THE AUTOMORPHISMS OF THE GROUP OF ROTATIONS AND ITS PROJECTIVE GROUP CORRESPONDING TO QUADRATIC FORMS OF ANY INDEX

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Let M be a vector space of dimension n over a field K of characteristic $\neq 2$ and f a non-degenerate quadratic form on M. The automorphisms of the orthogonal group $O_n(K,f)$, the rotation group $O_n^+(K,f)$, and their corresponding projective groups $PO_n(K,f)$, $PO_n^+(K,f)$, were determined by J. Dieudonné for n sufficiently large under the condition that the index of f is greater than zero (see **2**). It was shown by Rickart in (**4**) that for the group $O_n(K,f)$ the Dieudonné result still holds without this condition and J. Walter showed in (**5**) that the condition is also superfluous for the group $PO_n(K,f)$. In the present note, under the assumption that K has more than 5 elements, we give a characterization of the involutions of $O_n^+(K,f)$ of type $(2, n-2)$ or $(n-2, 2)$ without any restriction on the index of f which also works for the group $PO^+(K,f)$. This characterization simplifies the first part of the Dieudonné proofs of Theorems 16 and 18 of (**2**) and although it leaves out the case when K has only 3 or 5 elements, it makes it possible to extend his results to the case of a quadratic form of index 0 (see **3**, Chapter IV, §§ 5, 7). Hence the theorems mentioned above can be stated as follows.

THEOREM 1. *Every automorphism ϕ of the group of rotations $O_n^+(K,f)$, where K is a field of characteristic $\neq 2$ and $n \geqslant 5$, may be written in the form*

$$\phi(S) = \chi(S)\, TST^{-1}$$

where $\chi(S)$ is a representation of $O^+(K,f)$ in the multiplicative group $\{1, -1\}$ and T is a semi-similitude of f.

THEOREM 2. *Every automorphism of the projective group of rotations $PO^+(K,f)$, where K is a field of characteristic $\neq 2$ and $n \geqslant 5$, $n \neq 8$, is induced by an automorphism of $O^+(K,f)$.*

The proof of the lemma given below rests on the two following facts:

1. In a vector space of dimension 2 the group of rotations is commutative (see **1**, p. 121). Moreover, if K has more than 5 elements the commutator group $\Omega_2(K,f)$ of the orthogonal group contains elements which are not involutions. (If x_1, x_2 is an orthogonal basis of M with respect to f there exists an $\alpha \in K$, $\alpha \neq 0$, such that $x_1 + \alpha x_2$ and $x_1 - \alpha x_2$ are non-isotropic

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vectors and not orthogonal to each other. The product of the symmetries defined by $x_1 + \alpha x_2$ and $x_1 - \alpha x_2$ is an element of $\Omega_2(K, f)$ which is not an involution.)

2. If $n \geqslant 3$ the centre of the commutator group $\Omega_n(K, f)$ consists of either the identity transformation I or of I and $-I$. The same is true of the centralizer of $\Omega_n(K, f)$ in $O_n^+(K, f)$ (see **1**, Theorem 3.23).

LEMMA. *Let M be a vector space over a field K of characteristic $\neq 2$ which contains more than 5 elements. Let f be a non-degenerate quadratic form on M, U an involution of $O_n^+(K, f)$ and $(C_{O^+}(U))'$ the commutator of the centralizer of U in $O_n^+(K, f)$. Then the centre of $(C_{O^+}(U))'$ contains elements which are not involutions if and only if one of the subspaces of the involution U has dimension 2.*

Proof. Let M^+ and M^- be the plus and minus spaces of U and $n - 2p$ and $2p$ their respective dimensions. Let f^+ and f^- be the restrictions of f to these subspaces. Then the centralizer of U in $O_n^+(K, f)$ can be described as the subgroup of $O_{n-2p}(K, f^+) \times O_{2p}(K, f^-)$ consisting of the pairs $S_1 \times S_2$, where S_1 and S_2 are both rotations or neither is a rotation. Hence

$$(C_{O^+}(U))' = \Omega_{n-2p}(K, f^+) \times \Omega_{2p}(K, f^-)$$

and its centre contains elements which are not involutions if and only if $2p = 2$ or $n - 2p = 2$.

The same characterization can be given for the cosets in $PO_n^+(K, f)$ of the $(2, n-2)$ or $(n-2, 2)$ involutions among the cosets of the involutions of $O^+(K, f)$. The only modification in the proof comes from the fact that, when $2p = n - 2p$, there might exist elements of $O_n^+(K, f)$ which anticommute with U; therefore, the centralizer of the coset of U might be larger than the group of cosets determined by the subgroup of $O_{n-2p}(K, f^+) \times O_{2p}(K, f^-)$ defined above. At any rate the commutator of the centralizer of the coset of U consists of the cosets of $PO_n^+(K, f)$ defined by the elements of a group G, such that

$$O^+_{n-2p}(K, f^+) \times O^+_{2p}(K, f^-) \supseteq G \supseteq \Omega_{n-2p}(K, f^+) \times \Omega_{2p}(K, f^-)$$

and the result is still true.

REFERENCES

1. E. Artin, *Geometric algebra* (New York, 1957).
2. J. Dieudonné, *On the automorphisms of the classical groups*, Mem. Amer. Math. Soc., No. *2* (1951).
3. ——— *La géométrie des groupes classiques* (Berlin, 1955).
4. C. E. Rickart, *Isomorphic groups of linear transformations I and II*, Amer. J. Math., *72* (1950), 451–464, and *73* (1951), 697–716.
5. J. H. Walter, *Isomorphisms between projective unitary·groups*, Amer. J. Math., *77* (1955), 805–844.

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