POSITIVE VALUES OF INHOMOGENEOUS QUATERNARY OUADRATIC FORMS, II

VISHWA CHANDER DUMIR *

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In a previous paper [4] we showed that $\Gamma_{3,1} = \frac{16}{3}$. For the definition of $\Gamma_{r,s}$ for an indefinite quadratic form in $n = r+s$ variables of the type (r, s) see the above paper. Here we shall show that $\Gamma_{2,2} = 16$. More precisely we prove:

THEOREM. *Let $Q(x, y, z, t)$ be an indefinite quaternary quadratic form with determinant $D > 0$ and signature $(2, 2)$. Then given any real numbers x_0, y_0, z_0, t_0 we can find integers x, y, z, t such that*

$$0 < Q(x+x_0, y+y_0, z+z_0, t+t_0) \leqq (16\,|D|)^{\frac{1}{4}}. \tag{1.1}$$

Equality is necessary if and only if either

$$Q(x, y, z, t) \sim \rho Q_1 = \rho(xy+zt); \quad \text{or} \tag{1.2}$$

$$Q(x, y, z, t) \sim \rho Q_2 = \rho(x^2-y^2-z^2+t^2); \quad \text{or} \tag{1.3}$$

$$Q(x, y, z, t) \sim \rho Q_3 = \rho(x^2-y^2-2zt); \tag{1.4}$$

where $\rho \neq 0$. For Q_1 equality occurs if and only if

$$(x_0, y_0, z_0, t_0) \equiv (0, 0, 0, 0) \pmod 1, \text{ for } Q_2 \text{ if and only if}$$
$$(x_0, y_0, z_0, t_0) \equiv (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) \pmod 1 \text{ and for } Q_3 \text{ if and only if}$$
$$(x_0, y_0, z_0, t_0) \equiv (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0) \pmod 1.$$

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2. Some lemmas

In the course of the proof we shall use the following Lemmas:

LEMMA 1. *Let $Q(x, y, z, t)$ be an indefinite quaternary quadratic form of the type* (2,2) *and determinant $D > 0$. Then there exist integers x_1, y_1, z_1, t_1 such that*

$$0 < Q(x_1, y_1, z_1, t_1) \leqq (\tfrac{81}{16}D)^{\frac{1}{4}} \tag{2.1}$$

except when $Q(x, y, z, t) \sim \rho Q_1$, $\rho \neq 0$.

This is Theorem 1 of Oppenheim [6].

LEMMA 2. *Let $\varphi(y, z, t)$ be an indefinite ternary quadratic form with determinant $D < 0$, then we can find integers y_2, z_2, t_2 such that*

$$0 < \varphi(y_2, z_2, t_2) \leqq (\tfrac{9}{4}|D|)^{\frac{1}{3}} \tag{2.2}$$

except when $\varphi(y, z, t) \sim \rho(y^2+zt)$, $\rho > 0$.

This is a theorem due to Oppenheim [5].

LEMMA 3. *Let $\varphi(y, z, t)$ be an indefinite ternary quadratic form with determinant $D < 0$. Then given any real numbers y_0, z_0, t_0 we can find $(y, z, t) \equiv (y_0, z_0, t_0)$ (mod 1) such that*

$$0 < \varphi(y, z, t) \leqq (4|D|)^{\frac{1}{3}}. \tag{2.3}$$

This is the theorem of Barnes [1].

LEMMA 4. *Let $\varphi(y, z, t)$ be an indefinite ternary quadratic form with determinant $D \neq 0$, then given any real numbers y_0, z_0, t_0 we can find $(y, z, t) \equiv (y_0, z_0, t_0)$ (mod 1) satisfying*

$$|\varphi(y, z, t)| \leqq (\tfrac{27}{100}|D|)^{\frac{1}{3}}. \tag{2.4}$$

This is due to Davenport [3].

LEMMA 5. *Let $\psi(z, t)$ be an indefinite binary quadratic form with discriminant $\Delta^2 > 0$ and $\lambda > 0$ be a real number. Then given z_0, t_0 we can find $(z, t) \equiv (z_0, t_0)$ (mod 1) satisfying*

$$-\frac{\Delta}{4\lambda} \leqq \psi(z, t) < \frac{\lambda\Delta}{4}. \tag{2.5}$$

This is Theorem 1 of Blaney [2].

LEMMA 6. *Let $\psi(z, t)$ be an indefinite binary quadratic form with discriminant $\Delta^2 > 0$ and let $\infty \geqq \mu \geqq 3$ be a given real number. Then given z_0, t_0 we can find $(z, t) \equiv (z_0, t_0)$ (mod 1) satisfying*

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$$(2.6)\qquad -\frac{\mu\Delta}{\{(1+\mu)(9+\mu)\}^{\frac{1}{2}}} \leqq \psi(z,t) < \frac{\Delta}{\{(1+\mu)(9+\mu)\}^{\frac{1}{2}}}.$$

If $\mu = \infty$, *equality occurs if and only if*

$$(2.7)\qquad \begin{aligned} &\psi(z,t) \sim c\psi_1(z,t) = czt,\ (z_0, t_0) \equiv (0,0)\ (\mathit{mod}\ 1);\ \mathit{or}\\ &\psi(z,t) \sim c\psi_2(z,t) = c(z^2-t^2);\ (z_0,t_0) \equiv (\tfrac{1}{2},\tfrac{1}{2})\ (\mathit{mod}\ 1);\ c>0.\end{aligned}$$

This is Theorem 2 of Blaney [2].

LEMMA 7. *Let* $\psi(z,t)$ *be an indefinite binary quadratic form with discriminant* $\Delta^2 > 0$. *Then given* $\nu > 1$ *and any real numbers* z_0, t_0 *there exist* $(z,t) \equiv (z_0,t_0)$ (*mod* 1) *such that*

$$(2.8)\qquad -\frac{\nu^2\Delta}{\{(\nu-1)^3(\nu+3)\}^{\frac{1}{2}}} \leqq \psi(z,t) < -\frac{\Delta}{\{(\nu-1)^3(\nu+3)\}^{\frac{1}{2}}}.$$

This is Theorem 3 of Blaney [2].

3. Proof of the Theorem

Let

$$(3.1)\qquad m = \inf\{Q(x,y,z,t): x,y,z,t \text{ integers}, Q(x,y,z,t) > 0\}$$

3.1. CASE $m = 0$

LEMMA 8. *If* $m = 0$, *then the theorem is true.*

PROOF. Since $m = 0$; given ε_0 $(0 < \varepsilon_0 < 1)$ we can find integers x_1, y_1, z_1, t_1 such that

$$0 < Q(x_1,y_1,z_1,t_1) = \varepsilon < \varepsilon_0,\ (x_1,y_1,z_1,t_1) = 1.$$

By replacing Q by an equivalent form we can suppose $Q(1,0,0,0) = \varepsilon$. Then $Q(x,y,z,t)$ can be written as

$$Q(x,y,z,t) = \varepsilon(x+hy+gz+ut)^2 - \varphi(y,z,t);$$

where $\varphi(y,z,t)$ is an indefinite ternary quadratic form with determinant $-D/\varepsilon < 0$. By Lemma 3, we can find $(y,z,t) \equiv (y_0,z_0,t_0)$ (mod 1) such that

$$0 < \varphi(y,z,t) = \beta^2 \leqq \left(\frac{4D}{\varepsilon}\right)^{\frac{1}{3}}.$$

Let $\alpha = hy+gz+ut$ and choose $x \equiv x_0$ (mod 1) with

$$\frac{\beta}{\sqrt{\varepsilon}} < x+\alpha \leqq \frac{\beta}{\sqrt{\varepsilon}}+1,$$

so that

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$$0 < Q(x, y, z, t) = \varepsilon(x+\alpha)^2-\beta^2 \leqq \varepsilon+2\beta\sqrt{\varepsilon}$$

$$(3.2) \qquad \leqq \varepsilon+2\left(\frac{4D}{\varepsilon}\right)^{\frac{1}{8}}\varepsilon^{\frac{1}{2}}$$

$$< \varepsilon_0+2(4D)^{\frac{1}{8}}\varepsilon_0^{\frac{3}{8}}.$$

Since ε_0 can be chosen arbitrarily small, the right hand side of (3.2) can be made as small as we please and the lemma follows.

3.2. Proof continued

Lemma 9. *If $Q(x, y, z, t) \sim mQ_1 = m(xy+zt)$, then the theorem is true. Equality is needed for Q_1 if and only if $(x_0, y_0, z_0, t_0) \equiv (0, 0, 0, 0)$ (mod 1).*

Proof. Without loss of generality we can suppose that

$$Q = Q_1 = xy+zt.$$

Take any $(z, t) \equiv (z_0, t_0)$ (mod 1). Choose $y \equiv y_0$ (mod 1) with $0 < y \leqq 1$ and then take $x \equiv x_0$ (mod 1) to satisfy

$$0 < Q(x, y, z, t) = xy+zt \leqq y \leqq 1 = (16D)^{\frac{1}{4}}.$$

Equality can occur only if $y_0 \equiv 0$ (mod 1). By symmetry for equality we must have

$$x_0 \equiv y_0 \equiv z_0 \equiv t_0 \equiv 0 \text{ (mod 1)}.$$

Clearly equality is necessary when $(x_0, y_0, z_0, t_0) \equiv (0, 0, 0, 0)$ (mod 1). This completes the proof of the Lemma.

From now on we can suppose $m > 0$ and

$$(3.3) \qquad Q \nsim mQ_1 = m(xy+zt).$$

Then given $0 < \varepsilon_0 < \frac{1}{16}$, we can find integers x_1, y_1, z_1, t_1 to satisfy

$$Q(x_1, y_1, z_1, t_1) = \frac{m}{1-\varepsilon} \leqq (\tfrac{81}{16}D)^{\frac{1}{4}}; \quad 0 \leqq \varepsilon < \varepsilon_0;$$

by Lemma 1.

By definition of m we must have $(x_1, y_1, z_1, t_1) = 1$; since $1-\varepsilon > \frac{1}{4}$. By a suitable unimodular transformation we can suppose that $Q(1, 0, 0, 0) = m/1-\varepsilon$. $Q(x, y, z, t)$ can then be written as

$$Q(x, y, z, t) = \frac{m}{1-\varepsilon}\{(x+hy+gz+ut)^2-\varphi(y, z, t)\};$$

where $\varphi(y, z, t)$ is an indefinite ternary quadratic form of determinant

$$D_1 = -\frac{D}{\left(\frac{m}{1-\varepsilon}\right)^4} \leqq -\tfrac{16}{81}.$$

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Also, for integers x, y, z, t we have either $Q(x, y, z, t) \leqq 0$ or $Q(x, y, z, t) \geqq m$; i.e. either

$$(x+hy+gz+ut)^2-\varphi(y, z, t) \leqq 0 \quad \text{or}$$
$$(x+hy+gz+ut)^2-\varphi(y, z, t) \geqq 1-\varepsilon.$$

Because of homogeneity it suffices to prove

THEOREM A. *Let*

$$Q(x, y, z, t) = (x+hy+gz+ut)^2-\varphi(y, z, t); \tag{3.4}$$

where $\varphi(y, z, t)$ is an indefinite ternary quadratic form of determinant

$$D_1 = -D \leqq -\tfrac{16}{81}. \tag{3.5}$$

Let $0 < \varepsilon_0 < \frac{1}{16}$ be given arbitrarily small. Suppose that for integers x, y, z, t we have either

$$Q(x, y, z, t) \leqq 0 \quad \text{or} \quad Q(x, y, z, t) \geqq 1-\varepsilon \tag{3.6}$$

where $0 \leqq \varepsilon < \varepsilon_0 < \frac{1}{16}$. Let

$$d = (16D)^{\frac{1}{4}}, \tag{3.7}$$

so that from (3.5) we have $d \geqq \frac{4}{3}$. Then given any real numbers x_0, y_0, z_0, t_0 we can find $(x, y, z, t) \equiv (x_0, y_0, z_0, t_0)$ (mod 1) such that

$$0 < Q(x, y, z, t) \leqq d. \tag{3.8}$$

Equality holds in (3.8) if and only if $Q = Q_2$ or Q_3.

3.3. PROOF OF THEOREM A

LEMMA 10. *Let α, β, d be real numbers with $d \geqq 1$. Then for any real x_0 there exists $x \equiv x_0$ (mod 1) satisfying*

$$0 < (x+\alpha)^2-\beta^2 \leqq d, \tag{3.9}$$

provided

$$\beta^2 < \left(\frac{[d]}{2}\right)^2. \tag{3.10}$$

If d is not an integer (3.9) is true with strict inequality. If d is an integer a sufficient condition for (3.9) to be true with strict inequality is that

$$\beta^2 < \left(\frac{d-1}{2}\right)^2. \tag{3.11}$$

This is Lemma 6 of my paper [4].

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LEMMA 11. *Suppose we can find* $(y, z, t) \equiv (y_0, z_0, t_0)$ *(mod 1) satisfying*

$$(3.12) \qquad -(d-\tfrac{1}{4}) < \varphi(y, z, t) \begin{cases} \leqq \left(\dfrac{d-1}{2}\right)^2, & \text{if } d \text{ is an integer} \\ < \left(\dfrac{[d]}{2}\right)^2, & \text{if } d \text{ is not an integer.} \end{cases}$$

Then for any x_0 *there exists* $x \equiv x_0$ *(mod 1) such that*

$$(3.13) \qquad 0 < Q(x, y, z, t) \leqq d.$$

Further strict inequality in (3.12) *implies strict inequality in* (3.13).

PROOF. If $-(d-\frac{1}{4}) < \varphi(y, z, t) < 0$, choose $x \equiv x_0$ (mod 1) with $|x+hy+gz+ut| \leqq \frac{1}{2}$, so that

$$0 < Q(x, y, z, t) = (x+hy+gz+ut)^2-\varphi(y, z, t) < \tfrac{1}{4}+d-\tfrac{1}{4} = d.$$

If $\varphi(y, z, t) \geqq 0$, the result follows from Lemma 10 with $\alpha = hy+gz+ut$ and $\beta^2 = \varphi(y, z, t)$.

LEMMA 12. *If* $d > 6$, *then the theorem is true with strict inequality.*

PROOF. By Lemma 3, we can find $(y, z, t) \equiv (y_0, z_0, t_0)$ (mod 1) such that

$$0 < \varphi(y, z, t) < (4D)^{\frac{1}{3}} = (\tfrac{1}{4}d^4)^{\frac{1}{3}}.$$

Therefore (3.12) is satisfied with strict inequality if

$$(\tfrac{1}{4}d^4)^{\frac{1}{3}} < \left(\frac{d-1}{2}\right)^2$$

or

$$f(d) = d^3-7d^2+3d-1 > 0;$$

which is clearly true for $d \geqq 7$. If $6 < d < 7$, then (3.12) is satisfied if we have $(\frac{1}{4}d^4)^{\frac{1}{3}} < 9$ or $d^2 < 54$, which is true for $d < 7$. Thus (3.12) is satisfied and the result follows from Lemma 11.

LEMMA 13. *If* $3 < d \leqq 6$, *then again the theorem is true with strict inequality.*

PROOF. By Lemma 4, we can find $(y, z, t) \equiv (y_0, z_0, t_0)$ (mod 1) such that

$$|\varphi(y, z, t)| \leqq (\tfrac{27}{100}|D|)^{\frac{1}{3}} = (\tfrac{27}{1600}d^4)^{\frac{1}{3}}.$$

Now $(\frac{27}{1600}d^4)^{\frac{1}{3}} < d-\frac{1}{4}$, if

$$f(d) = \frac{(4d-1)^3}{d^4} > \tfrac{27}{25}.$$

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Since $f'(d) < 0$ for $d > 1$, $f(d)$ is a decreasing function of d. Therefore for $3 < d \leqq 6$, $f(d) \geqq f(6) = \frac{2\cdot 3^3}{6^4} > \frac{27}{25}$. Also

$$\left(\tfrac{27}{1600}d^4\right) < \begin{cases} \left(\dfrac{d-1}{2}\right)^2 & \text{if } 4 \leqq d \leqq 6 \\ \left(\dfrac{[d]}{2}\right)^2 & \text{if } 3 < d < 4 \end{cases}$$

can be easily verified to be true. Thus $\varphi(y, z, t)$ satisfies (3.12) and the result follows from Lemma 11.

LEMMA 14. *If* $\varphi(y, z, t) \sim \rho(y^2+zt)$, $\rho > 0$, $d \leqq 3$, *then again* (3.8) *holds with strict inequality.*

PROOF. Without loss of generality we can suppose

$$\varphi(y, z, t) = \rho(y^2+zt), \quad \rho > 0$$

so that

$$Q(x, y, z, t) = (x+hy+gz+ut)^2-\rho(y^2+zt).$$

By replacing x by $x+\alpha y+\beta z+\gamma t$ where α, β, γ are suitable integers we can suppose that

$$|h| \leqq \tfrac{1}{2}, \quad |g| \leqq \tfrac{1}{2}, \quad |u| \leqq \tfrac{1}{2}.$$

We first assert that $h = g = u = 0$. If $u \neq 0$, then

$$0 < Q(0, 0, 0, 1) = u^2 \leqq \tfrac{1}{4} < 1 - \varepsilon,$$

contrary to (3.6). Similarly $g = 0$. If $h \neq 0$, then

$$0 < Q(0, 1, 1, -1) = h^2 \leqq \tfrac{1}{4} < 1 - \varepsilon,$$

contrary to (3.6). Therefore,

$$Q(x, y, z, t) = x^2-\rho(y^2+zt).$$

Choose any $(x, y) \equiv (x_0, y_0) \pmod 1$. Choose $z \equiv z_0 \pmod 1$ with $0 < z \leqq 1$. Now choose $t \equiv t_0 \pmod 1$ to satisfy

$$0 < x^2-\rho y^2-\rho zt \leqq \rho z \leqq \rho = (4D)^{\frac{1}{3}} = (\tfrac{1}{4}d^4)^{\frac{1}{3}} < d,$$

since $d \leqq 3 < 4$. This proves the Lemma.

3.4. PROOF OF THEOREM A CONTINUED

From now on we can suppose that

$$\tfrac{4}{3} \leqq d \leqq 3; \quad \varphi(y, z, t) \not\sim \rho(y^2+zt), \quad \rho > 0. \tag{3.14}$$

By Lemma 2, we can find integers y_2, z_2, t_2 such that $(y_2, z_2, t_2) = 1$ and

(3.15) $$0 < a = \varphi(y_2, z_2, t_2) \leqq (\tfrac{9}{4}D)^{\frac{1}{3}} = (\tfrac{9}{64}d^4)^{\frac{1}{3}}.$$

By a unimodular transformation we can suppose that

(3.16) $$\varphi(y, z, t) = a\{(y+fz+vt)^2+\psi(z, t)\},$$

where $\psi(z, t)$ is an indefinite binary quadratic form with discriminant

(3.17) $$\Delta^2 = \frac{4D}{a^3} = \frac{d^4}{4a^3}.$$

Without loss of generality we can also suppose that

(3.18) $$|h| \leqq \tfrac{1}{2}, \quad |g| \leqq \tfrac{1}{2}, \quad |u| \leqq \tfrac{1}{2}, \quad |f| \leqq \tfrac{1}{2}, \quad |v| \leqq \tfrac{1}{2}.$$

In view of Lemma 11, if we can show that there exist $(y, z, t) \equiv (y_0, z_0, t_0)$ (mod 1) satisfying

(A) $$-(d-\tfrac{1}{4}) < \varphi(y, z, t) = a\{(y+fz+vt)^2+\psi(z, t)\} \begin{cases} < 1 \text{ if } 2 < d \leqq 3 \\ \leqq \tfrac{1}{4} \text{ if } \tfrac{4}{3} \leqq d \leqq 2 \end{cases}$$

then the proof of Theorem A will be complete except for the equality part.

LEMMA 15. *If* $2 < d \leqq 3$, *then again the theorem is true with strict inequality.*

PROOF. Since $d \leqq 3$, we have from (3.15)

$$0 < a \leqq (\tfrac{9}{64}d^4)^{\frac{1}{3}} \leqq \tfrac{9}{4}.$$

Let

$$\lambda = \frac{4-a}{a\Delta},$$

so that $\lambda > 0$. By Lemma 5, we can find $(z, t) \equiv (z_0, t_0)$ (mod 1) such that

$$-\frac{d^4}{16a^2(4-a)} = -\frac{a\Delta^2}{4(4-a)} = -\frac{\Delta}{4\lambda} \leqq \psi(z, t) < \frac{\lambda\Delta}{4} = \frac{1}{a} - \tfrac{1}{4}.$$

If

$$-\frac{4d-1}{4a} < \psi(z, t) < \frac{1}{a} - \tfrac{1}{4},$$

choose $y \equiv y_0$ (mod 1) with $|y+fz+vt| \leqq \tfrac{1}{2}$, so that

$$-(d-\tfrac{1}{4}) < \varphi(y, z, t) = a\{(y+fz+vt)^2+\psi(z, t)\} < a\left(\tfrac{1}{4}+\frac{1}{a}-\tfrac{1}{4}\right) = 1.$$

Thus (A) is satisfied and the result follows in this case. Let now

(3.19) $$\frac{4d-1}{4a} \leqq \beta = -\psi(z, t) \leqq \frac{d^4}{16a^2(4-a)}.$$

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(A) will be satisfied if we can find $y \equiv y_0 \pmod 1$ to satisfy

$$0 < (y+fz+vt)^2 - \left(\beta - \frac{4d-1}{4a}\right) < \frac{1}{a} + \frac{4d-1}{4a} = \frac{4d+3}{4a},$$

$$\frac{4d+3}{4a} > 1; \quad \text{since } a \leqq \tfrac{9}{4}, \quad d > 2.$$

In view of Lemma 10, this is possible if we have

$$0 \leqq \beta - \frac{4d-1}{4a} < \left(\frac{\frac{4d+3}{4a} - 1}{2}\right)^2.$$

This by (3.19) is possible if

$$\frac{d^4}{16a^2(4-a)} - \frac{4d-1}{4a} < \left(\frac{4d+3-4a}{8a}\right)^2.$$

A slight calculation shows that this is so if

$$(3.20) \qquad \begin{aligned} f(a, d) &= a(13-4d-4a)^2 + 4\{d^4 - (4d+3)^2\} < 0. \\ \frac{\partial f}{\partial a} &= (13-4d-4a)(13-4d-12a). \end{aligned}$$

Therefore, since $d \leqq 3$,

$$\max f(a, d) \leqq \max \left\{ f\left(\frac{13-4d}{12}, d\right), f\left((\tfrac{9}{64}d^4)^{\frac{1}{3}}, d\right) \right\},$$

for

$$a \leqq (\tfrac{9}{64}d^4)^{\frac{1}{3}} \leqq \max \left\{ f\left(\frac{13-4d}{12}, d\right), f(\tfrac{9}{4}, d) \right\}.$$

For $2 < d \leqq 3$,

$$\begin{aligned} f\left(\frac{13-4d}{12}, d\right) &= \frac{(13-4d)^3}{27} - 4(d^2+4d+3)(4d+3-d^2) \\ &\leqq \frac{(13-8)^3}{27} - 4(4+8+3)(4\cdot 3+3-9) \\ &< 0, \end{aligned}$$

and

$$\begin{aligned} f(\tfrac{9}{4}, d) &= 4\{9(1-d)^2 + d^4 - (4d+3)^2\} \\ &= 4d(d^3 - 7d - 42) \\ &< 0. \end{aligned}$$

Hence (3.20) is satisfied, so that (A) holds and the result follows.

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LEMMA 16. *If* $\frac{4}{3} \leqq d \leqq 2$, *then again the theorem is true.*

PROOF. We shall distinguish the following three subcases:

(i) $1 < a \leqq (\frac{9}{64}d^4)^{\frac{1}{3}}$

(ii) $0 < a < 1$

(iii) $a = 1$.

Proof of (i). Let $\nu > 1$ be a solution of

$$f(\nu) = \nu^4 - 6\nu^2 + 8\nu - 3 - \frac{4d^4}{a(a-1)^2} = 0.$$

Such a ν exists, since $f(1) < 0$, $f(\infty) > 0$. Then

$$\frac{\Delta}{\{(\nu-1)^3(\nu+3)\}^{\frac{1}{2}}} = \frac{a-1}{4a}.$$

By Lemma 7, we can find $(z, t) \equiv (z_0, t_0) \pmod 1$ to satisfy

$$-\nu^2 \frac{(a-1)}{4a} \leqq \psi(z, t) = -\beta < -\frac{a-1}{4a}.$$

If

$$\frac{a-1}{4a} < \beta < \frac{4d-1}{4a},$$

choose $y \equiv y_0 \pmod 1$ with $|y+fz+vt| \leqq \frac{1}{2}$, so that

$$-\left(\frac{4d-1}{4a}\right) < -\beta \leqq (y+fz+vt)^2 + \psi(z, t) = (y+fz+vt)^2 - \beta < \tfrac{1}{4} - \frac{a-1}{4a} = \frac{1}{4a}.$$

Thus (A) is satisfied and the result follows. Let now

$$\frac{4d-1}{4a} \leqq \beta \leqq \frac{\nu^2(a-1)}{4a}. \tag{3.21}$$

In order that (A) be satisfied, we want to find $y \equiv y_0 \pmod 1$ such that

$$0 < (y+fz+vt)^2 - \left(\beta - \frac{4d-1}{4a}\right) < \frac{d}{a}. \tag{3.22}$$

Since $1 < a \leqq (\frac{9}{64}d^4)^{\frac{1}{3}}$ and $d \leqq 2$, we have

$$\tfrac{1}{8} < \frac{a^3}{d^3} \leqq \frac{9d}{64} \leqq \tfrac{9}{32} < 1.$$

Therefore $1 < d/a < 2$, so that $[d/a] = 1$, d/a not an integer. By Lemma 10, (3.22) will be satisfied if we have

$$\beta - \frac{4d-1}{4a} < \tfrac{1}{4}.$$

This by (3.21) will be so if

$$\frac{\nu^2(a-1)}{4a} < \frac{4d+a-1}{4a}$$

or

$$\nu^2 < \frac{4d+a-1}{a-1} = \nu_0^2, \quad \text{say.}$$

Since $f'(\nu) = 4(\nu-1)^2(\nu+2) > 0$, $f(\nu)$ is an increasing function of ν, and $f(1) < 0$, it suffices to show that $f(\nu_0) > 0$; or

$$(4d+a-1)^2 - 6(a-1)(4d+a-1) + (8\nu_0-3)(a-1)^2 - \frac{4d^4}{a} > 0,$$

or

$$8a(a-1)^2(\nu_0-1) > 4d\{d^3-4ad+4a(a-1)\}. \tag{3.23}$$

Since $a > 1$, $\nu_0 > 1$, (3.23) is clearly satisfied if we have

$$g(a, d) = d^3-4ad+4a(a-1) \leqq 0 \tag{3.24}$$

$$\begin{aligned}\frac{\partial g}{\partial a} &= 4(2a-d-1)\\ &\leqq 4\{2(\tfrac{9}{64}d^4)^{\frac{1}{3}}-d-1\}\\ &= 4\{\tfrac{3}{2}d(\tfrac{d}{3})^{\frac{1}{3}}-d-1\}\\ &< 4\{\tfrac{3}{2}d-d-1\} \quad (\text{since } d \leqq 2)\\ &\leqq 0.\end{aligned}$$

Therefore for $1 < a \leqq (\frac{9}{64}d^4)^{\frac{1}{3}}$ and $\frac{4}{3} \leqq d \leqq 2$, we have

$$g(a, d) < g(1, d) = d^3-4d = d(d^2-4) \leqq 0.$$

Thus (3.24) is satisfied with strict inequality and the result follows. This proves the result in subcase (i).

Proof of (ii). Let

$$\mu = -5+\left\{16+\frac{4d^4}{a(1-a)^2}\right\}^{\frac{1}{2}} \tag{3.25}$$

be a root of

$$\frac{\Delta}{\{(1+\mu)(\mu+9)\}^{\frac{1}{2}}} = \frac{1-a}{4a}.$$

We have $\mu \geqq 3$, if $a(1-a)^2 \leqq d^4/12$, which is so, since

$$a(1-a)^2 \leqq \tfrac{1}{3}(1-\tfrac{1}{3})^2 = \tfrac{4}{27} < \frac{d^4}{12},$$

since $d \geqq \frac{4}{3}$. Thus by Lemma 6, we can find $(z, t) \equiv (z_0, t_0) \pmod 1$ such that

$$-\frac{\mu\Delta}{\{(1+\mu)(9+\mu)\}^{\frac{1}{2}}} \leqq \psi(z, t) < \frac{\Delta}{\{(1+\mu)(9+\mu)\}^{\frac{1}{2}}},$$

or

$$-\frac{\mu(1-a)}{4a} \leqq \psi(z, t) < \frac{1}{4a} - \tfrac{1}{4}.$$

If

$$-\frac{4d-1}{4a} < \psi(z, t) < \frac{1}{4a} - \tfrac{1}{4},$$

choose $y \equiv y_0 \pmod 1$, such that $|y+fz+vt| \leqq \frac{1}{2}$, so that

$$-(d-\tfrac{1}{4}) < \varphi(y, z, t) = a\{(y+fz+vt)^2+\psi(z, t)\} < \tfrac{1}{4}.$$

Thus (A) is satisfied and the result follows. Let now

$$\frac{4d-1}{4a} \leqq \beta = -\psi(z, t) \leqq \frac{\mu(1-a)}{4a}.$$

In order that (A) be satisfied we want to choose $y \equiv y_0 \pmod 1$ such that

$$0 < (y+fz+vt)^2 - \left(\beta - \frac{4d-1}{4a}\right) < \frac{d}{a}. \tag{3.26}$$

By Lemma 10, (3.26) will be satisfied if we have

$$\beta - \frac{4d-1}{4a} < \left(\frac{d-a}{2a}\right)^2.$$

This will be satisfied if we have

$$\mu\frac{(1-a)}{4a} - \frac{4d-1}{4a} < \left(\frac{d-a}{2a}\right)^2.$$

Substituting for μ from (3.25), a slight simplification shows that the above is true if

$$f(a, d) = 16a^3 - 4a^2(4-d) - 4ad(2-d)(1+d) - d^3 < 0,$$

for $0 < a < 1$.

By the rule of signs, for $\frac{4}{3} \leqq d \leqq 2$, $f(a, d)$ has at most one positive root. Since $f(\infty, d) > 0$ and

$$\begin{aligned} f(1, d) &= 16-4(4-d)-4d(2-d)(1+d)-d^3 \\ &= 3d^3-4d^2-4d \\ &= d(3d+2)(d-2) \\ &\leqq 0 \quad \text{for } \tfrac{4}{3} \leqq d \leqq 2. \end{aligned}$$

Thus for $0 < a < 1$, $\frac{4}{3} \leqq d \leqq 2$, we have

$$f(a, d) < 0.$$

The result then follows from Lemma 11.

Proof of (iii). $a = 1$.

By Lemma 6, with $\mu = \infty$, we can find $(z, t) \equiv (z_0, t_0) \pmod 1$ such that

$$-\frac{d^2}{2} = -\Delta \leqq -\beta = \psi(z, t) < 0 \tag{3.27}$$

by using (3.17).

We want to find $y \equiv y_0 \pmod 1$ to satisfy (A), i.e.

$$-\frac{4d-1}{4} < (y+fz+vt)^2-\beta \leqq \tfrac{1}{4}.$$

If $0 < \beta < \frac{1}{4}(4d-1)$, then the result follows by choosing $y \equiv y_0 \pmod 1$ with $|y+fz+vt) \leqq \frac{1}{2}$. Let now

$$\frac{4d-1}{4} \leqq \beta \leqq \frac{d^2}{2}. \tag{3.28}$$

(A) is equivalent to

$$0 < (y+fz+vt)^2 - \left(\beta - \frac{4d-1}{4}\right) \leqq d. \tag{3.29}$$

By Lemma 10, (3.29) will be satisfied if we have

$$\beta - \frac{4d-1}{4} \leqq \left(\frac{d-1}{2}\right)^2.$$

From (3.28), the above will be true if

$$\frac{d^2}{2} - \frac{4d-1}{4} \leqq \frac{d^2-2d+1}{4}$$

or

$$d \leqq 2,$$

which is so and hence the result follows from Lemma 11. This completes the proof of Lemma 16.

4. Case of equality

LEMMA 17. *Equality occurs if and only if* $Q \sim Q_2$ *or* Q_3.

PROOF. From Lemma 16, it follows that equality can occur only if

$$a = 1, \quad d = 2, \quad \Delta^2 = 4.$$

Also we must have equality in Lemma 6 when $\mu = \infty$, so that either

$$\psi(z, t) \sim c_1(z^2-t^2); \quad (z_0, t_0) \equiv (\tfrac{1}{2}, \tfrac{1}{2}) \pmod 1; \quad \text{or}$$
$$\psi(z, t) \sim c_2 zt; \quad (z_0, t_0) \equiv (0, 0) \pmod 1,$$

where $c_1, c_2 > 0$. Since $\Delta^2 = 4$, we have $c_1 = 1$, $c_2 = 2$. Without loss of generality we can suppose that either

$$\psi(z, t) = z^2-t^2; \quad (z_0, t_0) \equiv (\tfrac{1}{2}, \tfrac{1}{2}) \pmod 1; \quad \text{or}$$
$$\psi(z_2, t_2) = 2zt; \quad (z_0, t_0) \equiv (0, 0) \pmod 1.$$

We now discuss the two cases separately.

Case (i). $\psi(z, t) = z^2-t^2$; $(z_0, t_0) \equiv (\frac{1}{2}, \frac{1}{2}) \pmod 1$.
If equality is to occur in (A), then the inequalities

$$-\tfrac{7}{4} = -\frac{4d-1}{4a} < F(y, z, t) = \left(y+fz+vt+y_0+\frac{f}{2}+\frac{v}{2}\right)^2 + (z+\tfrac{1}{2})^2-(t+\tfrac{1}{2})^2 < \frac{1}{4a} = \tfrac{1}{4}$$

should have no solution in integers y, z, t.

$$-\tfrac{7}{4} < F(y, 0, 0) \leqq \left(y+y_0+\frac{f}{2}+\frac{v}{2}\right)^2 < \tfrac{1}{4}$$

is solvable for integer y unless

$$y_0+\frac{f}{2}+\frac{v}{2} \equiv \tfrac{1}{2} \pmod 1. \tag{4.1}$$

Similarly by considering $F(y, -1, 0)$ and $F(y, 0, -1)$ we find that if equality is to occur we must have

$$y_0-\frac{f}{2}+\frac{v}{2} \equiv \tfrac{1}{2} \pmod 1 \tag{4.2}$$

and

$$y_0+\frac{f}{2}-\frac{v}{2} \equiv \tfrac{1}{2} \pmod 1. \tag{4.3}$$

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From (4.1), (4.2), (4.3) and (3.18) we get

$$f = v = 0, \quad y_0 \equiv \tfrac{1}{2} \pmod 1.$$

Thus if equality is to occur we must have

$$\varphi(y, z, t) = y^2+z^2-t^2, \quad (y_0, z_0, t_0) \equiv (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) \pmod 1.$$

Again, if equality is to occur, the inequalities

$$0 < G(x, y, z, t) = \left(x+hy+gz+ut+x_0+\frac{h}{2}+\frac{g}{2}+\frac{u}{2}\right)^2-(y+\tfrac{1}{2})^2 -(z+\tfrac{1}{2})^2+(t+\tfrac{1}{2})^2 < 2,$$

should have no solution in integers x, y, z, t.

$$0 < G(x, 0, 0, 0) = \left(x+\frac{h}{2}+\frac{g}{2}+\frac{u}{2}+x_0\right)^2-\tfrac{1}{4} < 2$$

is solvable for integer x unless

(4.4) $$\frac{h}{2}+\frac{g}{2}+\frac{u}{2}+x_0 \equiv \tfrac{1}{2} \pmod 1.$$

Similarly by considering $G(x, 0, 0, -1)$, $G(x, 0, -1, 0)$ and $G(x, -1, 0, 0)$ we find that if equality is to occur we must have

(4.5) $$\frac{h}{2}+\frac{g}{2}-\frac{u}{2}+x_0 \equiv \tfrac{1}{2} \pmod 1,$$

(4.6) $$\frac{h}{2}-\frac{g}{2}+\frac{u}{2}+x_0 \equiv \tfrac{1}{2} \pmod 1,$$

(4.7) $$-\frac{h}{2}+\frac{g}{2}+\frac{u}{2}+x_0 \equiv \tfrac{1}{2} \pmod 1.$$

From (4.4), (4.5), (4.6), (4.7) and (3.18) we get

$$h = g = u = 0, \quad x_0 \equiv \tfrac{1}{2} \pmod 1.$$

Thus in case (i), equality can occur only if

$$Q = x^2-y^2-z^2+t^2 = Q_2, \quad (x_0, y_0, z_0, t_0) \equiv (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) \pmod 1.$$

We next show that equality is needed for this form. For this it suffices to show that for integers x, y, z, t we have either

$$(x+\tfrac{1}{2})^2-(y+\tfrac{1}{2})^2-(z+\tfrac{1}{2})^2+(t+\tfrac{1}{2})^2 \leqq 0 \text{ or } \geqq 2,$$

i.e.

$$X^2-Y^2-Z^2+T^2 \leqq 0 \text{ or } \geqq 8$$

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for odd integers X, Y, Z, T. This is clearly so, since

$$X^2-Y^2-Z^2+T^2 \equiv 1-1-1+1 \equiv 0 \pmod 8$$

for odd integers X, Y, Z, T.

This completes the proof of the lemma in this case.

Case (ii). $\psi(z, t) = 2zt$; $(z_0, t_0) \equiv (0, 0) \pmod 1$.

If equality is to occur in (A), then the inequalities

$$-\tfrac{7}{4} < F(y, z, t) = (y+fz+vt+y_0)^2+2zt < \tfrac{1}{4} \tag{4.8}$$

should have no solutions in integers y, z, t.

By considering $F(y, 0, 0)$, $F(y, 1, 0)$ and $F(y, 0, 1)$ we see that if equality is to occur we must have

$$y_0 \equiv \tfrac{1}{2} \pmod 1, \tag{4.9}$$

$$y_0+f \equiv \tfrac{1}{2} \pmod 1, \tag{4.10}$$

$$y_0+v \equiv \tfrac{1}{2} \pmod 1. \tag{4.11}$$

From (4.9), (4.10), (4.11) and (3.18) we get

$$f = v = 0, \quad y_0 \equiv \tfrac{1}{2} \pmod 1.$$

Thus if equality is to occur we must have

$$\varphi(y, z, t) = y^2+2zt, \quad (y_0, z_0, t_0) \equiv (\tfrac{1}{2}, 0, 0) \pmod 1.$$

Again, for equality, the inequalities

$$0 < G(x, y, z, t) = \left(x+hy+gz+ut+\frac{h}{2}+x_0\right)^2-(y+\tfrac{1}{2})^2-2zt < 2$$

should have no solution in integers x, y, z, t. By considering $G(x, 0, 0, 0)$, $G(x, 0, 0, 1)$, $G(x, 0, 1, 0)$ and $G(x, -1, 0, 0)$ we see that if equality is to occur we must have

$$x_0+\frac{h}{2} \equiv \tfrac{1}{2} \pmod 1, \tag{4.12}$$

$$x_0-\frac{h}{2} \equiv \tfrac{1}{2} \pmod 1, \tag{4.13}$$

$$x_0+\frac{h}{2}+g \equiv \tfrac{1}{2} \pmod 1, \tag{4.14}$$

and

$$x_0+\frac{h}{2}+u \equiv \tfrac{1}{2} \pmod 1. \tag{4.15}$$

From (4.12), (4.13), (4.14), (4.15), and (3.18) we have

$$h = g = u = 0, \quad x_0 \equiv \tfrac{1}{2} \pmod 1.$$

Thus equality can occur only if

$$Q(x, y, z, t) = x^2 - y^2 - 2zt = Q_3, \quad (x_0, y_0, z_0, t_0) \equiv (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0) \pmod 1.$$

We next show that equality is needed for this form. For this it suffices to show that for integers x, y, z, t we have either

$$(x+\tfrac{1}{2})^2 - (y+\tfrac{1}{2})^2 - 2zt \leqq 0 \quad \text{or } \geqq 2, \quad \text{i.e.}$$
$$(2x+1)^2 - (2y+1)^2 - 8zt \leqq 0 \text{ or } \geqq 8.$$

This is obviously true, since left hand side is $\equiv 0 \pmod 8$ for integers x, y, z, t. This completes the proof of the Lemma and the theorem follows.

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University of Illinois
Urbana, Illinois

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