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Moons, bridges, birds ... and nonexpansive mappings in Hilbert space

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In recent years some fixed point theorems have been proved for nonexpansive mappings in Hilbert space, which have non-convex domains. The purpose of this paper is to present a simple but very useful new result of that kind and to indicate some of its consequences.

1. Introduction

Throughout our discussion H will denote a Hilbert space, X will denote a nonempty bounded subset of H, and for a subset Y of H we use $\overline{\text{conv}}(Y)$ to denote the convex closure of Y. A mapping $f : X \to H$ such that $|f(x)-f(y)| \leq |x-y|$ for all $x, y \in X$, is called *nonexpansive*. X is said to have the *fixed point property for nonexpansive mappings*, if every nonexpansive self-map of X has a fixed point.

The precise generality of the class of sets having the fixed point property for nonexpansive mappings is not known, but it does include all closed bounded and convex sets ([2], [3], [5]), all contractible finite unions of closed bounded and convex sets ([6]), and all closed bounded and starshaped sets ([4]).

In this short note we prove - among others - that a non-empty bounded subset of H has the fixed point property for nonexpansive mappings, if it is Chebyshev with respect to its convex closure. Thus we establish the fixed point property for nonexpansive mappings for several sets which are not necessarily weakly closed or starshaped, and get a useful tool for

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constructing a lot of irregular sets having the fixed point property for nonexpansive mappings.

2. Main result

THEOREM. *Let* $f : X \to X$ *be nonexpansive. Then the following properties of* f *are equivalent:*

(1) f *has a fixed point in* X ;

(2) *there is* $x_0 \in X$ *such that for any*

$$y \in \overline{\operatorname{conv}}\left(\left\{f^n(x_0) : n \in Z^+\right\}\right)$$

there is a unique $x \in X$ *such that* $|y-x| = \operatorname{dist}(y, X)$.

Proof. (1) $\Rightarrow$ (2): obvious. (2) $\Rightarrow$ (1): In view of Kirszbraun's Theorem ([7]) there is a nonexpansive mapping $g : H \to \overline{\operatorname{conv}}(X)$ such that $g|_X = f$. Then the sequence $\left(\frac{1}{n+1}\sum_{k=0}^{n} g^k(x_0)\right)$ converges weakly to a fixed point y of g ([1]). Since $g^k(x_0) = f^k(x_0)$ for $k \in Z^+$, we have $y \in \overline{\operatorname{conv}}\left(\left\{f^n(x_0) : n \in Z^+\right\}\right)$, and hence there is a unique $x \in X$ such that $|y-x| = \operatorname{dist}(y, X)$. Then

$$\operatorname{dist}(y, X) \le |y-f(x)| = |g(y)-g(x)| \le |y-x| = \operatorname{dist}(y, X) ;$$

that is, $|y-f(x)| = \operatorname{dist}(y, X)$, and therefore $x = f(x)$.

3. Some consequences

1. Let $S = \{x \in H : |x| = 1\}$ and $f : S \to S$ be nonexpansive. Then f has a fixed point in S iff there is $x_0 \in S$ such that $0 \notin \overline{\operatorname{conv}}\left(\left\{f^n(x_0) : n \in Z^+\right\}\right)$.

2. Let X be Chebyshev with respect to its convex closure (that is, for any $y \in \overline{\operatorname{conv}}(X)$ there is a unique $x \in X$ such that $|y-x| = \operatorname{dist}(y, X)$). Then X has the fixed point property for nonexpansive mappings.

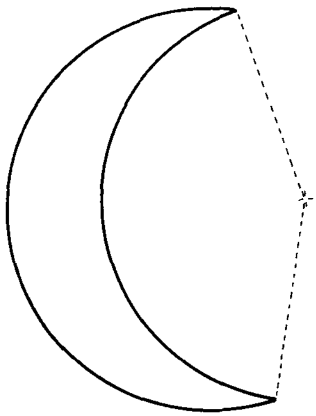
3. Let x^* be a linear functional of unit norm on H and

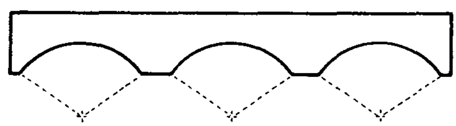
$0 < c < 1$. Then 2 implies that the set

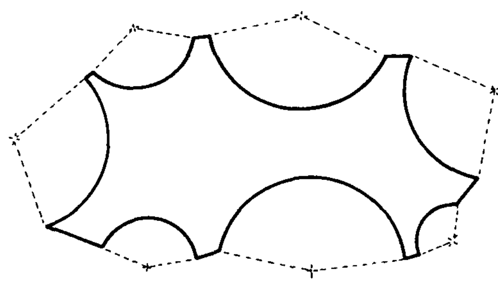
$$\{x \in H : |x| = 1 \text{ and } \operatorname{re} x^*(x) \geq c\}$$

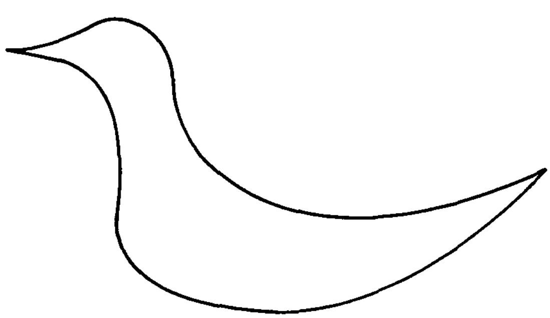
has the fixed point property for nonexpansive mappings. This has been established in [8], if $\sqrt{15}/4 < c < 1$.

4. The following (two-dimensional) figures indicate further (infinite-dimensional) subsets of H , which have the fixed point property for nonexpansive mappings by 2:









4. Two problems

PROBLEM 1. Does there exist a convex body Y in H such that the boundary of Y is Chebyshev with respect to its convex closure?

PROBLEM 2. Does there exist a convex body Y in H such that the boundary of Y has the fixed point property for nonexpansive mappings?

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