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A Hahn-Banach theorem for complex semifields

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The following form of the Hahn-Banach theorem is proved: Let X be a linear space over the complex semifield E and let $f : S \to E$ be a linear functional defined on a subspace S of X. If $p : X \to R^{\Delta}$ is a seminorm with the property that $|f(s)| << p(s)$ for all s in S, then f has a linear extension F to X with the property that $|F(x)| << p(x)$ for all x in X.

1. Introduction

Since the introduction of topological semifields by M.Ja. Antonovskiĭ, V.G. Boltjanskiĭ, and T.A. Sarymsakov [1], several Hahn-Banach type extension theorems for semifield valued linear functionals have been obtained. K. Iséki and S. Kasahara [2] obtained an extension theorem for semifield-valued linear functionals on a real linear space. A generalization of this result was obtained by M. Kleiber and W.J. Pervin [3] who showed that a Hahn-Banach type extension theorem for semifield-valued linear functionals defined on a linear space over a Tychonoff semifield could be obtained. In this paper it is noted that the Kleiber-Pervin result applies to linear spaces over arbitrary semifields and a general form of the Hahn-Banach theorem for complex semifields is obtained.

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2. Definitions and background

A topological semifield is a pair (E, K) where E is a topological ring and $K \subseteq E$ is the subring of "positive" elements satisfying the axioms in [1]. By $x << y$ we denote the usual partial order given by $y - x \in \bar{K}$. An idempotent e is said to be *irreducible* if $0 << e' << e$ implies that $e' = 0$ or $e' = e$. The set of all irreducible idempotents will be denoted by Δ. It is known that, for each $q \in \Delta$, qE is isomorphic to the reals.

The following embedding theorem has been proved by Antonovskiĭ, Boltjanskiĭ, and Sarymsakov [1]:

THEOREM. *Every semifield (E, K) is semifield isomorphic to a subsemifield of the semifield R^Δ of all real-valued functions defined on Δ. The isomorphism is given by $x \to \tilde{x}$ where $\tilde{x}(q)$ is the image of xq under the above mentioned isomorphism between qE and R.*

From this it follows that the "axis" of the semifield E, defined as the minimal subsemifield containing the identity 1 of E, corresponds to those elements of R^Δ all of whose coordinates are the same.

The following Hahn-Banach type theorem has been proved by Kleiber and Pervin [3]:

THEOREM. *Let $p : X \to R^\Delta$ be a sublinear functional on X, a linear space over E, and let f be an E-valued linear functional defined on a linear subspace S of X. If $f(s) << p(s)$ for every $s \in S$, then f has a linear extension F on X such that $F(x) << p(x)$ for all $x \in X$.*

Although the above result is stated in [3] for Tychonoff semifields, i.e., $E = R^\Delta$, the proof given yields the same result for arbitrary semifields since the element $a \in R^\Delta$ constructed in the proof would actually belong to E.

If E is a semifield, then the complexification $\mathcal{E} = E + iE$ is called a *complex semifield*. There is clearly a natural embedding of $\mathcal{E}$

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as a subring of C^{Δ}. If $a \in E \subseteq C^{\Delta}$ and if $\pi_q : E \to C$ is the q-th projection, then we define $|a| \in R^{\Delta}$ by $\pi_q(|a|) = |\pi_q(a)|$.

LEMMA. *For each* $w \in E$ *and* $q \in \Delta$, *there exists an element* $z = z(w, q) \in E$ *such that* $|z|$ *is the identity* 1 *of* E *and* $e_q zw = e_q|w|$.

Proof. $e_q w \in E$ and so, by the embedding theorem, $\pi_{q'}(e_q w) = 0$ if $q' \neq q$. Now $\pi_q(e_q w) \in C$ and so there exists a complex number $e^{i\theta}$ such that $e^{i\theta}\pi_q(e_q w) = |\pi_q(e_q w)|$. Let z be that element of C^{Δ} all of whose coordinates equal $e^{i\theta}$. Since the axis belongs to E, $z \in E$. Clearly $|z| = 1$ and z is the desired element.

3. Complex Hahn-Banach theorem

If X is a linear space over a complex semifield $E = E + iE$, then $p : X \to R^{\Delta}$ is called a *seminorm* if

1) $p(x) \gg 0$;

2) $p(x+y) \ll p(x) + p(y)$;

3) $p(ax) = |a|p(x)$

for every x, $y \in X$ and $a \in E$ where $\ll$ denotes the usual coordinatewise ordering of the semifield R^{Δ}.

THEOREM. *Let* X *be a linear space over the complex semifield* E *and let* $f : S \to E$ *be a linear functional defined on a subspace* $S \subseteq X$. *If* $p : X \to R^{\Delta}$ *is a seminorm with the property that* $|f(s)| \ll p(s)$ *for all* $s \in S$, *then* f *has a linear extension* F *to* X *with the property that* $|F(x)| \ll p(x)$ *for all* $x \in X$.

Proof. Let f_1 and f_2 be the real and imaginary parts of f. We will show first that f_1 and f_2 are (E-valued) linear functionals on S viewed as a linear space over E. For x, $y \in S$ and $a \in E$ we may write

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$$f_1(ax+y) + if_2(ax+y) = f(ax+y) = af(x) + f(y)$$
$$= [af_1(x) + f_1(y)] + i[af_2(x) + f_2(y)] .$$

Equating real and imaginary parts, we obtain the linearity of f_1 and f_2 .

For each $s \in S$ we have $f_1(s) \ll |f_1(s)| \ll |f(s)| \ll p(s)$. By the Kleiber-Pervin extension theorem, f_1 can be extended to an (E-valued) linear functional F_1 on X in such a way that $F_1(x) \ll p(x)$ for all $x \in X$. Since $p(-x) = p(x)$, it may be noted that $F(x) \ll p(x)$ for all x implies that $|F(x)| \ll p(x)$ for all $x \in X$. Now define $F : X \to \mathcal{E}$ by

$$F(x) = F_1(x) - iF_1(ix) .$$

We will show that F is the desired extension.

For each $s \in S$ we have

$$i\left(f_1(s)+if_2(s)\right) = if(s) = f(is) = f_1(is) + if_2(is)$$

so that $f_1(is) = -f_2(s)$. Consequently,

$$F(s) = F_1(s) - iF_1(is) = f_1(s) - if_1(is) = f_1(s) + if_1(s) = f(s)$$

so that F extends f . F is easily seen to be an $\mathcal{E}$-valued linear functional when X is viewed as a linear space over E . To complete the linearity argument, it suffices to show that $F(ix) = iF(x)$. But

$$F(ix) = F_1(ix) - iF_1(-x) = F_1(ix) + iF_1(x) = i\left(F_1(x)-iF_1(ix)\right) = iF(x)$$

as desired.

Finally, we must show that $|F(x)| \ll p(x)$ for all $x \in X$. Fix $x \in X$. For each $q \in \Delta$ we may select, by the Lemma with $w = F(x)$, an element $z \in \mathcal{E}$ such that $|z| = 1$ and $e_q zF(x) = e_q|F(x)|$. Now $F\left(e_q zx\right) = |F\left(e_q x\right)| \in E$ and so $|F\left(e_q zx\right)| = |F_1\left(e_q zx\right)| \ll p\left(e_q zx\right)$. Thus we have

$$e_q|F(x)| = |e_q F(x)| = |z|\,|F\left(e_q x\right)| = |F\left(e_q zx\right)| \ll p\left(e_q zx\right)$$
$$= |e_q z|p(x) = e_q p(x) .$$

Since we have shown $e_q|F(x)| \ll e_q p(x)$ for all $q \in \Delta$, it follows from the embedding theorem that $|F(x)| \ll p(x)$.

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