with D.C.C. in the strong sense that for any ring R with D.C.C., its $\mathfrak{Q}$ radical equals its $\mathfrak{N}$ radical. To this end we shall use Kurosh's upper and lower radical constructions.

Given any set of rings P, the *lower radical property* determined by P is defined as follows:

A ring is of first degree over P if it is a homomorphic image of some ring in P or if it is zero (this is to complete the definition in case P is vacuous). We say a ring R is of degree $\beta > 1$ over P if every non-zero homomorphic

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image of R contains a non-zero ideal which is a ring of degree $\beta - 1$ over P. If β is a limit ordinal we say a ring is of degree β over P if it is of some degree $\alpha < \beta$, over P. Then we consider all rings of any degree over P and we say a ring is radical if it is of some degree over P. This yields a radical property for which all rings in P are radical and it is less than or equal to any other radical property for which all rings in P are radical.

Given any set of rings Q with the following property:

(d) *Every non-zero ideal of a ring of Q can be homomorphically mapped onto some non-zero ring of Q;*

the *upper radical property* determined by Q is defined as follows:

We consider the class $\bar{Q}$, the set of all rings R such that every non-zero ideal of R can be homomorphically mapped onto some non-zero ring of Q. We then say a ring is radical if it cannot be homomorphically mapped onto a non-zero ring of $\bar{Q}$. This yields a radical property for which all rings in Q are semi-simple and it is bigger than or equal to any other radical property for which all rings in Q are semi-simple.

We make the following definitions:

$\mathfrak{L}$ = *the lower radical property determined by all the zero simple rings.*

$\mathfrak{D}$ = *the lower radical property determined by all nilpotent rings which are nil radicals of rings with D.C.C.*

$\mathfrak{B}$ = *the lower radical property determined by all nilpotent rings.*

$\mathfrak{N}$ = *the lower radical property determined by all nil rings.*

$\mathfrak{U}$ = *the upper radical property determined by all finite dimensional total matric rings over division rings* (*since this class consists only of simple rings it clearly has property D.*)

It is clear then that

$$\mathfrak{L} \leqslant \mathfrak{D} \leqslant \mathfrak{B} \leqslant \mathfrak{N} \leqslant \mathfrak{U}.$$

If $\mathfrak{Q}$ is any radical property that coincides with $\mathfrak{N}$ on rings with D.C.C. then all zero simple rings are $\mathfrak{Q}$-radical, for zero simple rings have D.C.C. and are $\mathfrak{N}$-radical. Therefore $\mathfrak{L} \leqslant \mathfrak{Q}$. On the other hand, all finite dimensional total matric rings over division rings have D.C.C., are nil semi-simple and are therefore $\mathfrak{Q}$ semi-simple. Thus $\mathfrak{Q} \leqslant \mathfrak{U}$. Consequently $\mathfrak{L} \leqslant \mathfrak{Q} \leqslant \mathfrak{U}$. However, every nilpotent ring is of course $\mathfrak{N}$-radical and if it is a nil radical of a ring with D.C.C. then it must also be $\mathfrak{Q}$ radical. Therefore $\mathfrak{D} \leqslant \mathfrak{Q} \leqslant \mathfrak{U}$.

THEOREM 1. *A general radical property $\mathfrak{Q}$ coincides with the nil radical $\mathfrak{N}$ on rings with D.C.C. if and only if $\mathfrak{D} \leqslant \mathfrak{Q} \leqslant \mathfrak{U}$.*

Proof. We have already proved half of this theorem. To prove the other half it is sufficient to show that both $\mathfrak{D}$ and $\mathfrak{U}$ coincide with $\mathfrak{N}$ on rings with

D.C.C. To see that $\mathfrak{N}$ and $\mathfrak{U}$ coincide let R be any ring with D.C.C. Let N be its $\mathfrak{N}$-radical and U be its $\mathfrak{U}$-radical. $N \subseteq U$. We consider R/N which is well known to be a finite direct sum of D_i's, where the D_i are finite dimensional total matrix rings over division rings. Now U/N is an ideal of R/N and must then be a finite direct sum of the D_i's that it contains. Then U can be homomorphically mapped, via U/N, onto one of the D_i which is $\mathfrak{U}$-semi-simple. However, U is $\mathfrak{U}$-radical and so is every homomorphic image of U and the only ring that is both radical and semi-simple is the ring consisting only of zero. Thus U/N must be zero, $U = N$.

To see that $\mathfrak{N}$ and $\mathfrak{D}$ coincide again let R be any ring with D.C.C. Let N be its $\mathfrak{N}$-radical and D its $\mathfrak{D}$-radical. $D \subseteq N$. Now N is nilpotent and it is a nil radical of a ring with D.C.C. and therefore N is $\mathfrak{D}$-radical, $N = D$. Q.E.D.

Kurosh makes the statement that a general radical property $\mathfrak{Q}$ coincides with the nil radical $\mathfrak{N}$ on rings with D.C.C. if and only if $\mathfrak{L} \leqq \mathfrak{Q} \leqq \mathfrak{U}$. However, he overlooked the fact that though this is true in the weak sense that every $\mathfrak{Q}$-radical ring with D.C.C. is $\mathfrak{L}$-radical, this is false in the strong sense as the following example shows:

Let A be the set of all $\alpha x + \beta e$ where α and β are rational numbers and where $x^2 = 0$, $e^2 = e$, $ex = xe = x$. This is a commutative ring which is a two-dimensional vector space over the rationals. The only non-zero proper ideal of A is $N = \{\alpha x\}$. Clearly N is the nil radical of A and A has D.C.C. We want to show that A is $\mathfrak{L}$-semi-simple. The $\mathfrak{L}$-radical of A is contained in N and thus it remains to show that N is $\mathfrak{L}$-semi-simple. Assume then that N contains some $\mathfrak{L}$-ideals, each of them being of some degree over the class of all zero simple rings. Let γ be the minimal ordinal such that N has an ideal I which is of degree γ. Clearly γ is not a limit ordinal. Since I is of degree γ, every non-zero homomorphic image of I must contain a non-zero ideal of degree $\gamma - 1$ and in particular I itself must contain a non-zero ideal J of degree $\gamma - 1$. However, since $N^2 = 0$, $JN = 0$ and J is therefore an ideal of N. Thus N contains a non-zero ideal of degree $\gamma - 1$, which contradicts the minimality of γ, unless $\gamma = 1$. Thus if N contains any $\mathfrak{L}$-ideals it must contain one of degree 1. However, any homomorphic image of a zero simple ring is a zero simple ring and thus the only rings of degree 1 are the zero simple rings themselves. However, any non-zero ideal of N is merely an additive subgroup and contains at least the infinite cyclic additive group generated by a non-zero element, and thus cannot be simple. Therefore N has no $\mathfrak{L}$-ideals and N is $\mathfrak{L}$-semi-simple.

Thus A is a ring with D.C.C. whose $\mathfrak{N}$-radical is not equal to its $\mathfrak{L}$-radical. This proves also that $\mathfrak{L} \lneqq \mathfrak{D}$.

What is true is that every $\mathfrak{N}$-radical ring R with D.C.C. is also $\mathfrak{L}$-radical. For if R is nilpotent, $R^m = 0 \neq R^{m-1}$. By D.C.C. R^{m-1} contains a minimal ideal I of R. If J is any ideal of I it is also an ideal of R for $JR \subseteq R^{m-1}R = 0$. Thus I is a zero simple ring. Thus R contains a zero simple ring. Similarly,

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every homomorphic image of R contains a zero simple ring and therefore R is of degree 2 over the zero simple rings and thus R is $\mathfrak{L}$-radical.

Of course in the example above, though N is contained in a ring with D.C.C., N itself does not have D.C.C. Curiously every homomorphic image of that N, which is not isomorphic to N, is $\mathfrak{L}$-radical.

It is clear that, in general, $\mathfrak{U} \neq \mathfrak{N}$ for the set of all rational numbers of the form $2m/(2n+1)$ is a Jacobson radical ring which is clearly $\mathfrak{U}$-radical, but is $\mathfrak{N}$-semi-simple. Also $\mathfrak{B} \neq \mathfrak{N}$, for $\mathfrak{N}$ is the Baer upper radical and $\mathfrak{B}$ is the Baer lower radical and Baer (**5**, § 2) has given an example where they are different. To see that $\mathfrak{B}$ is the Baer lower radical we first point out that $\mathfrak{B}$ is identical with the lower radical property determined by the zero ring on an infinite cyclic additive group W. Clearly if all nilpotent rings are radical then in particular W is radical. On the other hand, every nilpotent ring contains an ideal which is a zero ring on a cyclic additive group and this is a homomorphic image of W. Thus every nilpotent ring is of degree 2 over W and thus if W is radical so are all nilpotent rings. Kurosh has pointed out that the lower radical determined by W is precisely Baer's lower radical.

To see that all five radical properties are different in general, that is,

$$\mathfrak{L} \lneqq \mathfrak{D} \lneqq \mathfrak{B} \lneqq \mathfrak{N} \lneqq \mathfrak{U},$$

we must finally show that $\mathfrak{D} \neq \mathfrak{B}$.

THEOREM 2. *The radical property* $\mathfrak{D} \lneqq \mathfrak{B}$, *the Baer lower radical.*

Proof. We know that $\mathfrak{D} \leqslant \mathfrak{B}$ and the question as to their equality will be settled if we show that W, the zero ring on an infinite cyclic additive group, is $\mathfrak{D}$-semi-simple.

Suppose then that W contains some $\mathfrak{D}$-ideals. However, every non-zero ideal of W is isomorphic to W and thus if W has a non-zero $\mathfrak{D}$-ideal, it must be $\mathfrak{D}$-radical itself. Let α be the minimal ordinal such that W is of degree α over the class of all nilpotent rings which are nil radicals of rings with D.C.C. Then clearly α is not a limit ordinal. Every non-zero homomorphic image of W then contains a non-zero ideal of degree $\alpha - 1$ and in particular W contains such an ideal and therefore W itself is of degree $\alpha - 1$ which contradicts the minimality of α, unless $\alpha = 1$. Then W is a homomorphic image of a nil radical of a ring with D.C.C.

Let R be a ring with D.C.C., let A be its nil radical and let H be an ideal of A such that $A/H \cong W$. Then $A^2 \subseteq H$. We consider then the ring R/A^2. It also has D.C.C. Its nil radical is known to be A/A^2 and this can be homomorphically mapped onto W for $A/A^2/H/A^2 \cong A/H \cong W$. Thus we may assume without loss of generality that $A^2 = 0$.

Every element of A is of the form $mx + h$ where m is an integer, h is in H, and where x is a representative of the generator of the infinite cyclic additive group A/H. Note that if mx is in H then m must be zero.

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Consider the sequence of left ideals of R:

$$R > Rx > R\cdot 2x > R\cdot 2^2x > \ldots > R\cdot 2^nx > \ldots .$$

None of these can be zero for if $R\cdot 2^nx = 0$ then

$$\{2^nx\} > \{2^{n+1}x\} > \ldots > \{2^{n+r}x\} > \ldots$$

where $\{2^{n+r}x\}$ is the additive group generated by $2^{n+r}x$, is a properly descending chain of non-zero left ideals of R, which contradicts D.C.C.

Again by D.C.C. there must exist an integer n such that

$$R\cdot 2^nx = R\cdot 2^{n+1}x.$$

Thus for every element f in R there must exist an element g in R such that $f\cdot 2^nx = g\cdot 2^{n+1}x$.

We note that $R \neq A$ else A has D.C.C. and therefore $A/H \cong W$ has D.C.C. which is impossible. Therefore R contains an idempotent e such that every element a of R is: $a = ae + (a - ae)$ where $a - ae$ is in A (**4**, pp. 17–19). Thus $ax = aex$, since $A^2 = 0$. Then $a(x - ex) = 0$ for every a, $R(x - ex) = 0$.

If $ex = x + h$ then there must exist an element b in R such that $e\cdot 2^nx = b\cdot 2^{n+1}x$. However, bx is in A and therefore $bx = mx + h'$, where m is an integer. Then $e\cdot 2^nx = 2^nx + 2^nh = b\cdot 2^{n+1}x = 2^{n+1}mx + 2^{n+1}h'$. Therefore $x(2^n - m2^{n+1}) = 2^{n+1}h' - 2^nh$ which is in H. Thus $2^n - m2^{n+1} = 0$ which is impossible.

On the other hand, if $ex \neq x + h$ then $ex = qx + h_1$ where $q \neq 1$ and $x - ex \neq 0$. Then $\{x - ex\} > \{2(x - ex)\} > \{2^2(x - ex)\} > \ldots > \{2^n(x - ex)\} > \ldots$ is a descending chain of left ideals of R; where again $\{2^n(x-ex)\}$ is the additive group generated by $2^n(x - ex)$. Each is non-zero for if $2^n(x - ex) = 0$ then $2^n(x - qx - h_1) = 2^n(1 - q)x - 2^nh_1 = 0$. Thus $2^n(1 - q)x$ is in H which is impossible unless $q = 1$.

This is a properly descending chain for if $\{2^n(x - ex)\} = \{2^{n+1}(x - ex)\}$, then there must exist an integer k such that

$$2^n(x - ex) = k2^{n+1}(x - ex).$$

$$\therefore 2^n(x - qx - h_1) = k\cdot 2^{n+1}(x - qx - h_1).$$

$$\therefore x[2^n(1 - q) - k2^{n+1}(1 - q)] = h_1(2^n - k2^{n+1})$$

which is in H.

$$\therefore 2^n(1 - q)(1 - 2k) = 0$$

which is impossible for an integer k.

Thus in either case we have a contradiction. Therefore W is $\mathfrak{D}$-semi-simple and the theorem is proved.

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University of British Columbia