WIDTH SEQUENCES FOR SPECIAL CLASSES OF (0, 1)-MATRICES

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1. Introduction. The α-width $\epsilon_A(\alpha)$ of a (0, 1)-matrix A is the minimal number of columns that can be selected from A in such a way that all row sums of the resulting submatrix of A are at least α. This notion was introduced in **(2)** and further studied in **(3)**. In these papers the major emphasis was on the minimal α-width sequence for the class $\mathfrak{A}$ of (0, 1)-matrices generated from an arbitrary A by interchanges:

$$\tilde{\epsilon}(\alpha) = \min_{A \in \mathfrak{A}} \epsilon_A(\alpha). \tag{1.1}$$

The $\mathfrak{A}$ in (1.1) can also be viewed as the class of all (0, 1)-matrices having the same row and column sums as A. A formula for $\tilde{\epsilon}(\alpha)$, in terms of the given row and column sums that characterize $\mathfrak{A}$, was obtained in **(2)**. It was further shown in **(3)** that there is a single, easily constructed matrix $\tilde{A}$ in $\mathfrak{A}$ that has minimal α-width for all α.

The present paper continues the study of α-width, but with a shift in emphasis. Here we shall be mainly concerned with obtaining further information regarding the width sequence $\epsilon_B(\alpha)$ for a fixed matrix B of size b by v, having k 1's per row and r 1's per column, whose class parameters b, v, k, r satisfy the inequality

$$(b - r)(v - k - 1) \leqslant v - 1. \tag{1.2}$$

Insofar as possible, we relate this information to the maximal width sequence

$$\bar{\epsilon}(\alpha) = \max_{B \in \mathfrak{B}} \epsilon_B(\alpha) \tag{1.3}$$

for the class $\mathfrak{B}$ generated by B. A class $\mathfrak{B}$ with parameters satisfying (1.2) has special combinatorial interest. For example, taking

$$b = v = n^2 + n + 1, \qquad k = r = n^2,$$

gives such a $\mathfrak{B}$; it contains complements of finite projective planes of order n, when these exist. For another example, take

$$v \equiv 1, 3 \pmod 6, \qquad b = \tfrac{1}{6}v(v-1), \qquad k = v - 3, \qquad r = \tfrac{1}{6}(v-1)(v-3),$$

to obtain a class $\mathfrak{B}$ that contains complements of Steiner triple systems on v elements.

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The determination of the maximal width sequence (1.3) for such a class $\mathfrak{B}$ involves deep issues. For instance, in the first example mentioned, $\bar{\epsilon}(1) = 2$ or 3 according as a finite projective plane of order n does not or does exist **(6)**. Specifically, the complement of a finite plane has 1-width 3, whereas other matrices in the class have 1-width 2. This state of affairs is of course decidedly in contrast with the situation for the minimal width sequence (1.1) for $\mathfrak{B}$. Indeed,

$$\tilde{\epsilon}(\alpha) = \min_{B \in \mathfrak{B}} \epsilon_B(\alpha) = \langle \alpha b/r \rangle. \tag{1.4}$$

Here $\langle x \rangle$ denotes the smallest integer $\geqslant x$. The formula (1.4) uses only the fact that matrices in $\mathfrak{B}$ have constant row and column sums **(2)**.

Our main result concerning the width sequence for an arbitrary B in $\mathfrak{B}$ is presented in § 3 (Theorem 3.2). We call it the 2-jump theorem. It asserts that

$$\epsilon_B(\alpha + 1) - \epsilon_B(\alpha) = 1 \text{ or } 2.$$

This narrows the problem of determining the width sequence for B to that of determining $\epsilon_B(1)$ and the location of the 2-jumps. The 2-jump theorem also holds for the maximal width sequence $\bar{\epsilon}(\alpha)$ for $\mathfrak{B}$ (Corollary 3.3), although, in contrast with $\tilde{\epsilon}(\alpha)$, it is not true that a single matrix always produces the sequence $\bar{\epsilon}(\alpha)$.

In § 4 we investigate the manner in which an interchange applied to B may affect its width sequence. If B has a 1-jump at $\alpha + 1$, then an interchange may increase the $(\alpha + 1)$-width by 1, whereupon the new matrix has a 2-jump at $\alpha + 1$; if, on the other hand, B has a 2-jump at $\alpha + 1$, an interchange may decrease the $(\alpha + 1)$-width by 1, whereupon the new matrix has a 1-jump at $\alpha + 1$. No other changes are possible at $\alpha + 1$ (Theorem 4.1). The proofs of both the 2-jump theorem and the interchange theorem rely ultimately on the impossibility of certain configurations in the class $\mathfrak{B}$.

In § 5 the interchange theorem of § 4 is applied to establish the existence in $\mathfrak{B}$ of a matrix with α- and $(\alpha + 1)$-widths satisfying the necessary conditions

(i) $$\tilde{\epsilon}(\alpha) \leqslant \epsilon(\alpha) \leqslant \bar{\epsilon}(\alpha),$$

(ii) $$\tilde{\epsilon}(\alpha + 1) \leqslant \epsilon(\alpha + 1) \leqslant \bar{\epsilon}(\alpha + 1),$$

(iii) $$1 \leqslant \epsilon(\alpha + 1) - \epsilon(\alpha) \leqslant 2.$$

In other words, if integers $\epsilon(\alpha)$ and $\epsilon(\alpha + 1)$ are specified satisfying these three conditions, there is a B in $\mathfrak{B}$ with α- and $(\alpha + 1)$-widths $\epsilon(\alpha)$ and $\epsilon(\alpha + 1)$, respectively. Examples show this to be a best-possible result.

Some facts concerning widths and complementation are recorded in § 6. The width sequence for a matrix A having constant row sums determines the width sequence for its complement A'. Indeed, the sequences $\epsilon_A(\alpha) - \alpha$ and $\epsilon_{A'}(\alpha') - \alpha'$ are conjugate partitions (Theorem 6.1), and so are the class sequences $\tilde{\epsilon}_{\mathfrak{A}}(\alpha) - \alpha$, $\tilde{\epsilon}_{\mathfrak{A}'}(\alpha') - \alpha'$, and $\bar{\epsilon}_{\mathfrak{A}}(\alpha) - \alpha$, $\bar{\epsilon}_{\mathfrak{A}'}(\alpha') - \alpha'$ (Theorem

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6.2.). This section is also the natural place to point out the close connection between a certain width problem and the existence of ovals in a finite projective plane. Finding the second 2-jump in the width sequence for the complement of a plane is tantamount to determining the maximal number of points in the plane having the property that no three are collinear.

The width sequence for the complement of a Steiner triple system is studied in § 7. Going back to the triples, this problem becomes simply that of determining the 1-width of a Steiner triple system. It is shown that a lower bound for the 1-width of a triple system on v elements is $\frac{1}{2}(v - 1)$, and conditions are determined under which this bound is achieved (Theorem 7.1). For instance, if $v = 15$ there is a triple system having 1-width $\frac{1}{2}(v - 1) = 7$. But there is also a triple system on 15 elements that has the surprisingly large 1-width 9.

The concluding section collects some miscellaneous examples and remarks. We mention one. Examples are constructed of classes $\mathfrak{B}$ having the property that all 2-jumps in the maximal width sequence occur before the first 2-jump in the minimal width sequence. Thus the difference between $\underline{\epsilon}(\alpha)$ and $\bar{\epsilon}(\alpha)$ for classes under consideration can be as large as possible on trivial grounds.

2. The class $\mathfrak{B}$.

Let $\mathfrak{B}$ denote the class of all b by v (0, 1)-matrices having exactly k 1's in each row and r 1's in each column. Here k and r are positive integers and

$$bk = vr. \tag{2.1}$$

We further assume throughout the body of the paper that the parameters b, v, k, r satisfy the inequality

$$(b - r)(v - k - 1) \leqslant v - 1. \tag{2.2}$$

Superficially, the inequality (2.2) indicates that matrices in $\mathfrak{B}$ have a high density of 1's. More precisely, (2.2) asserts that if one passes to the complementary class $\mathfrak{B}'$ by replacing 1's by 0's and 0's by 1's, and computes the average value $\bar{\lambda}$ for inner products of distinct columns of a matrix B' in $\mathfrak{B}'$, then

$$\bar{\lambda} = \frac{(b - r)(v - k - 1)}{v - 1} \leqslant 1. \tag{2.3}$$

The main significance of the class $\mathfrak{B}$, or its complement $\mathfrak{B}'$, derives from the consideration of certain combinatorial configurations. A *balanced incomplete block design* is an arrangement of v elements into b sets in such a way that:

$D(1)$ Each set contains exactly k distinct elements.

$D(2)$ Each element occurs in exactly r sets.

$D(3)$ Each pair of distinct elements occurs in exactly λ sets

$$(0 < \lambda < r < b).$$

The parameters b, v, k, r, λ must then satisfy

$$bk = vr, \tag{2.4}$$

$$r(k-1) = \lambda(v-1), \tag{2.5}$$

$$b \geqslant v \quad \text{(Fisher inequality).}^1 \tag{2.6}$$

A block design may, of course, be represented by a b by v incidence matrix

$$B = (b_{ij}), \tag{2.7}$$

where $b_{ij} = 1$ if the jth element is in the ith set and $b_{ij} = 0$ otherwise. Henceforth, when we speak of designs, we have this representation in mind.

The complement of a block design with parameters

$$b, \quad v, \quad k, \quad r, \quad \lambda \tag{2.8}$$

is a block design with parameters[2]

$$b, \quad v, \quad k' = v - k, \quad r' = b - r, \quad \lambda' = \lambda + b - 2r. \tag{2.9}$$

If equality holds in (2.3), the class $\mathfrak{B}'$ contains block designs with $\lambda' = 1$, provided these exist for the specified parameter values of $\mathfrak{B}'$, and if this is the case the class $\mathfrak{B}$ contains designs with[3] $\lambda = 1 - (b - 2r)$.

For $b = v$, $k = r$, the design is *symmetric* (or a v, k, λ *configuration*). *Finite projective planes* are symmetric designs with parameters

$$v = n^2 + n + 1, \qquad k' = n + 1, \qquad \lambda' = 1 \qquad (n \geqslant 2), \tag{2.10}$$

and complementary parameters

$$v = n^2 + n + 1, \qquad k = n^2, \qquad \lambda = n^2 - n. \tag{2.11}$$

Steiner triple systems are designs with parameters[4]

$$b = \tfrac{1}{6}v(v-1),\ v \equiv 1, 3 \pmod 6,\ k' = 3,\ r' = \tfrac{1}{2}(v-1),\ \lambda' = 1\ (v \geqslant 7), \tag{2.12}$$

and complementary parameters

$$b,\quad v,\quad k = v - 3,\quad r = \tfrac{1}{2}(v-1)(v-3),\quad \lambda = \tfrac{1}{2}(v-3)(v-4). \tag{2.13}$$

Both (2.11) and (2.13) satisfy (2.2) with equality. For the parameters (2.12), Steiner triple systems always exist. The precise range of n in (2.10) for which planes exist is an open question, but the Bruck-Ryser non-existence theorem excludes infinitely many values of n **(1)**.

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[1]This inequality need not hold for a class satisfying (2.2). Indeed, if $b > v$, then (2.2) implies that $(v - k)(b - r - 1) < b - 1$, and hence the transposed class also satisfies our basic assumption.

[2]The trivial designs with $k = v - 1$ are exceptional in the sense that complementation gives $k' = 1$ and $\lambda' = 0$.

[3]The only exceptional case is $b = v = 3$ and $r' = k' = 2$, for which $\mathfrak{B}$ has $\lambda = 0$.

[4]The case $v = 3$ is included for Steiner triples and excluded for designs. This discrepancy is in all events trivial.

There is a close connection between the existence of a design in $\mathfrak{B}'$ (or in $\mathfrak{B}$) and the maximal 1-width of the class $\mathfrak{B}$. We state this as follows.

THEOREM 2.1. *The class $\mathfrak{B}'$ contains a block design if and only if the maximal 1-width of $\mathfrak{B}$ is* 3.

The proof is almost immediate. Let B' in $\mathfrak{B}'$ be a design. Then every pair of columns of B' has inner product 1, so that every pair of columns of the complementary design B has a row composed of 0's. It follows that $\epsilon_B(1) = 3$. On the other hand, if B' is not a design, then B' has a pair of columns with inner product 0, by virtue of (2.3), and hence B has a pair of columns containing at least one 1 per row. Thus $\epsilon_B(1) \leqslant 2$.

Although this connection between designs and widths is close to the surface, Theorem 2.1 provides motivation for studying widths in the class $\mathfrak{B}$. There are other connections, and other motivations, also.

As the proof of Theorem 2.1 shows, the complement of a finite plane has maximal 1-width. The plane itself, however, has minimal 1-width for its class.

THEOREM 2.2. *A v, k, λ configuration has (minimal) λ-width k.*

Proof. In a v, k, λ configuration, the inner product of each pair of rows is also equal to λ. Thus, singling out those columns corresponding to the 1's in some row, we see that the design has λ-width at most k. On the other hand, the formula (1.4) shows that the minimal λ-width for the class is given by

$$\tilde{\epsilon}(\lambda) = \left\langle \frac{\lambda v}{k} \right\rangle = \left\langle \frac{k(k-1)v}{k(v-1)} \right\rangle = k.$$

It should perhaps be pointed out here that a statement about widths for a matrix B in $\mathfrak{B}$ can be translated to one involving widths of B' in $\mathfrak{B}'$. This will be made clear in § 6. But notice, for example, that the content of Theorem 2.1 might also be phrased as follows: The $(k'-1)$-width of a design in $\mathfrak{B}'$ is $v-1$, whereas the $(k'-1)$-width of other matrices in $\mathfrak{B}'$ is at most $v-2$.

We have chosen, perhaps somewhat arbitrarily, to focus primary attention on widths in $\mathfrak{B}$ rather than $\mathfrak{B}'$.

3. The 2-jump theorem. Excluding from consideration the trivial class $\mathfrak{B}$ for which $b = r$, $v = k$, so that $\mathfrak{B}$ would consist of the single matrix J having all its entries 1, we have seen that for any B in $\mathfrak{B}$,

$$\epsilon_B(1) = 2 \text{ or } 3. \tag{3.1}$$

In this section we obtain certain information on higher α-widths for matrices in $\mathfrak{B}$.

We first state and prove a theorem concerning the 1-width of a general (0, 1)-matrix A. This theorem provides a crude upper bound on 1-width that is sufficient for our purposes in this and the following section.

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THEOREM 3.1. *Let A be an m by n $(0, 1)$-matrix having 1-width at least ϵ, and let σ denote the number of zeros in an arbitrary column of A. Let A be extended to an m by t $(0, 1)$-matrix A^*, all of whose row sums equal β. Then*

$$\binom{t-1-\beta}{\epsilon-2}_{\sigma} \geqslant \binom{n-1}{\epsilon-2}. \tag{3.2}$$

Proof. By permutations of the rows and the first n columns of A^*, we may take A^* in the form

$$A^* = \left[\begin{array}{c|c|c} 0 & & \\ \cdot & & \\ \cdot & W & X \\ \cdot & & \\ 0 & & \\ \hline 1 & & \\ \cdot & & \\ \cdot & * & * \\ \cdot & & \\ 1 & & \end{array}\right].$$

Here the first column of A^* has σ zeros in the initial positions and $m - \sigma$ ones in the remaining positions. The submatrix W is of size σ by $n - 1$. Now

$$\binom{t-1-\beta}{\epsilon-2}_{\sigma}$$

counts the sequences of $\epsilon - 2$ zeros formed from the rows of the matrix $[W, X]$. This number is greater than or equal to the number of such sequences formed from W. But A has 1-width at least ϵ. Hence $\epsilon - 1$ columns of A must contain a row of $\epsilon - 1$ zeros. This implies that the number of sequences of $\epsilon - 2$ zeros in the rows of W is greater than or equal to

$$\binom{n-1}{\epsilon-2}.$$

THEOREM 3.2. *For any B in $\mathfrak{B}$,*

$$\epsilon_B(\alpha+1) - \epsilon_B(\alpha) = 1 \text{ or } 2, \qquad \alpha = 1, 2, \ldots, k-1. \tag{3.3}$$

Proof. For any $(0, 1)$-matrix A, one has $\epsilon_A(\alpha + 1) - \epsilon_A(\alpha) \geqslant 1$. Hence to prove (3.3) for matrices in $\mathfrak{B}$, we assume a B in $\mathfrak{B}$ with

$$\epsilon_B(\alpha+1) - \epsilon_B(\alpha) \geqslant 3 \tag{3.4}$$

for some α, and obtain a contradiction.

We may take B in the form

$$B = \left[\begin{array}{c|c} W & X \\ \hline Y & Z \end{array}\right]. \tag{3.5}$$

Here W is of size e by $t = \epsilon_B(\alpha)$ and has row sums at least $\alpha + 1$. The matrix Y, termed a critical α-submatrix of B in **(2)**, is of size $e' = b - e$ by t and has row sums equal to α. Note that $e' > 0$, by the minimal property of α-width. The matrix Z is of size e' by $t' = v - t$ and has row sums equal to $\alpha' = k - \alpha$. The 1-width of Z is greater than or equal to 3, for otherwise (3.4) would be violated. Let z denote the number of 1's in some column of Z. Then by Theorem 3.1 we have

$$(t' - 1 - \alpha')(e' - z) \geqslant t' - 1,$$

or

$$r - z \geqslant r - e' + \frac{t' - 1}{t' - 1 - \alpha'}. \tag{3.6}$$

Next we assert that

$$\frac{t' - 1}{t' - 1 - \alpha'} > b - r. \tag{3.7}$$

This is equivalent to

$$t' - 1 > (b - r)(t' - 1 - \alpha'). \tag{3.8}$$

To prove (3.8), we first note that the configuration (3.5) implies that

$$(\alpha + 1)e + \alpha e' \leqslant rt,$$

whence

$$\alpha b + e \leqslant rt$$

and

$$(k - \alpha')b + e \leqslant (v - t')r.$$

But $bk = vr$, so that

$$\alpha' \geqslant rt'/b.$$

Hence to prove (3.8) it suffices to prove the sharper inequality

$$b(t' - 1) > (b - r)(t'(b - r) - b). \tag{3.9}$$

This reduces to

$$b(b - r - 1) > t'((b - r)^2 - b),$$

and thus to

$$vr(b - r - 1) > kt'((b - r)^2 - b).$$

We know that $t' < v$. Hence to prove (3.9) it suffices to prove that

$$r(b - r - 1) \geqslant k((b - r)^2 - b).$$

This reduces to

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$$bk - r \geqslant (b - r)(k(b - r) - r)$$

or

$$vr - r \geqslant (b - r)(vr - kr - r).$$

But this gives

$$v - 1 \geqslant (b - r)(v - k - 1), \tag{3.10}$$

which is our assumption (2.2) on the parameters of $\mathfrak{B}$. Hence (3.7) is valid.

But now (3.6) implies that

$$r - z > r - e' + b - r.$$

Hence

$$r - z \geqslant e + 1. \tag{3.11}$$

But $r - z$ is the number of 1's in a column of X in the configuration (3.5), and thus (3.11) is a contradiction. This proves Theorem 3.2.

Let B be an arbitrary matrix in $\mathfrak{B}$. We say that B has a *1-jump at* $\alpha + 1$ if

$$\epsilon_B(\alpha + 1) - \epsilon_B(\alpha) = 1, \qquad \alpha = 1, 2, \ldots, k - 1, \tag{3.12}$$

and a *2-jump at* $\alpha + 1$ if

$$\epsilon_B(\alpha + 1) - \epsilon_B(\alpha) = 2, \qquad \alpha = 1, 2, \ldots, k - 1. \tag{3.13}$$

It is a convenient technicality to extend this terminology by saying that B has a 1-jump at 1 if $\epsilon_B(1) = 2$ and a 2-jump at 1 if $\epsilon_B(1) = 3$ (even though this is inconsistent with the natural definition $\epsilon_A(0) = 0$). With this convention, an α-width sequence for B, namely

$$\epsilon_B(1), \epsilon_B(2), \ldots, \epsilon_B(k), \tag{3.14}$$

contains precisely $v - k - 1 = k' - 1$ 2-jumps. The problem of determining (3.14) is that of finding the location of these 2-jumps.

It is easy to see from Theorem 3.2 that both the minimal width sequence $\tilde{\epsilon}(\alpha)$ and the maximal width sequence $\bar{\epsilon}(\alpha)$ for $\mathfrak{B}$ have jumps at most 2. Of course we know this directly for $\tilde{\epsilon}(\alpha)$, since there is always a single matrix $\tilde{A}$ in an arbitrary class $\mathfrak{A}$ having all its α-widths minimal **(3)**, but there is no need to invoke this fact.

COROLLARY 3.3. *For the class* $\mathfrak{B}$,

$$\tilde{\epsilon}(\alpha + 1) - \tilde{\epsilon}(\alpha) = 1 \text{ or } 2, \qquad \alpha = 1, 2, \ldots, k - 1, \tag{3.15}$$

$$\bar{\epsilon}(\alpha + 1) - \bar{\epsilon}(\alpha) = 1 \text{ or } 2, \qquad \alpha = 1, 2, \ldots, k - 1. \tag{3.16}$$

We give a proof for (3.16). It suffices to contradict $\bar{\epsilon}(\alpha + 1) - \bar{\epsilon}(\alpha) \geqslant 3$. Thus, suppose

$$\bar{\epsilon}(\alpha) = \epsilon_{B_1}(\alpha) = t,$$

$$\bar{\epsilon}(\alpha + 1) = \epsilon_{B_2}(\alpha + 1) \geqslant t + 3,$$

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for matrices B_1, B_2 in $\mathfrak{B}$ and for some α. By Theorem 3.2, we must have

$$\epsilon_{B_2}(\alpha) \geqslant t + 1,$$

contradicting the maximality of $\bar{\epsilon}(\alpha) = t$.

In view of Corollary 3.3, we may apply the 1-jump, 2-jump terminology to both class sequences

$$\tilde{\epsilon}(1), \tilde{\epsilon}(2), \ldots, \tilde{\epsilon}(k), \tag{3.17}$$

$$\bar{\epsilon}(1), \bar{\epsilon}(2), \ldots, \bar{\epsilon}(k). \tag{3.18}$$

Each of these has $v - k - 1$ 2-jumps, and we say that the class $\mathfrak{B}$ has $v - k - 1$ 2-jumps. The 2-jumps of (3.17) can be determined from the formula

$$\tilde{\epsilon}(\alpha) = \langle \alpha b/r \rangle. \tag{3.19}$$

Roughly speaking, they are evenly spaced. But determining the 2-jumps in (3.18) involves intricate combinatorial properties of the class, as is apparent from Theorem 2.1.

We conclude this section with an example of a class $\mathfrak{B}$ and the width sequences for certain matrices in $\mathfrak{B}$. Let

$$b = v = 13 (= 3^2 + 3 + 1), \qquad k = r = 9 (= 3^2),$$

so that $\mathfrak{B}'$ contains the plane B_1' of order 3. Table I shows the width sequences for the matrices $\tilde{B}$, B_1, and

$$B_2 = \left[\begin{array}{cccccc|ccccccc}
1 & 1 & 0 & 0 & 0 & 0 & & & & & & & \\
1 & 1 & 0 & 0 & 0 & 0 & & & & & & & \\
0 & 0 & 1 & 1 & 0 & 0 & & & & & & & \\
0 & 0 & 1 & 1 & 0 & 0 & & & & J & & & \\
0 & 0 & 0 & 0 & 1 & 1 & & & & & & & \\
0 & 0 & 0 & 0 & 1 & 1 & & & & & & & \\
\hline
 & & & & & & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
 & & & & & & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
 & & & & & & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
 & & & J & & & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
 & & & & & & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
 & & & & & & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
 & & & & & & 0 & 0 & 1 & 0 & 1 & 1 & 0
\end{array}\right].$$

The matrix B_2 was constructed by O. Gross to show that successive 2-jumps are possible. Note that B_2 has a plane of order 2 in the lower right-hand corner. This helps in the calculation of the width sequence for B_2.

TABLE I

WIDTH SEQUENCES FOR THE MATRICES $\tilde{B}$, B_1, AND B_2

α	1	2	3	4	5	6	7	8	9
$\tilde{\epsilon}$	2	3	⑤	6	⑧	9	⑪	12	13
ϵ_{B_1}	③	4	⑥	7	8	9	⑪	12	13
ϵ_{B_2}	2	④	5	6	⑧	⑩	11	12	13

It is clear from Table I that the maximal width sequence for this class cannot be produced by a single matrix. This is not exceptional, but is rather the typical situation.

4. The effect of an interchange on widths.

An *interchange* is a transformation of the elements of a (0, 1)-matrix that changes a minor of type (a) below into one of type (b), or vice versa, and leaves all other elements fixed:

$$\text{(a)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{(b)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Given two matrices A_1 and A_2 in the class $\mathfrak{A}$ of all (0, 1)-matrices having specified row and column sums, one can pass from A_1 to A_2 by a finite sequence of interchanges **(5)**. In other words, a matrix A in $\mathfrak{A}$ generates the entire class by interchanges. We also recall that an interchange can alter the α-width of a matrix by at most 1 **(2)**.

Theorem 4.1, below, outlines the possible effects of a single interchange on the width sequence for a matrix in $\mathfrak{B}$. The theorem says, in short, that it may be possible to lower a 2-jump to a 1-jump, or, inversely, to raise a 1-jump to a 2-jump, but that it is impossible to raise a 2-jump or to lower a 1-jump. More precisely:

THEOREM 4.1. *Let B be a matrix in $\mathfrak{B}$, and suppose that B has a 1-jump at $\alpha + 1$. If an interchange applied to B increases its $(\alpha + 1)$-width by 1, then the transformed matrix has a 2-jump at $\alpha + 1$. Suppose that B has a 2-jump at $\alpha + 1$. If an interchange applied to B decreases its $(\alpha + 1)$-width by 1, then the transformed matrix has a 1-jump at $\alpha + 1$. These are the only ways that an interchange can change widths at $\alpha + 1$.*

Proof. Let B have a 1-jump at $\alpha + 1$. Suppose that an interchange applied to B yields a matrix B^* having $(\alpha + 1)$-width

$$\epsilon_{B^*}(\alpha + 1) = \epsilon_B(\alpha + 1) + 1 = \epsilon_B(\alpha) + 2.$$

By the 2-jump theorem, we then have either

$$\epsilon_{B^*}(\alpha) = \epsilon_B(\alpha) \qquad \text{or} \qquad \epsilon_{B^*}(\alpha) = \epsilon_B(\alpha) + 1.$$

But if the latter alternative holds, we should have

$$\epsilon_{B^*}(\alpha) = \epsilon_B(\alpha + 1),$$

and this contradicts the minimal property of α-width for the matrix B^*. To see this, note that the matrix B can be written in the form

$$B = \left[\begin{array}{c|c} W & * \\ \hline Y & * \end{array}\right], \tag{4.1}$$

where Y is a critical $(\alpha + 1)$-submatrix. That is, Y has $\epsilon_B(\alpha + 1)$ columns and has row sums equal to $\alpha + 1$. The matrix W, if present, has row sums $>\alpha + 1$. If the interchange raises the $(\alpha + 1)$-width of B, it is essential that a 1 in some column of Y be replaced by a 0 in the interchange. If, after the interchange, this column of the matrix

$$\begin{bmatrix} W \\ Y \end{bmatrix}$$

is ignored, the remaining columns have row sums $\geqslant\alpha$. Thus $\epsilon_{B^*}(\alpha) < \epsilon_B(\alpha + 1)$, and hence B^* has a 2-jump at $\alpha + 1$.

Let B have a 1-jump at $\alpha + 1$. One interchange applied to B cannot yield a matrix B^* having $(\alpha + 1)$-width $\epsilon_B(\alpha + 1) - 1$. Indeed, if this were the case, we would have $\epsilon_{B^*}(\alpha) = \epsilon_B(\alpha) - 1$, and the inverse interchange contradicts the previous assertion.

Let B have a 2-jump at $\alpha + 1$. We now prove that one interchange applied to B cannot yield a transformed matrix B^* having $(\alpha + 1)$-width $\epsilon_B(\alpha + 1) + 1$. If it could, then by the 2-jump theorem, the matrix B^* has α-width $\epsilon_B(\alpha) + 1$. Now the matrix B can be written in the form

$$B = \left[\begin{array}{c|c} W_1 & * \\ \hline Y_1 & * \end{array}\right], \tag{4.2}$$

where Y_1 is a critical α-submatrix. Thus Y_1 has row sums α and has $\epsilon_B(\alpha) = t$ columns, and W_1, if present, has row sums $>\alpha$. If, after the interchange, the α-width of B has been raised to $t + 1$, it is essential that the interchange replace a 1 in Y_1 with a 0. Then B^* can be written as

$$B^* = \left[\begin{array}{c|c|c} W & X & * \\ \hline Y & Z & Z^* \end{array}\right]. \tag{4.3}$$

Here Y has size e' by t, and each row of Y has sum α, except the last, which has sum $\alpha - 1$. W is of size $e = b - e'$ by t, with all row sums at least $\alpha + 1$. Note that $e > 0$, so that W is present. Let $\alpha + \alpha' = k$. Z is of size e' by $\alpha' + 1$. The last row of Z consists entirely of 1's and the 1-width of Z is $\geqslant 3$, since the $(\alpha + 1)$-width of B^* is $t + 3$. Let z be a column sum of Z and let

$t + t' = v$. Then by an application of Theorem 3.1 to the first $e' - 1$ rows of the matrix $[Z, Z^*]$, we obtain

$$(4.4) \qquad (t' - 1 - \alpha')(e' - z) \geqslant \alpha'.$$

Hence

$$(4.5) \qquad r - z \geqslant r - e' + \frac{\alpha'}{t' - 1 - \alpha'}.$$

We assert that

$$(4.6) \qquad \frac{\alpha'}{t' - 1 - \alpha'} > b - r - 1,$$

or, equivalently,

$$(4.7) \qquad \alpha' > (b - r - 1)(t' - 1 - \alpha').$$

Since the configuration (4.3) implies $\alpha' \geqslant rt'/b$, to prove (4.7) it suffices to establish the sharper inequality

$$(4.8) \qquad rt' > (b - r - 1)(t'(b - r) - b).$$

This reduces to

$$(4.9) \qquad b(b - r - 1) > t'((b - r)^2 - b),$$

an inequality that was shown to be valid in the proof of Theorem 3.2. Hence (4.6) holds, and (4.5) then implies that

$$(4.10) \qquad r - z > r - e' + b - r - 1 = e - 1.$$

Thus

$$(4.11) \qquad r - z \geqslant e.$$

But then, in the configuration (4.3), X must be a matrix of 1's. Thus, looking at a row sum of $[W, X]$, we have

$$(4.12) \qquad \alpha + 1 + \alpha' + 1 \leqslant k,$$

and this is a contradiction.

Finally, let B have a 2-jump at $\alpha + 1$, and suppose that one interchange applied to B yields a transformed matrix B^* with

$$\epsilon_{B^*}(\alpha + 1) = \epsilon_B(\alpha + 1) - 1 = \epsilon_B(\alpha) + 1.$$

Then either

$$\epsilon_{B^*}(\alpha) = \epsilon_B(\alpha) \qquad \text{or} \qquad \epsilon_{B^*}(\alpha) = \epsilon_B(\alpha) - 1.$$

We now know, however, that the latter alternative is impossible since the inverse interchange would contradict what we have just proved. Thus B^* has a 1-jump at $\alpha + 1$.

This completes the proof of Theorem 4.1.

It should perhaps be remarked that both of the possibilities outlined in Theorem 4.1 can actually occur, in view of the fact that one can pass through a class by interchanges. Note also that our 1-jump, 2-jump terminology for $\alpha + 1 = 1$ is consistent with Theorem 4.1.

In the next section we give an application of Theorem 4.1.

5. An existence theorem. It was observed in **(2)** that for an arbitrary class $\mathfrak{A}$, if $\epsilon(\alpha)$ is an integer in the interval $\tilde{\epsilon}(\alpha) \leqslant \epsilon(\alpha) \leqslant \bar{\epsilon}(\alpha)$, then there is an A in $\mathfrak{A}$ having α-width $\epsilon(\alpha)$. This follows from the facts: (i) an interchange can change an α-width by at most 1, (ii) one can pass through the class $\mathfrak{A}$ by interchanges. For the class $\mathfrak{B}$, the interchange theorem of §4 yields a stronger result:

THEOREM 5.1. *For the class* $\mathfrak{B}$, *let* $\epsilon(\alpha)$ *and* $\epsilon(\alpha + 1)$ *be integers satisfying*

$$\tilde{\epsilon}(\alpha) \leqslant \epsilon(\alpha) \leqslant \bar{\epsilon}(\alpha), \tag{5.1}$$

$$\tilde{\epsilon}(\alpha + 1) \leqslant \epsilon(\alpha + 1) \leqslant \bar{\epsilon}(\alpha + 1), \tag{5.2}$$

$$1 \leqslant \epsilon(\alpha + 1) - \epsilon(\alpha) \leqslant 2, \tag{5.3}$$

for some $\alpha = 1, 2, \ldots, k - 1$. *Then there is a* B *in* $\mathfrak{B}$ *having* α*-width* $\epsilon(\alpha)$ *and* $(\alpha + 1)$*-width* $\epsilon(\alpha + 1)$.

Proof. We first prove the theorem with (5.2) replaced by

$$\tilde{\epsilon}(\alpha + 1) < \epsilon(\alpha + 1) < \bar{\epsilon}(\alpha + 1). \tag{5.4}$$

Suppose that

$$\epsilon(\alpha + 1) = \epsilon(\alpha) + 1 \tag{5.5}$$

for all B in $\mathfrak{B}$ of $(\alpha + 1)$-width $\epsilon(\alpha + 1)$. Then by Theorem 4.1, B is not transformable by interchanges into a matrix of $(\alpha + 1)$-width $\tilde{\epsilon}(\alpha + 1)$, a contradiction.

Suppose that

$$\epsilon(\alpha + 1) = \epsilon(\alpha) + 2 \tag{5.6}$$

for all B in $\mathfrak{B}$ of $(\alpha + 1)$-width $\epsilon(\alpha + 1)$. Again by Theorem 4.1, B is not transformable by interchanges into a matrix of $(\alpha + 1)$-width $\bar{\epsilon}(\alpha + 1)$, a contradiction.

This proves the theorem with (5.2) replaced by (5.4).

Four cases remain:

$$\epsilon(\alpha + 1) = \tilde{\epsilon}(\alpha + 1), \qquad \tilde{\epsilon}(\alpha + 1) = \tilde{\epsilon}(\alpha) + 1, \tag{5.7}$$

$$\epsilon(\alpha + 1) = \tilde{\epsilon}(\alpha + 1), \qquad \tilde{\epsilon}(\alpha + 1) = \tilde{\epsilon}(\alpha) + 2, \tag{5.8}$$

$$\epsilon(\alpha + 1) = \bar{\epsilon}(\alpha + 1), \qquad \bar{\epsilon}(\alpha + 1) = \bar{\epsilon}(\alpha) + 1, \tag{5.9}$$

$$\epsilon(\alpha + 1) = \bar{\epsilon}(\alpha + 1), \qquad \bar{\epsilon}(\alpha + 1) = \bar{\epsilon}(\alpha) + 2. \tag{5.10}$$

If (5.7) holds, we must show the existence of a B in $\mathfrak{B}$ with $(\alpha + 1)$-width $\tilde{\epsilon}(\alpha + 1)$ and α-width $\tilde{\epsilon}(\alpha)$. In this case, every B of $(\alpha + 1)$-width $\tilde{\epsilon}(\alpha + 1)$ has α-width $\tilde{\epsilon}(\alpha)$.

If (5.8) holds, we must show the existence of a B in $\mathfrak{B}$ of $(\alpha + 1)$-width $\tilde{\epsilon}(\alpha + 1)$ and α-width $\tilde{\epsilon}(\alpha)$, and, in case $\tilde{\epsilon}(\alpha) < \bar{\epsilon}(\alpha)$, of a B in $\mathfrak{B}$ of $(\alpha + 1)$-width $\tilde{\epsilon}(\alpha + 1)$ and α-width $\tilde{\epsilon}(\alpha) + 1$. The first of these is immediate, since every matrix of α-width $\tilde{\epsilon}(\alpha)$ has $(\alpha + 1)$-width $\tilde{\epsilon}(\alpha + 1)$ if (5.8) holds. For the second, consider a B of α-width $\tilde{\epsilon}(\alpha) + 1$. If B has $(\alpha + 1)$-width $\tilde{\epsilon}(\alpha + 1)$, we are done. Suppose, then, that all such B have $(\alpha + 1)$-width $\tilde{\epsilon}(\alpha + 1) + 1$, and consequently have 2-jumps at $\alpha + 1$. But by Theorem 4.1 and the remarks preceding the theorem, there is an interchange transforming some such B into a matrix of $(\alpha + 1)$-width $\tilde{\epsilon}(\alpha + 1)$, whereupon the transformed matrix has a 1-jump at $\alpha + 1$. Hence there is a B of α-width $\tilde{\epsilon}(\alpha) + 1$ and $(\alpha + 1)$-width $\tilde{\epsilon}(\alpha + 1)$.

If (5.9) holds, we must show the existence of a B in $\mathfrak{B}$ of $(\alpha + 1)$-width $\bar{\epsilon}(\alpha + 1)$ and α-width $\bar{\epsilon}(\alpha)$, and in case $\tilde{\epsilon}(\alpha) < \bar{\epsilon}(\alpha)$, of a B in $\mathfrak{B}$ of $(\alpha + 1)$-width $\bar{\epsilon}(\alpha + 1)$ and α-width $\bar{\epsilon}(\alpha) - 1$. The first of these is immediate because in this case every matrix of α-width $\bar{\epsilon}(\alpha)$ has $(\alpha + 1)$-width $\bar{\epsilon}(\alpha + 1)$. Consider a B of α-width $\bar{\epsilon}(\alpha) - 1$. If this B has $(\alpha + 1)$-width $\bar{\epsilon}(\alpha + 1)$, then the theorem holds for (5.9). Suppose, then, that every B of α-width $\bar{\epsilon}(\alpha) - 1$ has $(\alpha + 1)$-width $\bar{\epsilon}(\alpha + 1) - 1$. Now there exists an interchange that transforms some B of $(\alpha + 1)$-width $\bar{\epsilon}(\alpha + 1) - 1$ into a B of $(\alpha + 1)$-width $\bar{\epsilon}(\alpha + 1)$. Then, by Theorem 4.1, there exists a B of $(\alpha + 1)$-width $\bar{\epsilon}(\alpha + 1)$ and α-width $\bar{\epsilon}(\alpha) - 1$.

If (5.10) holds, we must show that there is a B in $\mathfrak{B}$ of $(\alpha + 1)$-width $\bar{\epsilon}(\alpha + 1) = \bar{\epsilon}(\alpha) + 2$ and α-width $\bar{\epsilon}(\alpha)$. In this case, every B of $(\alpha + 1)$-width $\bar{\epsilon}(\alpha + 1)$ has α-width $\bar{\epsilon}(\alpha)$.

This completes the proof of Theorem 5.1.

It can be shown by examples that Theorem 5.1 is a best-possible result in the sense that there are classes $\mathfrak{B}$ for which one can specify three integers $\epsilon(\alpha)$, $\epsilon(\alpha + 1)$, $\epsilon(\alpha + 2)$ satisfying the obvious necessary conditions, but there is no matrix in the class having these as its α-, $(\alpha + 1)$-, and $(\alpha + 2)$-widths, respectively. For instance, the complement of the (unique) plane of order 3 has 1-width 3, 2-width 4, 3-width 6, while $\tilde{\epsilon}(1) = 2$, $\tilde{\epsilon}(2) = 3$, $\tilde{\epsilon}(3) = 5$ for its class. But there is no matrix in this class having 1-width 3, 2-width 4, and 3-width 5.

6. Widths and complements. Let A be an arbitrary b by v (0, 1)-matrix, and, for the moment, designate its largest row sum by k. Define

$$\mu_A(\beta), \qquad \beta = 0, 1, \ldots, k, \tag{6.1}$$

to be the maximal number of columns that can be selected from A in such a way that the resulting submatrix has row sums at most β. It was shown in **(3)** that the sequence (6.1) and the width sequence for the complement A' of A,

$$\epsilon_{A'}(\alpha), \qquad \alpha = 0, 1, \ldots, v - k, \tag{6.2}$$

determine each other in the following way. Let α be fixed in its interval and let β be the least integer in its interval for which

$$\mu_A(\beta) - \beta \geqslant \alpha. \tag{6.3}$$

Then, denoting this least β by $\beta(\alpha)$, we have

$$\epsilon_{A'}(\alpha) = \alpha + \beta(\alpha). \tag{6.4}$$

On the other hand, starting with the sequence (6.2) and fixing β, let $\alpha = \alpha(\beta)$ be the largest integer in its interval for which

$$\epsilon_{A'}(\alpha) - \alpha \leqslant \beta. \tag{6.5}$$

Then

$$\mu_A(\beta) = \alpha(\beta) + \beta. \tag{6.6}$$

Here we take $\epsilon_{A'}(0) = 0$ and, if A has no zero columns, $\mu_A(0) = 0$.
If the matrix A has constant row sums k, it is clear that

$$v - \mu_A(\beta) = \epsilon_A(k - \beta). \tag{6.7}$$

Hence, for constant row sums, the width sequence for a matrix A determines the width sequence for A'. We summarize the relationship between these sequences as follows:

THEOREM 6.1. *Let A be a b by v (0, 1)-matrix having constant row sums k, let A' be its complement with row sums $k' = v - k$, and let $\epsilon_A(\alpha)$ be the width sequence for A. Let α' be a fixed integer in the interval $0 \leqslant \alpha' \leqslant k'$, and let $\alpha(\alpha')$ be the largest integer α in the interval $0 \leqslant \alpha \leqslant k$ satisfying*

$$\epsilon_A(\alpha) - \alpha \leqslant k' - \alpha'. \tag{6.8}$$

Then

$$\epsilon_A(\alpha') - \alpha' = k - \alpha(\alpha'). \tag{6.9}$$

Hence the sequences $\epsilon_A(\alpha) - \alpha$ and $\epsilon_{A'}(\alpha') - \alpha'$ are conjugate partitions.

Proof. By (6.4), $\epsilon_{A'}(\alpha') - \alpha'$ is the least integer β in the interval $0 \leqslant \beta \leqslant k$ such that

$$\mu_A(\beta) - \beta \geqslant \alpha'.$$

By (6.7), this is the least integer β such that

$$v - \epsilon_A(k - \beta) - \beta \geqslant \alpha'.$$

Setting $\alpha = k - \beta$ establishes (6.9).

It follows that the non-decreasing sequences

(6.10) $$\epsilon_A(\alpha) - \alpha, \qquad \alpha = 1, 2, \ldots, k,$$

(6.11) $$\epsilon_{A'}(\alpha') - \alpha', \qquad \alpha' = 1, 2, \ldots, k',$$

are conjugate partitions of the integer

$$\sum_{\alpha=1}^{k} (\epsilon_A(\alpha) - \alpha) = \sum_{\alpha'=1}^{k'} (\epsilon_{A'}(\alpha') - \alpha').$$

To see this, construct a k by k' (0, 1)-array in which row α contains $\epsilon_A(\alpha)-\alpha$ ones occupying the last $\epsilon_A(\alpha) - \alpha$ positions:

	1	2	3	4	$5 = k'$
1	0	0	0	0	1
2	0	0	0	1	1
3	0	0	0	1	1
4	0	0	0	1	1
5	0	1	1	1	1
6	0	1	1	1	1
$k = 7$	0	1	1	1	1

Then column α' of this array contains $\epsilon_{A'}(\alpha') - \alpha'$ ones; that is, the sequences (6.10) and (6.11) are conjugate.

Notice also that row α of the array contains $\mu_A(k - \alpha) - (k - \alpha)$ zeros, while column α' contains $\mu_{A'}(k' - \alpha') - (k' - \alpha')$ zeros, and thus the non-decreasing sequences

(6.12) $$\mu_A(\beta) - \beta, \qquad \beta = 0, 1, \ldots, k - 1,$$

(6.13) $$\mu_{A'}(\beta') - \beta', \qquad \beta' = 0, 1, \ldots, k' - 1,$$

are conjugate partitions of the integer

$$\sum_{\beta=0}^{k-1} (\mu_A(\beta) - \beta) = \sum_{\beta=0}^{k'-1} (\mu_{A'}(\beta') - \beta').$$

Now let $\mathfrak{A}$ be the class of b by v (0, 1)-matrices having constant row sums k and specified column sums. Since there is a single matrix $\tilde{A}$ in $\mathfrak{A}$ having minimal width $\tilde{\epsilon}(\alpha)$ for all α, it follows from Theorem 6.1 that the complementary matrix $\tilde{A}'$ yields the minimal width sequence for $\mathfrak{A}'$, the sequences $\tilde{\epsilon}(\alpha) - \alpha$ for $\mathfrak{A}$ and $\tilde{\epsilon}(\alpha') - \alpha'$ for $\mathfrak{A}'$ being conjugate. The same connection also holds between the maximal width sequences for the two complementary classes.

THEOREM 6.2. *Let $\mathfrak{A}$ be the class of all b by v (0, 1)-matrices having constant row sums k and specified column sums, and let $\mathfrak{A}'$ be the complementary class with row sums $k' = v - k$. Then the sequences*

(6.14) $$\bar{\epsilon}_{\mathfrak{A}}(\alpha) - \alpha, \qquad \alpha = 1, 2, \ldots, k,$$

(6.15) $$\bar{\epsilon}_{\mathfrak{A}'}(\alpha') - \alpha', \qquad \alpha' = 1, 2, \ldots, k',$$

are conjugate.

Proof. Suppose (6.14) is given, and let α' be fixed but arbitrary in its interval. Determine the largest integer α such that

$$\bar{\epsilon}_{\mathfrak{A}}(\alpha) - \alpha \leqslant k' - \alpha'.$$

Hence for this α, we have

(6.16) $$\bar{\epsilon}_{\mathfrak{A}}(\alpha) - \alpha \leqslant k' - \alpha' < \bar{\epsilon}_{\mathfrak{A}}(\alpha + 1) - (\alpha + 1).$$

Select a matrix A in $\mathfrak{A}$ having maximal $(\alpha + 1)$-width. Then (6.16) implies that

(6.17) $$\epsilon_A(\alpha) - \alpha \leqslant k' - \alpha' < \epsilon_A(\alpha + 1) - (\alpha + 1),$$

and hence, by Theorem 6.1,

(6.18) $$\epsilon_{A'}(\alpha') - \alpha' = k - \alpha.$$

Thus the conjugate of (6.14) is dominated by (6.15).

Interchanging the roles of (6.14) and (6.15) in the argument shows that the conjugate of (6.15) is dominated by (6.14). But this implies that the conjugate of (6.14) dominates (6.15). Hence (6.14) and (6.15) are conjugate.

Returning now to the class $\mathfrak{B}$, we have, from the 2-jump theorem,

(6.19) $$\epsilon_B(\alpha + 1) - (\alpha + 1) - (\epsilon_B(\alpha) - \alpha) = 0 \text{ or } 1, \qquad \alpha = 1, 2, \ldots, k - 1,$$

for B in $\mathfrak{B}$. Using the conjugate relation between the sequences $\epsilon_B(\alpha) - \alpha$ and $\epsilon_{B'}(\alpha') - \alpha'$, it follows from (6.19) that

$$\epsilon_{B'}(\alpha' + 1) - (\alpha' + 1) - (\epsilon_{B'}(\alpha') - \alpha') \geqslant 1, \qquad \alpha' = 0, 1, \ldots, k' - 2$$

and hence that

(6.20) $$\epsilon_{B'}(\alpha' + 1) - \epsilon_{B'}(\alpha') \geqslant 2, \qquad \alpha' = 0, 1, \ldots, k' - 2.$$

That is, the 2-jump theorem for $\mathfrak{B}$ implies that jumps in the width sequence for a matrix in the complementary class are at least 2, except possibly for the last jump. The inequality (6.20) is valid for $\alpha' = k' - 1$ unless B' is a design, in which case the left-hand side of (6.20) is 1.

The connection between the width sequence for the complement of a projective plane and the existence of ovals in the plane should be mentioned. Let B' be a plane of order n. A set of $n + 1$ points of B' is an oval if no three are collinear. In the notation (6.1), B' has an oval if and only if[5]

$$\mu_{B'}(2) \geqslant n + 1.$$

[5]For a projective plane B' of order n, it is easy to verify the inequality $\mu_{B'}(\beta) \leqslant (\beta - 1)n + \beta$. If the plane has an oval, then $\mu_{B'}(2) = n + 1$ or $n + 2$, according as n is odd or even.

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The width sequence for B has n 2-jumps, the first occurring at $\alpha = 1$ (that is, $\epsilon_B(1) = 3$). The location of the second 2-jump spots the existence or non-existence of an oval, since, by (6.6), if the second 2-jump occurs at α, then

$$\mu_{B'}(2) = 2 + \alpha - 1 = \alpha + 1.$$

We know of no counterexample to the assertion that every matrix in a class $\mathfrak{B}$ with parameters

$$b = v = n^2 + n + 1, \qquad r = k = n^2,$$

has its second 2-jump occurring at or beyond $\alpha = n$. One may speculate that the existence of "ovals" could be established for all matrices in the class $\mathfrak{B}'$, and hence for planes. Our efforts in this direction have met with no success.

Observe that the value $\alpha = n$ is the location of the first 2-jump in the minimal width sequence for $\mathfrak{B}$. We remark that it is not true, for a general class $\mathfrak{B}$ with parameters satisfying (2.2), that all matrices in $\mathfrak{B}$ have their second 2-jump occurring at or beyond the first 2-jump in the minimal width sequence. Indeed, this assertion is false for classes containing complements of Steiner triples, as will be shown by an example in § 8.

7. The 1-widths of Steiner triple systems. In this section, we specialize the class $\mathfrak{B}$ to have parameters

(7.1) $b = \frac{1}{6}v(v-1),\ \ v \equiv 1, 3 \pmod 6,\ \ k = v - 3,\ \ r = \frac{1}{6}(v-1)(v-3).$

Thus $\mathfrak{B}'$ has parameters

(7.2) $$b,\ \ v,\ \ k' = 3,\ \ r' = \tfrac{1}{2}(v-1),$$

and contains Steiner triples on v elements, that is, a collection of triples that covers each pair of the v elements just once.

Each matrix in $\mathfrak{B}$ has $v - k - 1 = 2$ 2-jumps in its width sequence. If B' is a Steiner triple system, then B has its first 2-jump at $\alpha = 1$, its width sequence having the form

(7.3) $$\epsilon_B(1) = 3,\ \epsilon_B(2) = 4, \ldots, \epsilon_B(t-2) = t,$$
$$\epsilon_B(t-1) = t + 2, \ldots, \epsilon_B(v-3) = v.$$

From the conjugate relation between the sequences

$$\epsilon_B(\alpha) - \alpha, \qquad \alpha = 1, 2, \ldots, v-3, \qquad \text{and} \qquad \epsilon_{B'}(\alpha') - \alpha', \qquad \alpha' = 1, 2, 3,$$

it follows that

$$\epsilon_{B'}(1) - 1 = (v-3) - (t-2) = v - t - 1,$$

and hence that the integer t in (7.3) satisfies

(7.4) $$t = v - \epsilon_{B'}(1).$$

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Thus the location of the second 2-jump in (7.3) is determined by the 1-width of the triple system B'.

THEOREM 7.1. *The* 1-*width of a Steiner triple system* B' *on* v *elements satisfies*

$$\epsilon_{B'}(1) \geqslant \tfrac{1}{2}(v-1). \tag{7.5}$$

Equality holds if and only if the triple system contains a triple subsystem with parameters

$$\bar{b} = \frac{(v-1)(v-3)}{24}, \quad \bar{v} = \frac{v-1}{2}, \quad \bar{k} = 3, \quad \bar{r} = \frac{v-3}{4}. \tag{7.6}$$

Before proving Theorem 7.1, we point out that (7.5) is a considerable improvement on the lower bound given by $\tilde{\epsilon}(1)$ for $\mathfrak{B}'$, since

$$\tilde{\epsilon}(1) = \langle \tfrac{1}{3}v \rangle.$$

Proof. Let B' have 1-width $\frac{1}{2}(v+p)$, p an odd integer. Then we may take B' in the form

$$\left[\begin{array}{c|c} X_3 & O \\ \hline X_2 & Y_1 \\ \hline X_1 & Y_2 \end{array}\right]. \tag{7.7}$$
$$\underbrace{\qquad}_{\frac{1}{2}(v+p)} \quad \underbrace{\qquad}_{\frac{1}{2}(v-p)}$$

Here X_3 contains three 1's in each row, X_2 and Y_2 contain two 1's in each row, X_1 and Y_1 contain one 1 in each row, and O is a zero matrix. The matrices X_i have $\frac{1}{2}(v+p)$ columns. Let X_i have x_i rows, $i = 1, 2, 3$. Then

$$\begin{aligned} x_1 + x_2 + x_3 &= \tfrac{1}{6}v(v-1), \\ 2x_1 + x_2 &= \tfrac{1}{2}(v-1)\cdot\tfrac{1}{2}(v-p), \\ 4x_1 + x_2 &= \tfrac{1}{2}(2v-p-3)\cdot\tfrac{1}{2}(v-p), \end{aligned} \tag{7.8}$$

the last equation coming from the inner-product restriction on the last $\frac{1}{2}(v-p)$ columns of B'. The unique solution of this system is

$$\begin{aligned} x_1 &= \frac{(v-p-2)(v-p)}{8}, \\ x_2 &= \frac{(p+1)(v-p)}{4}, \\ x_3 &= \frac{v^2-4v+3p^2}{24}. \end{aligned} \tag{7.9}$$

Thus if $p < -1$, then $x_2 < 0$ and hence we conclude that

$$p \geqslant -1, \tag{7.10}$$

$$\epsilon_{B'}(1) \geqslant \tfrac{1}{2}(v-1). \tag{7.11}$$

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Suppose that equality holds in (7.11). Then in the configuration (7.7) with $p = -1$, we have $x_2 = 0$, and X_3 is a triple system on $\frac{1}{2}(v-1)$ elements. Conversely, suppose the given triple system has a subsystem with parameters (7.6). Let X_3 represent the subconfiguration and write

(7.12) $$B' = \left[\begin{array}{c|c} X_3 & O \\ \hline X & * \end{array}\right].$$

Then X cannot contain a row with two 1's. Nor can X contain a row of 0's. For the column sums of X are $\frac{1}{4}(v+1)$ and

$$\bar{b} + \frac{v+1}{4}\cdot\frac{v-1}{2} = \frac{(v-1)(v-3)}{24} + \frac{(v+1)(v-1)}{8} = \frac{v(v-1)}{6} = b.$$

Hence for the matrix B' of (7.12),

(7.13) $$\epsilon_{B'}(1) = \tfrac{1}{2}(v-1).$$

This proves the theorem.

We remark that, given a triple system on $\bar{v} \geqslant 3$ elements, it is always possible to construct a triple system on $v = 2\bar{v} + 1$ elements that contains the given one **(4)**. A second remark concerns the configuration (7.7). The matrix Y_2 is the incidence matrix for all pairs on $\frac{1}{2}(v-p)$ elements, and has constant column sums $\frac{1}{2}(v-p-2)$; each column sum of Y_1 is $\frac{1}{2}(p+1)$.

Some examples of triple systems and their 1-widths are tabulated below:

(a) $v = 7$. Unique system, $\epsilon(1) = 3$.

(b) $v = 9$. Unique system

1, 2, 3			
1, 4, 5	2, 4, 9	3, 4, 8	
1, 6, 8	2, 5, 6	3, 5, 7	4, 6, 7
1, 7, 9	2, 7, 8	3, 6, 9	5, 8, 9

with $\epsilon(1) = 5$. (The set $\{1, 2, 3, 4, 5\}$ intersects every triple.)

(c) $v = 13$. Two distinct systems. Each contains

1, 2, 3					
1, 4, 5	2, 4, 6				
1, 6, 7	2, 5, 7	4, 3, 8			
1, 8, 9	2, 8, 10	4, 7, 9	7, 3, 11		
1, 10, 11	2, 9, 12	4, 10, 13	7, 8, 13	8, 5, 11	6, 9, 11
1, 12, 13	2, 11, 13	4, 11, 12	7, 10, 12	8, 6, 12	3, 5, 12

In addition, one system contains 3, 6, 10; 3, 9, 13; 5, 6, 13; 5, 9, 10; the other contains 3, 6, 13; 3, 9, 10; 5, 6, 10; 5, 9, 13. The set $\{1, 2, 3, 4, 5, 6, 7\}$ intersects every triple for both systems and hence $\epsilon(1) = 7$ for both.

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(d) $v = 15$. Eighty distinct systems. One of these has 1-width 7, by Theorem 7.1 and the remark following its proof. There is another that has 1-width 9. We describe it as follows. Let

$$Z = \begin{bmatrix} 0&0&1&1&0\\ 0&0&0&1&1\\ 1&0&0&0&1\\ 1&1&0&0&0\\ 0&1&1&0&0\\ 0&1&0&0&1\\ 1&0&1&0&0\\ 0&1&0&1&0\\ 0&0&1&0&1\\ 1&0&0&1&0 \end{bmatrix}, \qquad E = \begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{bmatrix},$$

and form the 35 by 15 matrix

$$\begin{bmatrix} Z & E & O\\ O & Z & E\\ E & O & Z\\ I & I & I \end{bmatrix}.$$

Here I is the 5 by 5 identity. It is easily checked that this is a triple system. We omit a proof that it has 1-width 9, except to say that the partitioned form we have used to describe it is advantageous in making a proof. It can also be shown that none of the eighty systems has 1-width 10 or more.

It would be interesting to have more information concerning the variation in 1-width for Steiner triples. In this connection, we note that the triple system just described can be generalized. Take $v \equiv 3 \pmod 6$ and set $v = 3s$, s an odd integer. Let Z be the incidence matrix of all pairs on s elements; it is not hard to show that Z may be arranged to appear as

$$Z = \begin{bmatrix} Z_1\\ Z_2\\ \vdots\\ Z_{\frac{1}{2}(s-1)} \end{bmatrix}, \tag{7.14}$$

where each Z_i is the sum of two permutation matrices, and

$$\sum_i Z_i = J - I,$$

where J is the matrix of all 1's. Let

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$$E = \begin{bmatrix} I \\ I \\ \cdot \\ \cdot \\ \cdot \\ I \end{bmatrix} \tag{7.15}$$

consist of $\frac{1}{2}(s-1)$ identity matrices of order s. Then the matrix

$$\begin{bmatrix} Z & E & O \\ O & Z & E \\ E & O & Z \\ I & I & I \end{bmatrix} \tag{7.16}$$

is a Steiner triple system on $v = 3s$ elements. Does the system (7.16) have 1-width $2s - 1 = \frac{2}{3}v - 1$?

8. Some miscellaneous examples.

Perhaps the simplest non-trivial class $\mathfrak{B}$ with parameters satisfying (2.2) is obtained by taking

$$b = v \geqslant 3, \qquad r = k = v - 2. \tag{8.1}$$

This class has just one 2-jump, and its maximal width sequence can be determined explicitly. It is

$$\bar{\epsilon}(1) = 2, \ldots, \bar{\epsilon}(\langle\tfrac{1}{3}v\rangle - 1) = \langle\tfrac{1}{3}v\rangle, \bar{\epsilon}(\langle\tfrac{1}{3}v\rangle) = \langle\tfrac{1}{3}v\rangle + 2, \ldots, \bar{\epsilon}(v-2) = v, \tag{8.2}$$

the 2-jump occurring at $\langle\frac{1}{3}v\rangle$. To prove this, let the integer $v \geqslant 3$ be written in one of the three forms

$$v = 3s, \tag{8.3}$$

$$v = 3s - 1, \tag{8.4}$$

$$v = 3s - 2, \tag{8.5}$$

where s is an integer. By Theorem 6.2, it suffices to show, for the complementary class $\mathfrak{B}'$, that

$$\bar{\epsilon}(1) = 2s, 2s - 1, 2s - 2, \tag{8.6}$$

according as (8.3), (8.4), or (8.5) holds.

Assume (8.3). We first single out a matrix in $\mathfrak{B}'$ that has 1-width $2s$. Let

$$D_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \tag{8.7}$$

and form

$$\begin{bmatrix} D_2 & & & & \\ & D_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot & \\ & & & & & D_2 \end{bmatrix}, \tag{8.8}$$

with D_2 repeated s times. The matrix (8.8) is in $\mathfrak{B}'$ and has 1-width $2s$. Suppose there were a matrix in $\mathfrak{B}'$ that had 1-width $2s + 1$. Such a matrix must contain an identity submatrix I of size $2s + 1$, and hence can be written as

$$\left[\begin{array}{c|c} * & X \\ \hline I & Y \end{array}\right], \tag{8.9}$$

with I having $2s + 1$ rows and columns. Then the matrix Y contains $2s + 1$ ones, whereas the matrices X and Y together contain $2(s - 1) = 2s - 2$ ones, a contradiction.

Assume (8.4). Let J_2 be the 2 by 2 matrix of 1's. Then the matrix

$$\begin{bmatrix} D_2 & & & & & \\ & D_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & D_2 & \\ & & & & & & J_2 \end{bmatrix}, \tag{8.10}$$

with D_2 repeated $s - 1$ times, is in $\mathfrak{B}'$ and has 1-width $2s - 1$. An argument similar to the one given above shows that no matrix in $\mathfrak{B}'$ has 1-width $2s$.

If (8.5) holds, the matrix

$$\begin{bmatrix} D_2 & & & & & & \\ & D_2 & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & D_2 & & \\ & & & & & & J_2 & \\ & & & & & & & J_2 \end{bmatrix}, \tag{8.11}$$

with D_2 repeated $s - 2$ times, is in $\mathfrak{B}'$ and has 1-width $2s - 2$. As above, no matrix in $\mathfrak{B}'$ has larger 1-width.

A matrix in $\mathfrak{B}'$ can be viewed as the edge-vertex incidence matrix of a multigraph having degree two at each vertex. The 1-width of the matrix is the minimum number of vertices that touch all the edges. Roughly speaking, the proof given above says that to maximize this number over all such graphs on v vertices, it is necessary to form as many triangles (matrices D_2) as possible.

Our next example is the one mentioned at the end of § 6. It shows that classes corresponding to complements of Steiner triples contain matrices with both 2-jumps occurring before the first 2-jump in the minimal width sequence. Let the class $\mathfrak{B}'$ have parameters

$$(8.12)\qquad b = \tfrac{1}{6}(s^2 - 3s + 3)(s-1)(s-2),\quad v = s^2 - 3s + 3,\quad k' = 3,\quad r' = \tfrac{1}{2}(s-1)(s-2),$$

where s is an integer. Note that $v \equiv 1, 3 \pmod 6$ according as $s \equiv 1, 2, 4, 5 \pmod 6$ or $s \equiv 0, 3 \pmod 6$, so that $\mathfrak{B}'$ contains Steiner triples on v elements. For the class $\mathfrak{B}$, the first 2-jump in the minimal width sequence $\tilde{\epsilon}(\alpha)$ occurs at

$$(8.13)\qquad \alpha_1 = \left[\frac{b}{r'}\right] = \left[\frac{v}{k'}\right] = 1 + \left[\frac{s(s-3)}{3}\right],$$

brackets denoting the largest integer.

Let D_3^s be the incidence matrix of all triples on s elements. Thus D_3^s is of size $\frac{1}{6}s(s-1)(s-2)$ by s. Consider the following matrix in $\mathfrak{B}'$:

$$(8.14)\qquad B' = \left[\begin{array}{ccc|c} D_3^s & & & O \\ & \ddots & & \\ & & D_3^s & \\ \hline & O & & J \end{array}\right].$$

Here D_3^s occurs $s - 3$ times and J is a matrix of 1's of size $\frac{1}{2}(s-1)(s-2)$ by 3. The matrix B' has

$$(8.15)\qquad \mu_{B'}(1) = s - 2,\quad \mu_{B'}(2) = 2(s-2),\quad \mu_{B'}(3) = s^2 - 3s + 3,$$

whence it follows from (6.3), (6.4) that the width sequence for its complement B has its 2-jumps at

$$(8.16)\qquad \alpha_2 = s - 2 \qquad \text{and} \qquad \alpha_3 = 2s - 5.$$

Comparing (8.16) with (8.13) shows that, for $s \geqslant 7$, both of these occur before α_1.

Our final example is designed to answer the question: Can the difference $\bar{\epsilon}(\alpha) - \tilde{\epsilon}(\alpha)$ for a class $\mathfrak{B}$ be bounded above by anything interesting? It is of course clear that

$$(8.17)\qquad \bar{\epsilon}(\alpha) - \tilde{\epsilon}(\alpha) \leqslant k' - 1$$

for all α. The above example shows that equality can hold for $k' = 3$. We now show that equality can hold for $k' > 3$.

Let $D^s_{k'}$ be the incidence matrix of all k'-tuples of s elements, having $\binom{s}{k'}$ rows and s columns, and consider the class $\mathfrak{B}'$ generated by the matrix

$$B' = \begin{bmatrix} D^s_{k'} & & \\ & \ddots & \\ & & D^s_{k'} \end{bmatrix}, \tag{8.18}$$

where $D^s_{k'}$ occurs t times. The class $\mathfrak{B}'$ has parameters

$$b = t\binom{s}{k'}, \qquad v = st, \qquad k', \qquad r' = \binom{s-1}{k'-1}, \tag{8.19}$$

and we can satisfy the class inequality for $\mathfrak{B}$ by choosing t sufficiently large. The first 2-jump in the minimal width sequence for $\mathfrak{B}$ occurs at

$$\alpha_1 = \left[\frac{v}{k'}\right] = \left[\frac{st}{k'}\right]. \tag{8.20}$$

We assume that k' divides s, so that brackets may be dropped in (8.20).

The matrix (8.18) has

$$\mu_{B'}(\beta) = \begin{cases} \beta t, & \text{if } 0 \leqslant \beta \leqslant k' - 1, \\ st, & \text{if } \beta = k', \end{cases} \tag{8.21}$$

whence

$$\epsilon_B(\alpha) = \begin{cases} \alpha + \left\langle \dfrac{\alpha}{t-1} \right\rangle, & \text{if } \alpha \leqslant (k'-1)(t-1), \\ \alpha + k', & \text{if } \alpha > (k'-1)(t-1). \end{cases} \tag{8.22}$$

For $s > k'(k' - 1)$, we have

$$\alpha_1 - 1 = \frac{st}{k'} - 1 > (k'-1)(t-1),$$

and thus

$$\epsilon_B(\alpha_1 - 1) = \alpha_1 - 1 + k'. \tag{8.23}$$

Since α_1 is the position of the first 2-jump in $\tilde{\epsilon}(\alpha)$,

$$\tilde{\epsilon}(\alpha_1 - 1) = \alpha_1. \tag{8.24}$$

Hence equality holds in (8.17) for $\alpha = \alpha_1 - 1$.

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