A Generalisation of a Theorem of E. Toeplitz

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§ 1. *Introduction.*

In the *Mathematische Annalen*, **11** (1877), 440-444, E. Toeplitz has proved the following theorem: *If three quadrics are polar quadrics of three points with respect to a cubic surface, then their* (2, 2, 2) *invariant vanishes*; the invariant being of the second degree in the coefficients of each of the three quadrics.

The original proof, as given by Toeplitz, is complicated, but its underlying principles are revealed very clearly by combining the notation of the symbolic invariant theory with that of matrices. This brings out at once the interesting fact that the property is general, and applies to quadrics and cubics in space of, not merely three, but *any* odd number of dimensions.

§ 2. *Notation.*

Let the non-symbolic forms of three quadrics in a space of $m-1$ dimensions be $f_1 = \Sigma\, a_{ij}\, x_i\, x_j$, $f_2 = \Sigma\, b_{ij}\, x_i\, x_j$, and $f_3 = \Sigma\, c_{ij}\, x_i\, x_j$, where $i, j = 1, 2, \ldots, m$.

Symbolically these are

$$\begin{aligned} f_1 &\equiv a_x^2 = a'^2_x = a''^2_x = \ldots\ldots = (a_1 x_1 + a_2 x_2 + \ldots\ldots + a_m x_m)^2, \\ f_2 &\equiv b_x^2 = b'^2_x = b''^2_x = \ldots\ldots, \\ f_3 &\equiv c_x^2 = c'^2_x = c''^2_x = \ldots\ldots, \end{aligned}$$

where $\quad a_i\, a_j = a_{ij} = a_j\, a_i = a_{ji}$, etc.

§ 3. *Statement of the theorem.*

The necessary and sufficient condition that the three quadrics, a_x^2, b_x^2, c_x^2, in the $(m-1)$ manifold be the polar quadrics of three points ξ, η, ζ with respect to a cubic primal t_x^3 is that

$$(A_n\, B_n)\,(B_n\, C_n)\,(C_n\, A_n) = 0$$

where $\quad$ (1) $\quad m = 2n$,

and $\quad$ (2) $\quad A_n = aa'\, a'' \ldots\ldots a^{(n-1)}$, a convolution of n equivalent symbols a.

§ 4. *Proof.*

The polar quadric of a point $\xi = \{\xi_1, \xi_2, \ldots, \xi_m\}$ with respect to t_x^3 is $t_x^2 t_\xi$, and this is to be the same quadric as a_x^2. Hence $t_x^2 t_\xi = \pi a_x^2$. Similarly, $t_x^2 t_\eta = \rho b_x^2$ and $t_x^2 t_\zeta = \chi c_x^2$, where π, ρ, χ are scalar.

Polarising these three equations with respect to an arbitrary point y, we get the following relations:

$$t_x t_y t_\xi = \pi a_x a_y; \quad t_x t_y t_\eta = \rho b_x b_y; \quad t_x t_y t_\zeta = \chi c_x c_y \tag{1}$$

whence we see that

$$\rho b_x b_\zeta = \chi c_x c_\eta ; \quad \chi c_x c_\xi = \pi a_x a_\zeta; \quad \pi a_x a_\eta = \rho b_x b_\xi.$$

In matrix notation, the first of these becomes $\rho x' Q \zeta = \chi x' R \eta$, where the accent attached to x denotes *transposition*; that is, if x is a column, then x' is a row vector. This is indicated by the kind of brackets we use:

$$x' = [x_1, x_2, \ldots, x_m],$$
$$\zeta = \{\zeta_1, \zeta_2, \ldots, \zeta_m\}.$$

Again, $$Q = (b_{ij}) = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdot & & \cdot \\ b_{31} & \cdot & \cdot & & \cdot \\ \cdot & & & & \\ b_{m1} & \cdot & \cdot & \cdots & b_{mm} \end{bmatrix}.$$

Hence we have the following equations:

$$x'\left(Q\frac{\zeta}{\chi} - R\frac{\eta}{\rho}\right) = 0, \tag{2}$$

$$x'\left(R\frac{\xi}{\pi} - P\frac{\zeta}{\chi}\right) = 0, \tag{3}$$

$$x'\left(P\frac{\eta}{\rho} - Q\frac{\xi}{\pi}\right) = 0. \tag{4}$$

Eliminating the $3m$ scalar quantities $\frac{\xi_i}{\pi}$, $\frac{\eta_i}{\rho}$, $\frac{\zeta_i}{\chi}$, $(i = 1, 2, \ldots, m)$, from the equations (2), (3), (4), we get a vanishing determinant of $3m$ rows and columns, conveniently represented as follows:

$$|X| = \begin{vmatrix} 0 & -R & Q \\ R & 0 & -P \\ -Q & P & 0 \end{vmatrix} = 0.$$

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We shall shew that this skew-symmetric determinant, when expanded and expressed in symbolic form, is equal to

$$\left(\frac{1}{n!}\right)^6 . \left((A_n B_n)(B_n C_n)(C_n A_n) \right)^2.$$

Conversely, if this invariant vanishes, the argument can be reversed; we should get consistent equations (1), giving possible values for the coefficients of the cubic form.

Pre-multiplying the matrix X of the above equations by KH where

$$H = \begin{bmatrix} 1 & QP^{-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad K = \begin{bmatrix} 1 & 0 & RP^{-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we have
$$KHX = \begin{bmatrix} QP^{-1}R - RP^{-1}Q & 0 & 0 \\ R & 0 & -P \\ -Q & P & 0 \end{bmatrix}.$$

But K, H are evidently unimodular matrices; that is, $|K| = |H| = 1$.

Hence
$$|X| = |QP^{-1}R - RP^{-1}Q| . |P|^2. \tag{5}$$

The sign in front of the determinant is settled by the fact that the matrix P is of even order $2n$. Again, the matrix $S = QP^{-1}R - RP^{-1}Q$ is skew symmetric and of even order: hence its determinant is a perfect square.

Further, we can easily shew that $|S|$ is an invariant of the three quadrics.

Let the matrix equation $x = M\bar{x}$ denote a linear transformation from the variables x to the new variables $\bar{x}$; and let $x'Px = \bar{x}'\bar{P}\bar{x}$. Hence $\bar{x}'M'PM\bar{x} = \bar{x}'\bar{P}\bar{x}$: or $M'PM = \bar{P}$. Similarly for Q and R.

Therefore
$$\begin{aligned} \bar{Q}\bar{P}^{-1}\bar{R} &= (M'QM)(M^{-1}P^{-1}M'^{-1})(M'RM) \\ &= M'QP^{-1}RM. \end{aligned}$$

Hence
$$\begin{aligned} \bar{S} &= \bar{Q}\bar{P}^{-1}\bar{R} - \bar{R}\bar{P}^{-1}\bar{Q} \\ &= M'(QP^{-1}R - RP^{-1}Q)M \\ &= M'SM: \end{aligned}$$

or
$$|\bar{S}| = |S| . |M|^2.$$

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The transformation also replaces $|P|$ by $|P|.|M|^2$, and hence replaces $|X|$ by $|S||M|^2.|P|^2|M|^4$; that is

$$|\bar{X}| = |S||P|^2|M|^6,$$

which shews that $|X|$ is an invariant of weight six. Its square root is an invariant of weight three and may be calculated from a convenient canonical form.

Since any two general quadrics can be reduced to forms giving matrices of the unit and diagonal types, let us leave R general, and take P as the unit matrix I, and Q as the diagonal matrix diag $(b_1, b_2, b_3, \ldots, b_m)$. Hence we get simply for (5), $|X| = |QR - RQ|$. But in this case,

$$QR = \begin{bmatrix} b_1 c_{11}, & b_1 c_{12}, & \ldots & b_1 c_{1m} \\ b_2 c_{12}, & . & , \ \ldots & . \\ b_3 c_{13}, & . & , \ \ldots & . \\ . \ , & . & , \ \ldots & . \\ b_m c_{1m}, & . & , \ \ldots & b_m c_{mm} \end{bmatrix}.$$

Hence $$|X| = |c_{ij}(b_i - b_j)|, \text{ where } i, j = 1, 2, \ldots, m.$$

We shall assume the lemma (proved hereafter in § 6) that, when this determinant is expanded and its terms re-arranged, we get

$$\sqrt{|X|} = (-)^{n/2} \Sigma b_i b_j \ldots . b_k (C_{ij \ldots . k})_{rs \ldots . t},$$

or

$$(-)^{(n-1)/2} \Sigma b_i b_j \ldots . b_k (C_{ij \ldots . k})_{rs \ldots . t},$$

according as $n = \frac{1}{2}m$ is even or odd. The following properties are to be noted.

(i) There are $\dfrac{m!}{n!\ n!}$ terms in each summation.

(ii) For our present purpose, the sign in front of Σ is of no importance, and we shall write $\sqrt{|X|} = \pm \Sigma b_i b_j \ldots . b_k (C_{ij \ldots . k})_{rs \ldots . t}$.

(iii) Again, $(ij \ldots . k)$ and $(rs \ldots . t)$ are each sequences of n letters and are algebraic complements for the sequence $(123 \ldots . m)$.

(iv) The sign convention is determinantal; that is, $(ij \ldots . krs \ldots . t)$ is regarded as positive, whereas $(ji \ldots . krs \ldots . t)$, for example. is negative.

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(v)
$$(C_{ij\ldots k})_{rs\ldots t} = \begin{vmatrix} c_{ir} & c_{is} & \ldots & c_{it} \\ c_{jr} & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ c_{kr} & \cdot & \ldots & c_{kt} \end{vmatrix}.$$

Symbolically then

$$(C_{ij\ldots k})_{rs\ldots t} = \frac{1}{n!}(c\,c'\,c'' \ldots c^{(n-1)})_{ij\ldots k}\,(c\,c'\,c'' \ldots c^{(n-1)})_{rs\ldots t}$$
$$= \frac{1}{n!} C_{ij\ldots k}\, C_{rs\ldots t},$$

and the symbol for the product $b_i b_j \ldots b_k$ is $\frac{1}{n!} B^2_{ij\ldots k}$, similarly.

Hence $$(n!)^2 \sqrt{|X|} = \pm \Sigma B^2_{ij\ldots k} C_{ij\ldots k} C_{rs\ldots t}. \tag{6}$$

Now

$$(A_n B_n)(B_n C_n)(C_n A_n) = \Sigma A_{ij\ldots k} B_{rs\ldots t} \Sigma B_{ij\ldots k} C_{rs\ldots t} \Sigma C_{ij\ldots k} A_{rs\ldots t}.$$

But for the case we have taken, where A_n is derived from the unit, and B_n from a diagonal, matrix, we have

(i) $A_{ij\ldots k} A_{rs\ldots t} = 0$ unless $(ij\ldots k) = (rs\ldots t)$;

(ii) $A^2_{ij\ldots k} = n!$;

(iii) $B_{ij\ldots k} B_{rs\ldots t} = 0$ unless $(ij\ldots k) = (rs\ldots t)$.

Hence
$$(A_n B_n)(B_n C_n)(C_n A_n) = \Sigma B^2_{ij\ldots k} C_{ij\ldots k} C_{rs\ldots t}.$$

Comparing this result with (6), we see that

$$\sqrt{|X|} = \pm \left(\frac{1}{n!}\right)^3 (A_n B_n)(B_n C_n)(C_n A_n)$$

or $$|X| = \left(\frac{1}{n!}\right)^6 \cdot ((A_n B_n)(B_n C_n)(C_n A_n))^2. \tag{7}$$

§ 5. *Special Cases.*

(i) *Binary Case*, $m = 2$.

The binary invariant $(ab)(bc)(ca)$ vanishes if a_x^2, b_x^2, c_x^2 (that is, three point-pairs on a line), are first polars of three points with respect to a binary cubic, t_x^3 (which, of course, represents three points on the same line): or, equivalently, if the three pairs of points are in *involution*.

(ii) *Quaternary Case*, $m = 4$.

In this case we get for (7),

$$|X| = \frac{1}{64}(\,(AB)\,(BC)\,(CA)\,)^2,$$

which is the result given by Toeplitz.

(iii) $m = 2n - 1$.

Since $|X| = |c_{ij}(b_i - b_j)|$, that is, is a skew symmetric determinant, which vanishes if m is odd, there is no definite result in the $(2n - 1)$-ary case.

§ 6. *Proof of Lemma.*

Let $|X| = |c_{pq}(b_p - b_q)| = |a_{pq}|$, where $p, q = 1, 2, \ldots, m$. Then we require to shew that

$$\sqrt{|X|} = (-)^{n/2}\,\Sigma\, b_i\, b_j \ldots\, b_k\,(C_{ij\ldots k})_{rs\ldots t},$$

or

$$(-)^{(n-1)/2}\,\Sigma\, b_i\, b_j \ldots\, b_k\,(C_{ij\ldots k})_{rs\ldots t},$$

according as n is even or odd. The summation sign and the sign conventions have already been explained.

Now $\sqrt{|X|}$ is a *Pfaffian*; and Pfaffians of order m can be calculated from those of order $m - 2$ in the following way.[1]

If $\sqrt{|X|} = [1, 2, \ldots, m] = \sqrt{|a_{pq}|} = \sqrt{|c_{pq}(b_p - b_q)|}$, then (8),
$[1, 2, \ldots, m] = a_{12}[3, 4, \ldots, m] + a_{13}[4, \ldots, m, 2] + \quad + a_{1m}[2, 3, \ldots, m-1]$
where, after the suffix 1 has been selected, the others follow cyclically.

Again,
$$[1, 2] = a_{12},$$
$$[1, 2, 3, 4] = a_{12}\,a_{34} + a_{13}\,a_{42} + a_{14}\,a_{23},$$
and so on.

We shall assume the lemma true for the $m - 2$ case; further, we shall take $n - 1$ to be even; that is,

$$[1, 2, \ldots, m - 2] = (-)^{(n-1)/2}\,\Sigma\, b_1\, b_2 \ldots\, b_{n-1}\,(C_{12\ldots, n-1})_{n, \ldots, 2n-2}.$$

[1] Scott and Mathews, *Theory of Determinants* (1904), p. 95.

But from (8), $[1, 2, \ldots, m]$

$$= (-)^{(n-1)/2} c_{12} (b_1 - b_2) \Sigma b_3 b_4 \ldots b_{n+1} (C_{34, \ldots, n+1})_{n+2, \ldots, 2n} + \ldots$$
$$+ (-)^{(n-1)/2} c_{1n} (b_1 - b_n) \Sigma b_{n+1} b_{n+2} \ldots b_{2n-1} (C_{n+1, \ldots, 2n-1})_{2n, 2, \ldots, n-1} + \ldots$$
$$+ (-)^{(n-1)/2} c_{1m} (b_1 - b_m) \Sigma b_2 b_3 \ldots b_n (C_{23, \ldots, n})_{n+1, \ldots, 2n-1}.$$

Here the factor $b_1 b_2 b_3 \ldots b_n$ occurs in n of the above terms and the sum of the coefficients is easily shewn to be $(-)^{(n-1)/2} (C_{12, \ldots, n})_{n+1, \ldots, 2n}$ if n is odd. But this is true since we took $n-1$ to be even. Similarly for the other terms, and so we get

$$[1, 2, \ldots, m] = (-)^{(n-1)/2} \Sigma b_1 b_2 b_3 \ldots b_n (C_{123 \ldots n})_{n+1, \ldots, m}.$$

The other case, when $n-1$ is odd, is proved in a similar manner.

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