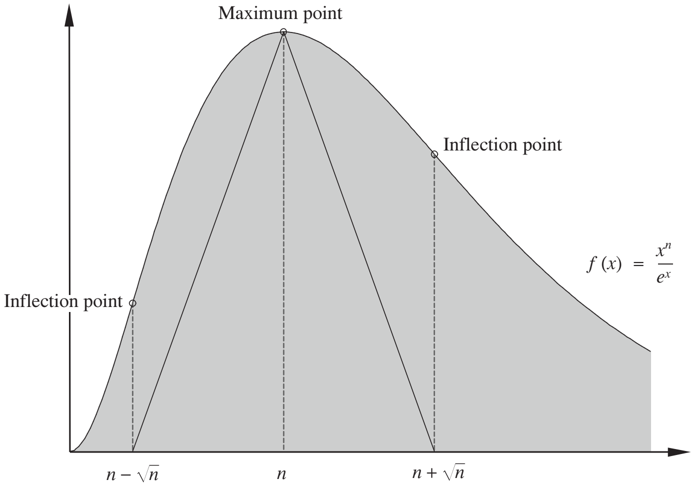
108.12 Proof without words: a lower bound for *n*!



$$n! = \Gamma(n+1) = \int_0^\infty f(x)\,dx > \frac{1}{2}\left(2\sqrt{n}\right)f(n) = \left(\frac{n}{e}\right)^n \sqrt{n}$$

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MEHDI HASSANI
Department of Mathematics,
University of Zanjan, University Blvd.,
45371-38791, Zanjan, Iran
e-mail: *mehdi.hassani@znu.ac.ir*

108.13 Indeterminate exponentials without tears

Every calculus student learns how to solve indeterminate limits of the form $f(n)^{g(n)}$ where $f(n) \to 1$ and $g(n) \to \infty$; most quickly learn to hate and fear this process. It is error-prone, full of tedious algebra, and requires careful attention to L'Hôpital's rule. Here is a typical "fairly simple" example.

$$\begin{aligned}\lim_{n\to\infty} \ln\left(\frac{n+4}{n}\right)^{3n+1} &= \lim_{n\to\infty} \frac{\ln\left(\frac{n+4}{n}\right)}{\frac{1}{3n+1}} \\ &= \lim_{n\to\infty} \frac{\left(\frac{n}{n+4}\right)\left(\frac{-4}{n^2}\right)}{-\frac{3}{(3n+1)^2}} \quad \text{using L'Hôpital's rule} \\ &= \lim_{n\to\infty}\left(\frac{-4}{n(n+4)} \times \frac{-(3n+1)^2}{3}\right) \\ &= \lim_{n\to\infty} \frac{4(3n+1)^2}{3n(n+4)} = 12\end{aligned}$$

and so

$$\lim_{n\to\infty}\left(\frac{n+4}{n}\right)^{3n+1} = e^{12}.$$

What tedium! And this is the short version, suppressing details on the two derivatives (perhaps two quotient rules, perhaps something slightly better). Of course, this may be tedious for students, but some people who are experts use simpler and shorter ways. Indeed, replacing n by $4k$ converts the limit to $\lim_{k \to \infty} \left(\frac{k+1}{k}\right)^{12k+1}$, equivalently $\left(\lim_{k \to \infty} \left(\frac{k+1}{k}\right)^{k}\right)^{12} = e^{12}$. So the problem reduces to the familiar limit.

Here, we are interested in formulating these methods as a general formula for calculating indeterminate limits. We prove the following theorem.

Theorem: Suppose that $f(n)$ is a function with $\lim_{n \to \infty} f(n) = 1$, and $g(n)$ is a function with $\lim_{n \to \infty} g(n) = \infty$. Then

$$\lim_{n \to \infty} f(n)^{g(n)} = e^{\lim_{n \to \infty} g(n)(f(n) - 1)}.$$

We present two proofs for this theorem. In the first proof we assume that the function $f(n)$ is differentiable and then L'Hôpital's rule is used. The second proof needs neither L'Hôpital's rule, nor the hypothesis that $f(n)$ is differentiable, nor interpolation with cubic splines.

First proof: After the use of L'Hôpital's rule,

$$\begin{aligned}\lim_{n \to \infty} \ln f(n)^{g(n)} &= \lim_{n \to \infty} g(n) \ln f(n) \\ &= \lim_{n \to \infty} g(n)\left[f(n) - 1\right] \frac{\ln f(n)}{f(n) - 1} \\ &= \lim_{n \to \infty} g(n)\left[f(n) - 1\right]\end{aligned}$$

and so, if $\lim_{n \to \infty} g(n)\left[f(n) - 1\right] = L$, then $\lim_{n \to \infty} f(n)^{g(n)} = e^{L}$.

Second proof:
Now begin with $\lim_{x \to 0} \frac{1}{x} \ln(1 + x) = 1$. Replacing x by $f(n) - 1$ shows that

$$\lim_{n \to \infty} f(n)^{g(n)} = \lim_{n \to \infty} e^{g(n) \ln(1 + (f(n) - 1))} = e^{\lim_{n \to \infty} g(n)(f(n) - 1)}.$$

With this theorem our "fairly simple" example becomes truly fairly simple:

$$\lim_{n \to \infty} \left(\frac{n + 4}{n}\right)^{3n + 1} = e^{\lim_{n \to \infty} (3n + 1)\left(\frac{n+4}{n} - 1\right)} = e^{\lim_{n \to \infty} \frac{4}{n}(3n + 1)} = e^{\lim_{n \to \infty} 12 + \frac{4}{n}} = e^{12}.$$

This theorem can be applied to the famous Euler's Limit $\lim_{n \to \infty} \left(\frac{n+1}{n}\right)^{n} = e$, and, to some extensions thereof, such as (from [1])

$$\lim_{n \to \infty} \left(\frac{A_{n+1}}{A_n}\right)^{\frac{A_n}{A_{n+1} - A_n}} = e,$$

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where A_n is a strictly increasing sequence of positive numbers satisfying the asymptotic formula $A_{n+1} \sim A_n$.

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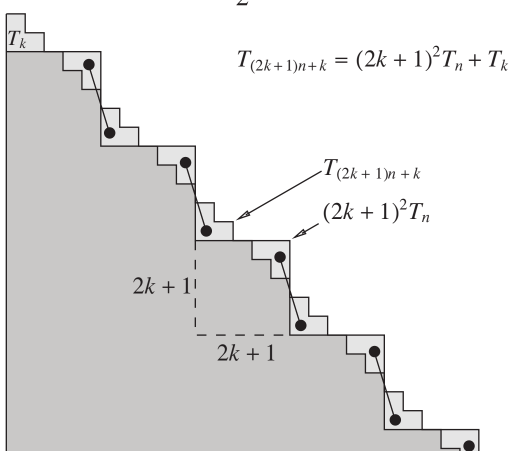
Reference

1. R. Farhadian, A generalization of Euler's limit, *Amer. Math. Monthly*. **129** (2022) p. 384.

REZA FARHADIAN
Department of Statistics,
Razi University, Kermanshah, Iran
e-mail: *farhadian.reza@yahoo.com*
VADIM PONOMARENKO
Department of Mathematics and Statistics,
San Diego State University, San Diego, USA
e-mail: *vponomarenko@sdsu.edu*

108.14 A triangle number identity

The triangle number $T_n = \dfrac{n(n + 1)}{2}, n \geqslant 1$.



PAUL STEPHENSON
Bohmerstrrasse 66, 45144,
Essen, Germany
e-mail: *pstephenson1@me.com*