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RESEARCH ARTICLE

Star-shaped order for distributions characterized by several parameters and some applications

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Abstract

The star-shaped ordering between probability distributions is a common way to express aging properties. A well-known criterion was proposed by Saunders and Moran [(1978). On the quantiles of the gamma and F distributions. *Journal of Applied Probability* 15(2): 426–432], to order families of distributions depending on one real parameter. However, the lifetime of complex systems usually depends on several parameters, especially when considering heterogeneous components. We extend the Saunders and Moran criterion characterizing the star-shaped order when the multidimensional parameter moves along a given direction. A few applications to the lifetime of complex models, namely parallel and series models assuming different individual components behavior, are discussed.

1. Introduction

In many applications where two random variables represent the lifetime of different systems, it is of interest to study their aging properties. This will allow to determine which system is performing better with respect to some given property: the aging rate, lifetime expectancy, skewness of lifetimes, etc. For this purpose, stochastic ordering between random variables provides a convenient way to describe such comparisons. These orderings may be defined through relations between distributions, survival or failure rate functions of the relevant random variables. The monographs by Shaked and Shantikumar [18] or Marshall and Olkin [16], give a good account of various stochastic orders and their applications.

We will be interested in the star-shaped order, introduced by Barlow and Proschan [4] and defined by a monotonicity property on a suitable transformation on the distribution functions, as expressed by Definition 1 below. It can be easily seen that the definition is equivalent to allowing at most one crossing between the distribution functions of scaled lifetimes, as referred in Proposition C.11 of Marshall and Olkin [16] or 4.B.2 in Shaked and Shantikumar [18]. It follows from this characterization that the star-shaped order may, thus, be interpreted as a comparison of the lifetime aging rate for systems that started functioning simultaneously. From the practical point of view, since the distribution functions of the lifetime variables under comparison often do not have an explicit formula, it may be technically difficult to verify the star-shaped ordering. Thus, it becomes relevant to establish equivalent conditions for which the monotonicity of the referred function holds. Using the sign technique referred to in Marshall and Olkin [16] or Shaked and Shantikumar [18], Arab and Oliveira [1] analyzed the ordering relationships within the Gamma and the Weibull families of distributions, later extended to the comparison of lifetimes of parallel systems in Arab *et al.* [2]. However, when the underlying distributions depend on a large number of parameters, this sign analysis becomes rather hard to control and often does not allow for a conclusion. An alternative approach may be based on a criterion proposed by Saunders and Moran [17] when the distribution functions depend on a single real parameter. This criterion turned out to be useful to exhibit order relations within parametric families of distributions (see [9] or [11,12] among many others). As what concerns the lifetimes of more complex systems, the Saunders and Moran's criterion was used by Kochar and Xu [10] to obtain a characterization for parallel systems each one formed by two types of components with exponentially distributed lifetimes. More recently, Arab *et al.* [2] proved that the lifetimes of parallel systems with homogeneous and independent exponential components get smaller (or age faster) with respect to the star-shaped order, as the number of components increases. We note that, for the case of series and parallel systems with more than two heterogeneous and, especially, nonexponentially distributed components, not much work seems to have been done regarding the star-shaped comparability. Since these are complex models, depending on more than one parameter, the Saunders and Moran's [17] result cannot be used. Comparability with respect to star-shaped ordering for multiparameter models has recently been addressed by Belzunce *et al.* [5] and Arriaza *et al.* [3], who proved sufficient conditions allowing to order within a few parametric families of distributions, including the generalized Gamma distribution and types I and II Beta generalized distribution. We propose a different approach, exploring extensions of the Saunders and Moran's criterion to order models depending on several parameters.

This paper is structured as follows. In Section 2, we present the extension of the Saunders and Moran's [17] criterion to families of distributions depending on multidimensional parameters. In Section 3, we discuss a few applications of the obtained criterion to complex systems with heterogeneous components, describing conditions on the parameters so that the star-shaped comparability holds when the lifetimes of the components satisfy suitable proportionality assumptions, including the popular proportional hazard rate (PHR) and proportional reversed hazard rate (PRHR) models.

2. A criterion for the star-shaped order

Let $\mathcal{F}$ denote the family of distribution functions vanishing at 0 with support contained in $[0, +\infty)$. Let X be a nonnegative random variable with distribution function $F_X \in \mathcal{F}$, density function f_X, and survival function $\bar{F}_X$. We start by defining the star-shaped order relation, following Shaked and Shantikumar [18].

Definition 1. *Let X and Y be two nonnegative random variables with distribution functions $F_X, F_Y \in \mathcal{F}$, respectively. The random variable X (or its distribution F_X) is said to be smaller than Y (or its distribution F_Y) in the star-shaped order, denoted by $X \leq_* Y$ (or $F_X \leq_* F_Y$), if $(1/x)F_Y^{-1}(F_X(x))$ is increasing with $x > 0$ (or equivalently, $F_Y^{-1}(u)/F_X^{-1}(u)$ is increasing with $u \in (0, 1)$).*

Remark 2. *It is easy to verify that the star-shaped order is scale invariant, implying that in case of families of distributions that have a scale parameter, we are able to choose the parameter in a convenient way.*

The decision about the star-shaped order often relies on sign variation techniques, as follows from (4.B.2) from Shaked and Shantikumar [18]. Expectedly, the sign variation analysis raises technical difficulties, especially when dealing with distributions involving a large number of parameters, such as parallel systems, series systems, or order statistics. Saunders and Moran [17] proved a more tractable condition for the star-shaped order to hold, providing a full characterization of such relation within a family of distributions.

Theorem 3 ([17]). *Let $\{F_a : a \in I \subseteq \mathbb{R}\}$ be a family of distributions such that $F_a \in \mathcal{F}$ with density f_a which does not vanish on any subinterval of its support. Then $F_a^{-1}(\alpha)/F_a^{-1}(\beta)$ decreases (resp., increases) with respect to $a \in J \subseteq I$, for each fixed $\alpha > \beta$, if and only if $D(a,x) = (1/xf_a(x))(\partial F_a/\partial a)(x)$ increases (resp., decreases) with x on the support of F_a, for every fixed $a \in J \subseteq I$.*

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The simple criterion for the star-shaped relationships that follows is derived as a direct consequence of the monotonicity result presented above.

Theorem 4. *Let $\{F_a : a \in I \subseteq \mathbb{R}\}$ be a family of distributions as in Theorem 3. Then $F_a \leq_* F_b$, for every $a \leq b$ such that $a, b \in J \subseteq I$ if and only if $D(a, x) = (1/xf_a(x))(\partial F_a/\partial a)(x)$ is decreasing with x on the support of F_a, for every $a \in J \subseteq I$.*

Proof. Take $a, b \in J \subseteq I$, such that $a \leq b$. For $\alpha \geq \beta$, we have that

$$F_a \leq_* F_b \Leftrightarrow \frac{F_b^{-1}(\beta)}{F_a^{-1}(\beta)} \leq \frac{F_b^{-1}(\alpha)}{F_a^{-1}(\alpha)} \Leftrightarrow \frac{F_a^{-1}(\alpha)}{F_a^{-1}(\beta)} \leq \frac{F_b^{-1}(\alpha)}{F_b^{-1}(\beta)},$$

which is equivalent to $F_a^{-1}(\alpha)/F_a^{-1}(\beta)$ being increasing with respect to a. Taking into account Theorem 3, the conclusion follows. □

Note that Theorem 4 states a necessary and sufficient condition for the star-shaped order to hold between distributions F_a for every $a \in J$. However, in general, distributions may depend on more than one parameter, as happens for parallel or series systems with heterogeneous components and, in general, coherent systems. Hence, it is natural to seek for extensions of Theorem 4 to families of distributions indexed by higher dimensional parameters.

To state our result, we need to introduce some notations. Let $\mu \in I \subseteq \mathbb{R}^n$ and $v \in \mathbb{R}^n$, and consider $\mu + tv$, $t \in \mathbb{R}$, the line that goes through μ and has vector v. We will denote by $L_{(\mu,v)} = \{\lambda_t \in I \subseteq \mathbb{R}^n : \lambda_t = \mu + tv, t \in \mathbb{R}\}$. Moreover, given a family of distributions F_λ, $\nabla F_\lambda(x)$ stands for the gradient of $F_\lambda(x)$ with respect to the parameter λ and by $\langle v, \nabla F_\lambda(x)\rangle$, we denote the inner product between v and $\nabla F_\lambda(x)$.

Theorem 5. *Let $\{F_\lambda : \lambda \in I \subseteq \mathbb{R}^n\}$ be a family of distributions such that $F_\lambda \in \mathcal{F}$ and has density function f_λ which does not vanish on any subinterval of its support. Let $\mu \in I$, $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ and $J \subseteq I$. Then $F_{\lambda_t} \leq_* F_{\lambda_{t'}}$, for every $\lambda_t, \lambda_{t'} \in L_{(\mu,v)} \cap J$ and $t \leq t'$, if and only if $R(x) = \langle v, \nabla F_\lambda(x)\rangle / xf_\lambda(x)$ is decreasing with x on the support of F_λ, for every $\lambda \in L_{(\mu,v)} \cap J$.*

Proof. We want to prove that $G_t \leq_* G_{t'}$, for $t \leq t'$, where $G_t(x) = F_{\lambda_t}(x)$, for every x on the support of F_λ. By Theorem 4, this is equivalent to $(1/xg_t(x))(\partial G_t/\partial t)(x)$ being decreasing with x, where $g_t(x) = G_t'(x) = f_\lambda(x)$. Therefore, we may conclude that $F_{\lambda_t} \leq_* F_{\lambda_{t'}}$ if and only if $\langle v, \nabla F_\lambda(x)\rangle / xf_\lambda(x)$ is decreasing with x, for every $\lambda \in L_{(\mu,v)} \cap J$. □

Remark 6. *Note that one could think, of comparing distributions whose parameters belong to some general parametric curve, instead of straight lines, which would lead to an obvious extension of Theorem 5.*

In the case of families of distributions with two-dimensional parameters, the following version, using the slope of the line $L_{(\mu,v)}$, is convenient.

Proposition 7. *Let $\{F_\lambda : \lambda \in I \subseteq \mathbb{R}^2\}$ be a family of distributions such that $F_\lambda \in \mathcal{F}$ and has density function f_λ which does not vanish on any subinterval of its support. Let $\mu \in I$, $v = (v_1, v_2) \in \mathbb{R}^2$ and $J \subseteq I$. If $v_1 < 0$ (resp., $v_1 > 0$), then $F_{\lambda_t} \leq_* F_{\lambda_{t'}}$, for every $\lambda_t, \lambda_{t'} \in L_{(\mu,v)} \cap J$ where $t \leq t'$, if and only if $Q(x) = (1/xf(x))((\partial F_\lambda/\partial\lambda_1)(x) + k(\partial F_\lambda/\partial\lambda_2)(x))$ is increasing (resp., decreasing) with x on the support of F_λ, for every $\lambda \in L_{(\mu,v)} \cap J$, where $k = v_2/v_1$.*

Proof. According to Theorem 5, we have that, $F_{\lambda_t} \leq_* F_{\lambda_{t'}}$, for every $\lambda_t, \lambda_{t'} \in L_{(\mu,v)} \cap J$, where $t \leq t'$, if and only if $R(x) = (1/xf_\lambda(x))((\partial F_\lambda/\partial\lambda_1)(x)v_1 + (\partial F_\lambda/\partial\lambda_2)(x)v_2)$ is decreasing with $x > 0$, for every $\lambda \in L_{(\mu,v)} \cap J$. Factorizing $R(x)$ by v_1 and taking into account the sign of v_1, the conclusion follows. □

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3. Applications

We now apply the results proved in the previous section to prove comparability, with respect to the star-shaped order, for some models that are popular in reliability theory. Throughout this section, $X_1, \ldots, X_n$ will represent the lifetimes of the components of a complex system. The lifetime of a parallel system is $X_{(n)} = \max(X_1, \ldots, X_n)$, while the lifetime of a series system is given by $X_{(1)} = \min(X_1, \ldots, X_n)$.

3.1. Parallel systems with dependent components

First, we provide a condition for the star-shaped order to hold between parallel systems, for which their lifetime components are dependent and identically distributed. We say that the joint distribution of $(X_1, \ldots, X_n)$ follows an n-dimensional FGM (Farlie–Gumbel–Morgenstern, cf. [14]) distribution if

$$F_{(X_1,\ldots,X_n)}(x_1, \ldots, x_n) = \prod_{i=1}^{n} F(x_i)\left(1 + \sum_{1\leq j<k\leq n} a_{jk}\bar{F}(x_j)\bar{F}(x_k)\right), \tag{1}$$

where $|\sum_{1\leq j<k\leq n} a_{jk}| \leq 1$. Then, the distribution function of $X_{(n)}$ is given by

$$F_c(x) = F^n(x)(1 + c\bar{F}^2(x)), \tag{2}$$

where $c = \sum_{1\leq j<k\leq n} a_{jk} \in [-1, 1]$. Note that the constant c describes the strength of dependence among the random variables, while its sign reveals the direction of the dependence, that is, if $c > 0$ ($c < 0$), the components are positively (negatively) dependent.

Proposition 8. *Let $\{F_c : c \in [-1, 1]\}$ be a family of distributions defined as in (2). Then $F_a \leq_* F_b$, whenever $-1 \leq a < b \leq n/(n+2)$, if $Q_1(x) = F(x)\bar{F}(x)/xf(x)$ is decreasing with x on the support of F_c.*

Proof. According to Theorem 4, we need to prove that $D(c, x) = (1/xf_c(x))(\partial F_c/\partial c)(x)$ is decreasing with $x > 0$, for $-1 \leq c \leq n/(n+2)$. After simplifications, we have $D(c, x) = Q_1(x)h(x)$, where $h(x) = \bar{F}(x)/(n(1+c) - 2c(n+1)F(x) + c(n+2)F^2(x))$. Since $(\partial F_c/\partial c)(x) \geq 0$, we have that $D(c, x) \geq 0$. Now, taking into account that obviously $Q_1(x) \geq 0$, it follows that $h(x) \geq 0$, for $x > 0$. If $c = 0$, then h is decreasing, and the conclusion follows. Finally, assume that $|c| \leq 1$ and $c \neq 0$. Given that F is increasing and nonnegative, the monotonicity of h is the same as the monotonicity of the companion function

$$\tilde{h}(x) = \frac{1 - x}{n(1+c) - 2c(n+1)x + c(n+2)x^2},$$

for $x \in (0, 1)$. Differentiating, it is easily seen that $\tilde{h}'$ has the same sign as $N(x) = -n(1+c) + 2c(n+1) - 2c(n+2)x + c(n+2)x^2$. When $c > 0$, $N(x) \leq N(0) = c(n+2) - n \leq 0$, while when $c < 0$, $N(x) \leq N(1) \leq 0$. Thus, $\tilde{h}'(x) \leq 0$, implying that h is decreasing. Taking into account that Q_1 is a positive decreasing function, the conclusion follows. □

Remark 9. *Moreover, Proposition 8 implies that, with respect to systems with independent components, negatively dependent components results in faster aging of parallel systems, while positive dependence mean slower aging rates.*

Remark 10. *The decreasingness assumption on Q_1 is a technical assumption in Proposition 8. We provide some additional information showing that it is satisfied by Gamma distributions with shape parameter smaller than 1. Note that verifying that Q_1 is decreasing is equivalent to proving that, for every $c > 0$, $H(x) = \bar{F}(x)F(x) - cxf(x)$ changes sign at most once, and if the sign change occurs, it is in the order "+, −", as x goes from 0 to $+\infty$, where F is the distribution function of a Gamma distribution*

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with shape parameter $\alpha < 1$ and scale parameter $\lambda = 1$. To conclude about the sign variation of H, we need to study its derivative.

$$H'(x) = \frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)}(-F(x) + \bar{F}(x) - c\alpha + cx) = \frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)}V(x).$$

As $x > 0$, the sign of H' is the same as the sign of V. Differentiating again, we obtain $V'(x) = -2(x^{\alpha-1}e^{-x}/\Gamma(\alpha)) + c$, and $V''(x) = 2(x^{\alpha-2}e^{-x}/\Gamma(\alpha))K(x)$, where $K(x) = -\alpha + 1 + x$. Thus, K is increasing. Furthermore, $K(0) = -\alpha + 1$ and $\lim_{x\to+\infty} K(x) = +\infty$. Since $\alpha < 1$, it follows that $K(x) > 0$, for $x > 0$ therefore, $V''(x) > 0$, for $x > 0$ thus, V' is increasing. Moreover, $\lim_{x\to 0^+} V'(x) = -\infty$ and $\lim_{x\to+\infty} V'(x) = c$; hence, V' has sign variation "$-,+$", for $x > 0$. Consequently, V has monotonicity "$\searrow\nearrow$". Now, as $\lim_{x\to 0^+} V(x) = 1 - c\alpha$, and $\lim_{x\to+\infty} V(x) = +\infty$, the sign variation of V may be "$+,-,+$", "$-,+$", or "$+$". In the first case, H has monotonicity "$\nearrow\searrow\nearrow$", so, given that $\lim_{x\to 0^+} H(x) = 0$ and $\lim_{x\to+\infty} H(x) = 0$, H has sign variation "$+,-$". If V has sign variation "$-,+$", H will have monotonicity "$\searrow\nearrow$", implying that $H(x) < 0$, for $x > 0$. Finally, the case where $V(x) \geq 0$ cannot happen, since $\lim_{x\to 0^+} H(x) = 0$ and $\lim_{x\to+\infty} H(x) = 0$. Thus, H changes sign at most once in the order "$+,-$", and therefore Q_1 is increasing.

Remark 11. *For other families of distributions, such as Pareto (with support $(0, +\infty)$) or Weibull distributions, it is easy to verify that Q_1 is also decreasing, regardless of the value of the shape parameter, proceeding analogously as in Remark 10.*

3.2. Complex systems based on PHR and PRHR models

We now prove some ordering relationships for two models that have received extensive usage when modeling lifetime or survival time data: the PHR model, introduced by Cox [6], and the PRHR model, introduced by Gupta *et al.* [7]. The PRHR model was used, for example, by Tsodikov *et al.* [19] to describe a stochastic model of spontaneous carcinogenesis, where the progression time of the tumor was modeled by a PRHR model, while Lane *et al.* [15] modeled bank failure through a PHR model. We recall the definition of these models: a PHR (resp., PRHR) model with baseline distribution F has a distribution function satisfying $\bar{F}_a(x) = \bar{F}^a(x)$ (resp., $F_a(x) = F^a(x)$), for $a > 0$. Hence, the PHR and PRHR models introduce a family of distributions depending on one parameter. We first characterize the star-shaped ordering for each of these models as a straightforward consequence of the Saunders and Moran's criterion, Theorem 4.

Proposition 12. *Let $F \in \mathcal{F}$ with a density that does not vanish on any subinterval of its support, be some baseline distribution. Then $F_a \leq_* F_b$, for every $0 < a \leq b$ if and only if $\bar{g}(x) = \ln(\bar{F}(x))\bar{F}(x)/xf(x)$ is increasing with x on the support of F, in the case of the PHR model, or $g(x) = \ln(F(x))F(x)/xf(x)$ is decreasing with x on the support of F, in the case of the PRHR model.*

Proof. Consider the case of PHR model. According to Theorem 4, we have $D(a, x) = (1/xf_a(x))(\partial F_a/\partial a)(x) = -(1/a)\bar{g}(x)$. Hence, since $a > 0$, the conclusion follows. The PRHR case follows analogously. □

Remark 13. *The sample maxima and minima from independent, identically distributed random variables are typical examples of PHR and PRHR models.*

Remark 14. *It can be verified that the function g and $\bar{g}$ considered in the previous proposition are monotone for several families of distributions popular in reliability or aging models, such as the Gamma, Weibull, or Pareto (with support $(0, +\infty)$). For Weibull and Pareto distributions, the monotonicity of both functions does not depend on the value of the shape parameter. However, the same does not happen for Gamma distributions. Although g is increasing, for every shape parameter $\alpha > 0$, the function $\bar{g}$ is increasing if $\alpha < 1$ and decreasing if $\alpha > 1$. Verifying this may be accomplished as in Remark 10.*

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3.3. Complex systems with heterogeneous components

Throughout this subsection, we characterize the star-shaped ordering of a few different types of heterogeneous systems, looking both at parallel and series systems. A first model looks at parallel systems with components whose lifetimes distributions are subject to different scale changes. Let $F \in \mathcal{F}$ be a distribution function, with density function that does not vanish on any subinterval of its support and consider random variables X_i with distribution function $F_i(x) = F(\lambda_i x)$, where $\lambda_i > 0$, for $i = 1, \ldots, n$. The distribution function of $X_{(n)}$ is

$$F_\lambda(x) = \prod_{i=1}^{n} F(\lambda_i x), \quad \lambda = (\lambda_1, \ldots, \lambda_n). \tag{3}$$

First, recall below a definition concerning a special class of functions.

Definition 15 ([8]). *A function $f\colon \mathbb{R}^2 \to \mathcal{R}$ is said to be totally positive of order 2 (TP2) if for every real numbers $x_1 < x_2$ and $y_1 < y_2$, $f(x_1, y_2) f(x_2, y_1) \le f(x_1, y_1) f(x_2, y_2)$.*

Proposition 16. *Let $\{F_\lambda\colon \lambda \in (0, +\infty)^n, \lambda_1 < \lambda_2 < \cdots < \lambda_n\}$ be a family of distributions defined as in (3). Let $\mu, \nu \in (0, +\infty)^n$. If $G(a, x) = F(ax)/f(ax)$ is TP2 for $a > 0$, then $F_{\lambda_t} \le_* F_{\lambda_{t'}}$, for every $\lambda_t, \lambda_{t'} \in L_{(\mu,\nu)} \cap J$ where $t \le t'$ and $J = \{\lambda \in (0, +\infty)^n\colon \nu_i \lambda_j - \lambda_i \nu_j \le 0, i < j, i, j = 1, \ldots, n\}$.*

Proof. According to Theorem 5, we need to prove that $R(x) = (1/x f_\lambda(x)) \sum_{i=1}^{n} \nu_i (\partial F_\lambda / \partial \lambda_i)(x)$ is decreasing with x, where f_λ is the density function of F_λ. We have that

$$\frac{\partial F_\lambda}{\partial \lambda}(x) = x f(\lambda_i x) \prod_{\substack{j=1 \\ i \ne j}}^{n} F(\lambda_j x) \quad \text{and} \quad f_\lambda(x) = \sum_{i=1}^{n} \lambda_i f(\lambda_i x) \prod_{\substack{j=1 \\ i \ne j}}^{n} F(\lambda_j x).$$

Thus, $R(x) = \sum_{i=1}^{n} \nu_i P_i(x) / \sum_{i=1}^{n} \lambda_i P_i(x)$, where $P_i(x) = f(\lambda_i x) \prod_{j=1, i\ne j}^{n} F(\lambda_j x)$. Differentiating R, we get that the sign of R' is the same as the sign of

$$K(x) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (\nu_i \lambda_j - \lambda_i \nu_j)(P_i'(x) P_j(x) - P_i(x) P_j'(x)).$$

Since, for $i < j$ and $i, j = 1, \ldots, n$, we have that $\nu_i \lambda_j - \lambda_i \nu_j \le 0$, we need to prove that $P_i'(x) P_j(x) - P_i(x) P_j'(x) \ge 0$, for $i < j$. The function $P_i'(x) P_j(x) - P_i(x) P_j'(x)$ is the numerator of the derivative of $L(x) = P_i(x)/P_j(x) = f(\lambda_i x) F(\lambda_j x) / f(\lambda_j x) F(\lambda_i x)$. Given that, for $i < j$, $\lambda_i < \lambda_j$ and $G(a, x)$ is TP2, for every $a > 0$, it follows that L is increasing. Hence, the proof is concluded. □

Remark 17. *The Gamma distribution is one of the common families in reliability or aging models that verify the TP2 property. In fact, proving that $G(a, x)$ is TP2 is the same as proving that $K(x) = f(ax)F(bx)/f(bx)F(ax) = (a/b)^{\alpha-1}(e^{bx}F(bx)/e^{ax}F(ax))$ is increasing with x, for $0 < a < b$. Again, this is verified in an analogous way as in Remark 10. Other families of distributions that satisfy the TP2 property assumed in Proposition 16 are, for example, Weibull and Pareto (with support $(0, +\infty)$).*

The following corollary complements the ordering result proved in Kochar and Xu [10], where only two types of components with exponential lifetimes were allowed in each parallel system.

Corollary 18. *Let $\{F_\lambda\colon \lambda \in (0, +\infty)^n, \lambda_1 < \lambda_2 < \cdots < \lambda_n\}$ be a family of distributions defined as in (3), with $F(x) = 1 - e^{-x}$. Let $\mu, \nu \in (0, +\infty)^n$. Then $F_{\lambda_t} \le_* F_{\lambda_{t'}}$, for every $\lambda_t, \lambda_{t'} \in L_{(\mu,\nu)} \cap J$ where $t \le t'$ and $J = \{\lambda \in (0, +\infty)^n\colon \nu_i \lambda_j - \lambda_i \nu_j \le 0, i, j = 1, \ldots, n, i < j\}$.*

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Proof. Taking into account Proposition 16, we only need to prove that $G(a,x) = F(ax)/f(ax)$ is TP2, for $a > 0$. But this is equivalent to proving that, for $a < b$,

$$K(x) = \frac{f(ax)F(bx)}{f(bx)F(ax)} = \frac{e^{bx}-1}{e^{ax}-1}$$

is increasing with $x > 0$, which is easily seen to be true. □

We now have a look into complex systems based on components whose lifetimes follow a PRHR model. Assume that the lifetimes X_i have distribution function $F_i(x) = F^{\lambda_i}(x)$, for some baseline function $F \in \mathcal{F}$ with density f that does not vanish on any interval of the support of F, where $\lambda_i > 0$, for every $i = 1, \ldots, n$. Then, the distribution function of $X_{(1)}$ is

$$F_\lambda(x) = 1 - \prod_{i=1}^{n}(1 - F^{\lambda_i}(x)), \lambda = (\lambda_1, \ldots, \lambda_n). \tag{4}$$

Proposition 19. *Let $\{F_\lambda \colon \lambda \in (0,+\infty)^n, \lambda_1 < \lambda_2 < \cdots < \lambda_n\}$ be a family of distributions defined as in (4). Let $\mu, \nu \in (0,+\infty)^n$. If $g(x) = \ln(F(x))F(x)/xf(x)$ is decreasing with x on the support of F, then $F_{\lambda_t} \le_* F_{\lambda_{t'}}$, for every $\lambda_t, \lambda_{t'} \in L_{(\mu,\nu)} \cap J$ where $t \le t'$ and $J = \{\lambda \in (0,+\infty)^n \colon \nu_i\lambda_j - \lambda_i\nu_j \le 0, i < j, i,j = 1,\ldots,n\}$.*

Proof. Taking into account Theorem 5, we need to prove that $R(x) = (1/xf_\lambda(x)) \sum_{i=1}^n \nu_i (\partial F_\lambda/\partial\lambda_i)(x)$ is decreasing for every $\lambda \in J$. We have that

$$\frac{\partial F_\lambda}{\partial \lambda_i}(x) = \ln(F(x))F^{\lambda_i}(x) \prod_{\substack{j=1\\ i\neq j}}^{n}(1 - F^{\lambda_i}(x)),$$

and

$$f_\lambda(x) = f(x)\sum_{i=1}^{n}\lambda_i F^{\lambda_i - 1}(x)\prod_{\substack{j=1\\ i\neq j}}^{n}(1 - F^{\lambda_i}(x)).$$

Hence, $R(x) = g(x)h(x)$, where $h(x) = \sum_{i=1}^n \nu_i P_i(x)/\sum_{i=1}^n \lambda_i P_i(x)$, with $P_i(x) = F^{\lambda_i - 1}(x)\prod_{j=1, i\neq j}^{n}(1 - F^{\lambda_j}(x))$. It is easily seen that the first term in $R'(x) = g'(x)h(x) + g(x)h'(x)$ is negative. Observe that, since $\nu_i > 0$, for every $i = 1, \ldots, n$, it follows that $h(x) \ge 0$. Hence, to prove that R is decreasing, it is enough to establish that h is increasing, given that g is a negative decreasing function. The sign of h' is easily seen to be the same as the sign of $\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(\nu_i\lambda_j - \lambda_i\nu_j)(P_i'(x)P_j(x) - P_i(x)P_j'(x))$. Given that $\nu_i\lambda_j - \lambda_i\nu_j \le 0$, it remains to prove that $P_i'(x)P_j(x) - P_i(x)P_j'(x) \le 0$, for every $i, j = 1, \ldots, n$, $i < j$. Observe that $P_i'(x)P_j(x) - P_i(x)P_j'(x)$ is the numerator of the derivative of $K(x) = P_i(x)/P_j(x) = F^{\lambda_i - \lambda_j}(x)((1 - F^{\lambda_j}(x))/(1 - F^{\lambda_i}(x)))$. Hence, we need to prove that K is decreasing. Given that F is increasing and nonnegative, the monotonicity of K is the same as the monotonicity of the companion function

$$\tilde{K}(x) = x^{\lambda_i - \lambda_j}\frac{1 - x^{\lambda_j}}{1 - x^{\lambda_i}} = \frac{x^{-\lambda_j} - 1}{x^{-\lambda_i} - 1}, \quad \text{for } x \in (0,1),$$

which is easily seen to be decreasing on $(0, 1)$. □

Assuming the components follow a PHR model, we may derive a similar result about $X_{(n)}$. Consider that X_i has survival function $\bar{F}_i(x) = \bar{F}^{\lambda_i}(x)$, where $\lambda_i > 0$, for every $i = 1, \ldots, n$ and F is a baseline

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distribution as above. Then, the distribution function of $X_{(n)}$ is, with $\lambda = (\lambda_1, \ldots, \lambda_n)$,

$$F_\lambda(x) = \prod_{i=1}^{n} (1 - \bar{F}^{\lambda_i}(x)). \tag{5}$$

Proposition 20. *Let $\{F_\lambda : \lambda \in (0, +\infty)^n, \lambda_1 < \lambda_2 < \cdots < \lambda_n\}$ be a family of distributions defined as in (5). Let $\mu, \nu \in (0, +\infty)^n$. If $\bar{g}(x) = \ln(\bar{F}(x))\bar{F}(x)/xf(x)$ is increasing with x on the support of F, then $F_{\lambda_t} \leq_* F_{\lambda_{t'}}$, for every $\lambda_t, \lambda_{t'} \in L_{(\mu,\nu)} \cap J$ where $t \leq t'$ and $J = \{\lambda \in (0, +\infty)^n : \nu_i\lambda_j - \lambda_i\nu_j \leq 0, i < j, i, j = 1, \ldots, n\}$.*

Proof. The proof is analogous to that of Proposition 19, taking into account that

$$\frac{\partial F_\lambda}{\partial \lambda_i}(x) = -\ln(\bar{F}(x))\bar{F}^{\lambda_i}(x) \prod_{\substack{j=1 \\ i \neq j}}^{n} (1 - \bar{F}^{\lambda_i}(x)),$$

$$f_\lambda(x) = f(x) \sum_{i=1}^{n} \lambda_i \bar{F}^{\lambda_i - 1}(x) \prod_{\substack{j=1 \\ i \neq j}}^{n} (1 - \bar{F}^{\lambda_i}(x)),$$

where f is the density function of F, and $R(x) = -\bar{g}(x)h(x)$, where $h(x) = \sum_{i=1}^{n} \nu_i P_i(x) / \sum_{i=1}^{n} \lambda_i P_i(x)$, with $P_i(x) = \bar{F}^{\lambda_i - 1}(x) \prod_{j=1, i \neq j}^{n} (1 - \bar{F}^{\lambda_j}(x))$. Thus, we now need to prove that $h'(x) \leq 0$, in order to conclude that $R'(x) \leq 0$. Since $\nu_i\lambda_j - \lambda_i\nu_j \leq 0$, for $i, j = 1, \ldots, n$, $i \neq j$, this follows after proving that $P_i'(x)P_j(x) - P_i(x)P_j'(x) \geq 0$, for $i < j$, which is easily achieved analogously as in the proof of Proposition 19. □

The previous result is an extension of Theorem 3.3 in Kochar and Xu [13], where the authors considered one of the systems to be formed by homogeneous components.

Remark 21. *The conclusions in Propositions 19 and 20 may reverse the direction of the ordering, if the monotonicities assumed for g and $\bar{g}$ are reversed and we redefine the set as $J = \{\lambda \in (0, +\infty)^n : \nu_i\lambda_j - \lambda_i\nu_j \geq 0, i, j = 1, \ldots, n, i < j\}$.*

Remark 22. *As referred in Remark 10, it can verified that the functions g and $\bar{g}$ considered in Propositions 19 and 20, respectively, are monotone for several families of distributions popular in reliability or aging models, such as the Gamma, Lomax, Weibull, Pareto, or power.*

3.4. Parallel systems with homogeneous distributions

Arab *et al.* [2] proved in their Corollary 7.2, that parallel homogeneous systems with components that have exponential lifetimes age faster as the number of components increases. We may prove this also holds when the components have exponentiated Weibull lifetimes $X_1, \ldots, X_n$, whose distribution function is given by $F(x) = (1 - e^{-(\lambda x)^\beta})^\alpha$, for $x > 0$, where $\alpha, \beta > 0$ are shape parameters and $\lambda > 0$ is a scale parameter. The distribution function of $X_{(n)}$ is given by

$$F_X(x) = (1 - e^{-(\lambda x)^\beta})^a, \tag{6}$$

where $a = \alpha n$, for $n \geq 1$. Taking into account Remark 2, we may, without loss of generality, consider $\lambda = 1$.

Proposition 23. *Let $\{F_{(\beta,a)} : a > 0, \beta > 0\}$ be a family of distributions defined as in (6). Let $(\beta', a'), (\tilde{\beta}, \tilde{a}) \in (0, +\infty)^2$, such that $\beta' \geq \tilde{\beta}$, and $\nu = (\tilde{\beta}, \tilde{a}) - (\beta', a')$. If $\beta' = \tilde{\beta}$, then $F_{a'} \leq_* F_{\tilde{a}}$,* *for every $a' \geq \tilde{a}$. If $\beta' > \tilde{\beta}$, then $F_{\lambda_t} \leq_* F_{\lambda_{t'}}$, for every $\lambda_t, \lambda_{t'} \in L_{((\beta',a'),v)}$ where $t \leq t'$, if and only if $\lambda_t, \lambda_{t'} \in J = \{(\beta, a) \in (0,+\infty)^2 \colon (\tilde{a} - a')/(\tilde{\beta} - \beta') \geq -a/\beta\}$.*

Proof. If $\beta' = \tilde{\beta}$, then the family of distributions $F_{(\beta,a)}$ follows a PRHR model. Thus, taking into account Proposition 12, since

$$\frac{\ln(F(x))F(x)}{xf(x)} = \frac{(e^x - 1)\ln(1 - e^{-x})}{x}$$

is increasing with $x > 0$, the conclusion follows. Consider now $\beta' > \tilde{\beta}$. By Proposition 7, we need to prove that

$$Q(x) = \frac{1}{xf_{(\beta,a)}(x)}\left(\frac{\partial F_{(\beta,a)}}{\partial \beta}(x) + k\frac{\partial F_{(\beta,a)}}{\partial a}(x)\right)$$

is increasing with $x > 0$, for every $(\beta, a) \in L_{((\beta',a'),v)}$, if and only if $k = (\tilde{a} - a')/(\tilde{\beta} - \beta') \geq -a/\beta$. We begin by studying the case where $k \geq 0$. Taking into account Lemma 8 in Arab and Oliveira [1], we need to prove that, for every $c \in \mathbb{R}$, $Q(x) - c$ changes sign at most once, as x goes from 0 to $+\infty$, and if the sign change occurs it is in the order "$-, +$". Note $Q(x) - c$ and $H(x) = (1 - e^{-x^\beta})^{a-1}P(x)$, where

$$P(x) = (ae^{-x^\beta}x^\beta \ln(x) + k(1 - e^{-x^\beta})\ln(1 - e^{-x^\beta}) - ca\beta x^\beta e^{-x^\beta})$$

have, for each $x > 0$, the same sign. Hence, it is enough to characterize the sign of P. We look at the sign of P', that coincides, for $x > 0$, with the one of $V(x) = (-ax^\beta \ln(x) + a\ln(x) + a/\beta + k\ln(1 - e^{-x^\beta}) + k - ca\beta + ca\beta x^\beta)$. Differentiating V, we have $V'(x) = x^{\beta-1}K(x)$, where $K(x) = -a\beta\ln(x) - a + a/x^\beta + k\beta/(e^{x^\beta} - 1) + ca\beta^2$. Thus, $K'(x) = -a\beta/x - a\beta/x^{\beta+1} - k\beta^2 x^{\beta-1}e^{x^\beta}/(e^{x^\beta} - 1)^2$. Since $k \geq 0$, we have that $K'(x) \leq 0$, implying that K is decreasing. Given that, $\lim_{x\to 0^+} K(x) = +\infty$ and $\lim_{x\to+\infty} K(x) = -\infty$, it follows that the sign variation of K, which is the same as sign variation of V', is "$+, -$". Therefore, V has monotonicity "$\nearrow\searrow$". Moreover, $\lim_{x\to 0^+} V(x) = \lim_{x\to+\infty} V(x) = -\infty$, which implies that V has sign variation "$-, +, -$" or "$-$". In the first case, we have that P has monotonicity "$\searrow\nearrow\searrow$". Since $\lim_{x\to 0^+} P(x) = \lim_{x\to+\infty} P(x) = 0$, the sign variation of P, which coincides with the sign variation of H, is "$-, +$". In the second case, where $V(x) \leq 0$, then P would be decreasing. But this is impossible, given the behavior of P near 0 and at $+\infty$. Suppose now that $-(-a/\beta) \leq k = (\tilde{a} - a')/(\tilde{\beta} - \beta') \leq 0$. Differentiating Q, we obtain

$$Q'(x) = \frac{((\beta k x^\beta - \beta k)e^{x^\beta} + \beta k)\ln(1 - e^{-x^\beta}) + (\beta k + a)x^\beta}{x^{\beta+1}a\beta}.$$

Some elementary calculus arguments show that $Q'(x) \geq 0$, implying that Q is increasing with $x > 0$. If $k \leq -a/\beta$, Q cannot be a monotone function, since $\lim_{x\to 0^+} Q(x) = \lim_{x\to+\infty} Q(x) = +\infty$. Therefore, the proof is concluded. □

Remark 24. *According to Proposition 23, given $(\tilde{\beta}, \tilde{a})$ and (β', a'), we do not only have that $F_{(\beta,a)} \leq_* F_{(\tilde{\beta},\tilde{a})}$, for $(\beta, a) \in J \cap L_{((\beta',a'),v)}$, but we also have that the distributions depending on the parameters in the set $J \cap L_{((\beta',a'),v)}$ are ordered, with respect to the star-shaped order. That is, if k is the slope of the line going through $(\tilde{\beta}, \tilde{a})$ and (β', a'), then every point (β, a) in this line, and above the line $-k\beta'$ (condition given by the set J) defines distributions comparable with each other and with $F_{(\tilde{\beta},\tilde{a})}$. However, it does not allow us to decide about star-shaped comparability between $F_{(\beta,a)}$ and $F_{(\tilde{\beta},\tilde{a})}$, when $(\beta, a) \notin J$. Nevertheless, if we keep changing the value of k (and, therefore, the position of the point*

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(β', a')), it follows from Proposition 23 that the set of points (β, a) for which we have $F_{(\beta,a)} \leq_* F_{(\tilde{\beta},\tilde{a})}$, is given by the set $\{(\beta, a) \in (0, +\infty)^2 : \beta \geq \tilde{\beta}$ and $a \geq \tilde{a}\beta/(\tilde{\beta} - 2\beta)\}$.

4. Conclusion

Saunders and Moran [17] introduced a criterion to obtain the star-shaped ordering between distributions within a family depending on a single parameter. Although this criterion was used by many authors to exhibit star-shaped comparability within parametric families of distributions, Saunders and Moran's [17] result cannot be used for the comparison of complex systems, such as parallel or series systems with heterogeneous components, whose distributions depend on more than one parameter. In this paper, we extend the characterization given by Saunders and Moran [17] for the star-shaped order for families of distributions that involve multiple parameters. Based on this new criterion, sufficient conditions are obtained for comparing, with respect to the star-shaped order, parallel and series systems with heterogeneous components. Moreover, the new characterization is used to establish star-shaped comparability between parallel systems with homogeneous components, when the multidimensional parameter move along lines that depend on the scale parameters of the individual lifetime distributions.

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