Canad. J. Math. Vol. **64** (4), 2012 pp. 869–891
http://dx.doi.org/10.4153/CJM-2011-055-5

Balayage of Semi-Dirichlet Forms

Ze-Chun Hu and Wei Sun

Abstract. In this paper we study the balayage of semi-Dirichlet forms. We present new results on balayaged functions and balayaged measures of semi-Dirichlet forms. Some of the results are new even in the Dirichlet forms setting.

1 Introduction

Balayage is an important notion in potential theory that plays a crucial role in both classical potential theory and its probabilistic counterpart (cf. *e.g.*, Bliedtner–Hansen [2] and Doob [5]). In [25–27], Silverstein discussed the balayage of symmetic Dirichlet forms. In [17], LeJan extended Silverstein's results to non-symmetric Dirichlet forms, and in [18], he studied some properties of balayaged Dirichlet forms, among other things. In this paper, we study the balayage of semi-Dirichlet forms and focus on balayaged functions and balayaged measures. We refer the reader to Ma–Overbeck–Röckner [19], Ma–Röckner [20] and Fukushima–Oshima–Takeda [10] for descriptions of the (semi-)Dirichlet form theory and the notation and terminology of this paper. The reader is also referred to the recent review paper of Ma–Sun [22] for a brief introduction to semi-Dirichlet forms. Note that a semi-Dirichlet form is not merely a mathematical generalization of a Dirichlet form. We refer the reader to [19], Overbeck–Röckner–Schmuland [23], and Röckner–Schmuland [24] for many interesting examples of semi-Dirichlet forms. We also refer the reader to Fukushima–Uemura [11] for a recent construction of jump-type Hunt processes using semi-Dirichlet forms.

Let E be a metrizable Lusin space (*i.e.*, E is topologically isomorphic to a Borel subset of a complete separable metric space) and let m be a σ-finite positive measure on its Borel σ-algebra $\mathcal{B}(E)$. Let $\mathbf{M} = (\Omega, \mathcal{F}, \big(\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta}\big)$ be a *right* (continuous strong Markov) *process* with state space E, life time ζ, and cemetery Δ. Define $P_t f(x) := E_x[f(X_t)]$ for $f \in \mathcal{B}_b(E)$ and $x \in E$, where $\mathcal{B}_b(E)$ denotes the set of all bounded measurable functions on E. Suppose that for each $t > 0$ the restriction of P_t to $\mathcal{B}_b(E) \cap L^2(E; m)$ can be uniquely extended to a contraction operator T_t on $L^2(E; m)$. Then one can check that the semigroup $(T_t)_{t>0}$ is strongly continuous on

Received by the editors September 23, 2010; revised March 23, 2011.
Published electronically August 15, 2011.
We are grateful to the financial support of NNSFC (Grant No. 10801072) and NSERC (Grant No. 311945-2008).
AMS subject classification: **31C25**, 60J45.
Keywords: balayage, semi-Dirichlet form, potential theory.

$L^2(E;m)$. Suppose that $(T_t)_{t>0}$ is analytic. We set

$$D(\mathcal{E}) := \{u \in L^2(E;m) \mid \sup_{t>0} \tfrac{1}{t}(u - T_t u, u) < \infty\},$$

$$\mathcal{E}(u,v) := \lim_{t\to 0} \tfrac{1}{t}(u - T_t u, v), \ \forall u, v \in D(\mathcal{E}).$$

Hereafter $(\cdot\,,\,\cdot)$ and $\|\cdot\|_2$ denote the usual inner product and norm of $L^2(E;m)$. Then $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular semi-Dirichlet form, and $\mathbf{M}$ is properly associated with $(\mathcal{E}, D(\mathcal{E}))$ in the sense that $P_t f$ is an $\mathcal{E}$-quasi-continuous m-version of $T_t f$ for all $f \in \mathcal{B}_b(E) \cap L^2(E;m)$ and all $t > 0$. On the other hand, if a semi-Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$ is quasi-regular, then it is properly associated with a right process $\mathbf{M}$.

In this paper, we assume that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular semi-Dirichlet form on $L^2(E;m)$. By [19, Proposition 3.6(iii)], every $u \in D(\mathcal{E})$ has an $\mathcal{E}$-quasi-continuous m-version denoted by $\widetilde{u}$. We write $f \le g$ or $f = g$ for $f, g \in L^2(E;m)$ if the inequality or equality holds m-$a.e.$ on E.

Proposition 1.1 ([19, Proposition 2.19]) *Let h be a function on E that has an $\mathcal{E}$-quasi-continuous m-version denoted by $\widetilde{h}$. Define for $A \subset E$,*

$$\mathcal{L}_{h,A} := \{w \in D(\mathcal{E}) \mid \widetilde{w} \ge \widetilde{h}\ \mathcal{E}\text{-}q.e.\ on\ A\}.$$

Suppose that $\mathcal{L}_{h,A} \neq \varnothing$. Let $\alpha > 0$.

(i) *There exists a unique $h_A^\alpha \in \mathcal{L}_{h,A}$ such that for all $w \in \mathcal{L}_{h,A}$*

$$\mathcal{E}_\alpha(h_A^\alpha, w) \ge \mathcal{E}_\alpha(h_A^\alpha, h_A^\alpha). \tag{1.1}$$

(ii) *$\mathcal{E}_\alpha(h_A^\alpha, w) \ge 0$ for all $w \in D(\mathcal{E})$ with $\widetilde{w} \ge 0$ $\mathcal{E}$-q.e. on A. In particular, h_A^α is α-excessive and $\mathcal{E}_\alpha(h_A^\alpha, w) = 0$ for all $w \in D(\mathcal{E})_{A^c}$, where*

$$D(\mathcal{E})_{A^c} := \{u \in D(\mathcal{E}) \mid \widetilde{u} = 0\ \mathcal{E}\text{-}q.e.\ on\ A\}$$

and $A^c := E - A$.

(iii) *h_A^α is the smallest function u on E such that $u \wedge h_A^\alpha$ is an α-excessive function in $D(\mathcal{E})$ and $\widetilde{u} \ge \widetilde{h}$ $\mathcal{E}$-q.e. on A. In particular, $(0 \le)\, h_A^\alpha \le h$ (m-a.e. on E) if and only if $h \wedge h_A^\alpha$ is an α-excessive function in $D(\mathcal{E})$. In this case $\widetilde{h_A^\alpha} = \widetilde{h}$ $\mathcal{E}$-q.e. on A.*

(iv) *Let g be a function on E that has an $\mathcal{E}$-quasi-continuous m-version denoted by $\widetilde{g}$. If $\mathcal{L}_{g,A} \neq \varnothing$ and $\widetilde{g} \ge \widetilde{h}$ $\mathcal{E}$-q.e. on A, then $g_A^\alpha \ge h_A^\alpha$ (m-a.e. on E).*

(v) *Suppose that $B \subset A \subset E$. Then $(h_B^\alpha)_A^\alpha = h_B^\alpha$ (m-a.e. on E). If $h \wedge h_A^\alpha$ is an α-excessive function in $D(\mathcal{E})$, then $(h_A^\alpha)_B^\alpha = h_B^\alpha$ (m-a.e. on E).*

We call h_A^α the α-*balayaged* (or α-reduced) function of h on A. We define the *balayaged operator* $S(\cdot\,,\,\cdot\,,\,\cdot)$ on $(0,\infty) \times 2^E \times D(\mathcal{E})$ by

$$S(\alpha, A, h) := h_A^\alpha, \tag{1.2}$$

where 2^E is the family of all subsets of E. To simplify notation, we also write $S_A^\alpha h$ for $S(\alpha, A, h)$ in the sequel. In Section 2, we investigate some properties of the balayaged operator $S(\cdot\,,\,\cdot\,,\,\cdot)$ and, in particular, answer the following questions:

1. Fix α and A. Is h_A^α a continuous operator with respect to (w.r.t.) the function h?
2. Fix α and h. Is h_A^α a continuous mapping w.r.t. the set A in some sense?
3. Fix A and h. For $0 < \alpha < \beta$, what is the relation between h_A^α and h_A^β and what is the relation between $(h_A^\alpha)_A^\beta$ and $(h_A^\beta)_A^\alpha$? What about the limit of h_A^β as $\beta \to \alpha$?

In Section 3, we discuss the balayage of measures. It is known that any quasi-regular semi-Dirichlet form is quasi-homeomorphic to a *regular* semi-Dirichlet form (cf. Hu–Ma–Sun [13, Theorem 3.8]). For simplicity, we assume in Section 3 that $(\mathcal{E}, D(\mathcal{E}))$ is a regular semi-Dirichlet form on $L^2(E;m)$. Recall that $(\mathcal{E}, D(\mathcal{E}))$ is regular if the following conditions hold:

(i) E is a locally compact separable metric space and m is a positive Radon measure on E with $\mathrm{supp}[m] = E$.
(ii) $C_0(E) \cap D(\mathcal{E})$ is dense in $D(\mathcal{E})$ with respect to the $\mathcal{E}_1$-norm.
(iii) $C_0(E) \cap D(\mathcal{E})$ is dense in $C_0(E)$ with respect to the uniform norm $\| \|_\infty$.

Hereafter $C_0(E)$ denotes the set of all continuous functions on E with compact supports. A positive Radon measure μ on E is said to be *of finite energy integral* (w.r.t. $(\mathcal{E}, D(\mathcal{E}))$) if there exists a positive constant C such that

$$\int_E |v(x)|\mu(dx) \leq C\mathcal{E}_1(v,v)^{1/2}, \ \ \forall v \in C_0(E) \cap D(\mathcal{E}).$$

We denote by S_0 the family of all positive Radon measures of finite energy integral. Let $\mu \in S_0$ and $\alpha > 0$. Then there exists a unique $U_\alpha\mu \in D(\mathcal{E})$ such that

$$\mathcal{E}_\alpha(U_\alpha\mu, v) = \int_E v(x)\mu(dx), \ \ \forall v \in C_0(E) \cap D(\mathcal{E}).$$

We call $U_\alpha\mu$ an α-potential.

Lemma 1.2 *Let $u \in D(\mathcal{E})$ and $\alpha > 0$. Then the following conditions are equivalent:*

(i) *u is α-excessive.*
(ii) *u is an α-potential.*
(iii) $\mathcal{E}_\alpha(u,v) \geq 0, \ \forall v \in D(\mathcal{E}), \ v \geq 0.$
(iv) $\mathcal{E}_\alpha(u,v) \geq 0, \ \forall v \in C_0(E) \cap D(\mathcal{E}), \ v \geq 0.$

Proof The equivalence of (i) and (iii) is from [19, Theorem 2.4]. (ii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (iv) are trivial. Suppose that (iv) is satisfied. By [13, Lemma 2.4] and following the proof of [10, Theorem 2.2.1 (iv) $\Rightarrow$ (i)], we can prove (ii). Let $v \in D(\mathcal{E}), v \geq 0$. Then we can choose a sequence $v_n \in C_0(E) \cap D(\mathcal{E})$ that is $\mathcal{E}_1$-convergent to v. By [19, Remark 2.2(iii)] and [20, I. Lemma 2.12], we know that $v_n^+ \to v$ weakly in $D(\mathcal{E})$ as $n \to \infty$. Then $\mathcal{E}_\alpha(u,v) = \lim_{n\to\infty} \mathcal{E}_\alpha(u, v_n^+) \geq 0$, which proves (iii). ∎

Let $B \subset E$. Then, by Proposition 1.1, we know that the α-balayaged function $(U_\alpha\mu)_B^\alpha$ of $U_\alpha\mu$ on B is α-excessive. By Lemma 1.2, there exists a unique measure $\mu_B^\alpha \in S_0$ such that $(U_\alpha\mu)_B^\alpha = U_\alpha\mu_B^\alpha$. Then μ_B^α is called the α-*balayage* (or α-sweeping out) of μ on B. We define the *balayaged operator* $T(\,\cdot\,,\,\cdot\,,\,\cdot\,)$ on $(0,\infty) \times 2^E \times S_0$ by

$$T(\alpha, B, \mu) := \mu_B^\alpha. \tag{1.3}$$

https://doi.org/10.4153/CJM-2011-055-5 Published online by Cambridge University Press

In Section 3, we first give a characterization of μ_B^α and then investigate some properties of $T(\cdot,\cdot,\cdot)$. In particular, we answer the following questions:

4. Fix α and B. Is μ_B^α a continuous mapping w.r.t. the measure μ?
5. Fix α and μ. Is μ_B^α a continuous mapping w.r.t. the set B in some sense?
6. Fix B and μ. For $0<\alpha<\beta$, what is the relation between μ_B^α and μ_B^β and what is the relation between $(\mu_B^\alpha)_B^\beta$ and $(\mu_B^\beta)_B^\alpha$? What about the limit of μ_B^β as $\beta\to\alpha$?

Before ending this introduction, let us comment on the motivation and potential application of this work. Time change is one of the most basic transformations for Markov processes. Recently many remarkable results have been obtained for the time changes of symmetric Markov processes and Markov processes in weak duality (cf. Fukushima–He–Ying [9], Chen–Fukushima–Ying [3, 4] and Fitzsimmons–Getoor [8]). It was shown by Fitzsimmons [7, Theorem 5.7] that if the right process **M** is associated with a semi-Dirichlet form $(\mathcal{E},D(\mathcal{E}))$, then the time-changed process of **M** is also associated with a semi-Dirichlet form. We call this latter semi-Dirichlet form the balayaged semi-Dirichlet form of $(\mathcal{E},D(\mathcal{E}))$. A direct motivation of this paper is to give a complete characterization of the balayaged semi-Dirichlet form. However, the problems caused by the SPV integrability in the Beurling–Deny decomposition of semi-Dirichlet forms and the non-Markovian property of the dual forms (cf. Hu–Ma–Sun [12–15]) make the complete characterization very difficult. We hope that the results obtained in this paper can help us better understand the balayage of semi-Dirichlet forms.

2 Balayage of Functions

In this section we investigate some properties of the balayaged operator $S(\cdot,\cdot,\cdot)$ defined in (1.2). Let $(\mathcal{E},D(\mathcal{E}))$ be a quasi-regular semi-Dirichlet form on $L^2(E;m)$. Denote by $(T_t)_{t>0}$, $(G_\alpha)_{\alpha>0}$ and $(L,D(L))$ (resp. $(\widehat{T}_t)_{t>0}$, $(\widehat{G}_\alpha)_{\alpha>0}$ and $(\widehat{L},D(\widehat{L}))$) the semigroup, resolvent and generator (resp. co-semigroup, co-resolvent and co-generator) associated with $(\mathcal{E},D(\mathcal{E}))$. Since $(\mathcal{E},D(\mathcal{E}))$ satisfies the sector condition, for any $\alpha>0$, there exists a constant $K_\alpha>0$ (called the continuity constant) such that

$$|\mathcal{E}_\alpha(u,v)|\le K_\alpha\mathcal{E}_\alpha(u,u)^{1/2}\mathcal{E}_\alpha(v,v)^{1/2},\ \ \forall u,v\in D(\mathcal{E}). \tag{2.1}$$

Moreover, by (2.1) we can show that for any $\beta\ge\alpha>0$,

$$|\mathcal{E}_\beta(u,v)|\le (K_\alpha+1)\mathcal{E}_\beta(u,u)^{1/2}\mathcal{E}_\beta(v,v)^{1/2},\ \ \forall u,v\in D(\mathcal{E}). \tag{2.2}$$

Let $A\subset E$ and $h\in D(\mathcal{E})$. By (1.1) and (2.1), we have

$$\mathcal{E}_\alpha(h_A^\alpha,h_A^\alpha)\le\mathcal{E}_\alpha(h_A^\alpha,h)\le K_\alpha\mathcal{E}_\alpha(h_A^\alpha,h_A^\alpha)^{1/2}\mathcal{E}_\alpha(h,h)^{1/2}.$$

Then

$$\mathcal{E}_\alpha(h_A^\alpha,h_A^\alpha)\le K_\alpha^2\mathcal{E}_\alpha(h,h). \tag{2.3}$$

Furthermore, by (2.2) we can show that for any $\beta \geq \alpha > 0$,

$$\mathcal{E}_\beta(h_A^\beta, h_A^\beta) \leq (K_\alpha + 1)^2 \mathcal{E}_\beta(h, h). \tag{2.4}$$

Recall that a function $u \in L^2(E; m)$ is called *α-excessive* (resp. *α-coexcessive*) if $e^{-\alpha t} T_t u \leq u$ (resp. $e^{-\alpha t} \widehat{T}_t u \leq u$) for all $t > 0$.

Lemma 2.1 (cf. [19, 20]) *Let $u \in L^2(E; m)$ and $\alpha > 0$. If u is α-excessive, then $u \geq 0$. Furthermore, we have*

(i) *u is α-excessive if and only if $\beta G_{\beta+\alpha} u \leq u$ for all $\beta > 0$;*
(ii) *if $u \in D(\mathcal{E})$, then u is α-excessive if and only if $\mathcal{E}_\alpha(u, v) \geq 0$ for all $v \in D(\mathcal{E})$, $v \geq 0$;*
(iii) *if $f \in L^2(E; m)$, $f \geq 0$, then $G_\alpha f$ is α-excessive;*
(iv) *if u is α-excessive, then u is β-excessive for all $\beta > \alpha$;*
(v) *if $u, v \in L^2(E; m)$ are α-excessive, then $u \wedge v$ is α-excessive;*
(vi) *if $u \in L^2(E; m)$ is α-excessive, then $u \wedge 1$ is α-excessive.*

2.1 Operator $S(\alpha, A, \cdot)$

In this subsection we fix $\alpha > 0$, $A \subset E$ and consider the operator $S(\alpha, A, \cdot)$, *i.e.*, $S_A^\alpha(\cdot)$ on $D(\mathcal{E})$.

Theorem 2.2 (i) *S_A^α is sub-Markovian: if $h \in D(\mathcal{E})$ with $0 \leq h \leq 1$, then $0 \leq S_A^\alpha h \leq 1$.*
(ii) *S_A^α is sub-additive: if $h_1, h_2 \in D(\mathcal{E})$, then $S_A^\alpha(h_1 + h_2) \leq S_A^\alpha h_1 + S_A^\alpha h_2$.*
(iii) *S_A^α is continuous on $D(\mathcal{E})$ w.r.t. the $\mathcal{E}_\alpha$-norm.*

Proof (i) Let $h \in D(\mathcal{E})$ with $0 \leq h \leq 1$. By Proposition 1.1(ii), h_A^α is α-excessive. Then $h_A^\alpha \geq 0$ by Lemma 2.1. Since $1 \wedge h_A^\alpha$ is α-excessive by Lemma 2.1(vi) and $1 \geq \widetilde{h}$ $\mathcal{E}$-*q.e.* on A, hence $1 \geq h_A^\alpha$ by Proposition 1.1(iii). Therefore $0 \leq S_A^\alpha h \leq 1$.

(ii) Let $h_1, h_2 \in D(\mathcal{E})$. By Proposition 1.1(iii), $(\widetilde{(h_1)_A^\alpha + (h_2)_A^\alpha}) \geq \widetilde{h}_1 + \widetilde{h}_2 = \widetilde{h_1 + h_2}$ $\mathcal{E}$-*q.e.* on A. By Proposition 1.1(ii), $(h_1)_A^\alpha + (h_2)_A^\alpha$ is α-excessive. Then $((h_1)_A^\alpha + (h_2)_A^\alpha) \wedge (h_1 + h_2)_A^\alpha$ is also α-excessive. Therefore we obtain by Proposition 1.1(iii) that $(h_1 + h_2)_A^\alpha \leq (h_1)_A^\alpha + (h_2)_A^\alpha$, *i.e.*, , $S_A^\alpha(h_1 + h_2) \leq S_A^\alpha h_1 + S_A^\alpha h_2$.

(iii) Let $\{h_n\}_{n \geq 1}$ be a sequence in $D(\mathcal{E})$ such that h_n converges to $h \in D(\mathcal{E})$ w.r.t. the $\mathcal{E}_\alpha$-norm as $n \to \infty$, *i.e.*, ,

$$\lim_{n\to\infty} \mathcal{E}_\alpha(h_n - h, h_n - h) = 0. \tag{2.5}$$

By (ii), we get

$$(h_n)_A^\alpha - h_A^\alpha \leq (h_n - h)_A^\alpha, \ h_A^\alpha - (h_n)_A^\alpha \leq (h - h_n)_A^\alpha. \tag{2.6}$$

By (2.6), Lemma 2.1(ii), the sector condition, and (2.3), we get

$$
\begin{aligned}
(2.7) \qquad 0 &\le \mathcal{E}_\alpha\big((h_n)_A^\alpha - h_A^\alpha, (h_n)_A^\alpha - h_A^\alpha\big)\\
&= \mathcal{E}_\alpha\big((h_n)_A^\alpha, (h_n)_A^\alpha - h_A^\alpha\big) + \mathcal{E}_\alpha(h_A^\alpha, h_A^\alpha - (h_n)_A^\alpha)\\
&\le \mathcal{E}_\alpha((h_n)_A^\alpha, (h_n - h)_A^\alpha) + \mathcal{E}_\alpha(h_A^\alpha, (h - h_n)_A^\alpha)\\
&\le K_\alpha \mathcal{E}_\alpha((h_n)_A^\alpha, (h_n)_A^\alpha)^{1/2} \mathcal{E}_\alpha((h_n - h)_A^\alpha, (h_n - h)_A^\alpha)^{1/2}\\
&\quad + K_\alpha \mathcal{E}_\alpha(h_A^\alpha, h_A^\alpha)^{1/2} \mathcal{E}_\alpha((h - h_n)_A^\alpha, (h - h_n)_A^\alpha)^{1/2}\\
&\le K_\alpha^3 \mathcal{E}_\alpha(h_n, h_n)^{1/2} \mathcal{E}_\alpha(h_n - h, h_n - h)^{1/2}\\
&\quad + K_\alpha^3 \mathcal{E}_\alpha(h, h)^{1/2} \mathcal{E}_\alpha(h - h_n, h - h_n)^{1/2},
\end{aligned}
$$

which together with (2.5) implies that

$$\lim_{n\to\infty} \mathcal{E}_\alpha(S_A^\alpha h_n - S_A^\alpha h, S_A^\alpha h_n - S_A^\alpha h) = 0. \qquad \blacksquare$$

2.2 Operator $S(\alpha, \cdot, h)$

In this subsection we fix $\alpha > 0$, $h \in D(\mathcal{E})$ and consider the operator $S(\alpha, \cdot, h)$.

Proposition 2.3 *Let A, B be two subsets of E. Then for any $\alpha > 0$ and $h \in D(\mathcal{E})$, we have*

(i) *if $B \subset A$, then $h_B^\alpha \le h_A^\alpha \le h_B^\alpha + h_{A-B}^\alpha$;*
(ii) *$|h_A^\alpha - h_B^\alpha| \le h_{A-B}^\alpha + h_{B-A}^\alpha$;*
(iii) *$\mathcal{E}_\alpha(h_A^\alpha - h_B^\alpha, h_A^\alpha - h_B^\alpha) \le 2K_\alpha^2 \mathcal{E}_\alpha(h,h)^{1/2} \mathcal{E}_\alpha(h_{A-B}^\alpha + h_{B-A}^\alpha, h_{A-B}^\alpha + h_{B-A}^\alpha)^{1/2}$.*

Proof Part (i) is a direct consequence of Proposition 1.1(ii) and (iii).

(ii) By (i), we get

$$h_{A\cap B}^\alpha \le h_A^\alpha \le h_{A\cap B}^\alpha + h_{A-B}^\alpha, \quad h_{A\cap B}^\alpha \le h_B^\alpha \le h_{A\cap B}^\alpha + h_{B-A}^\alpha,$$

which implies that (ii) holds.

(iii) By (ii), Lemma 2.1(ii), the sector condition, and (2.3), we get

$$
\begin{aligned}
&\mathcal{E}_\alpha(h_A^\alpha - h_B^\alpha, h_A^\alpha - h_B^\alpha)\\
&\quad = \mathcal{E}_\alpha(h_A^\alpha, h_A^\alpha - h_B^\alpha) + \mathcal{E}_\alpha(h_B^\alpha, h_B^\alpha - h_A^\alpha)\\
&\quad \le \mathcal{E}_\alpha(h_A^\alpha, h_{A-B}^\alpha + h_{B-A}^\alpha) + \mathcal{E}_\alpha(h_B^\alpha, h_{A-B}^\alpha + h_{B-A}^\alpha)\\
&\quad \le K_\alpha\big[\mathcal{E}_\alpha(h_A^\alpha, h_A^\alpha)^{1/2} + \mathcal{E}_\alpha(h_B^\alpha, h_B^\alpha)^{1/2}\big] \mathcal{E}_\alpha(h_{A-B}^\alpha + h_{B-A}^\alpha, h_{A-B}^\alpha + h_{B-A}^\alpha)^{1/2}\\
&\quad \le 2K_\alpha^2 \mathcal{E}_\alpha(h,h)^{1/2} \mathcal{E}_\alpha(h_{A-B}^\alpha + h_{B-A}^\alpha, h_{A-B}^\alpha + h_{B-A}^\alpha)^{1/2}. \qquad \blacksquare
\end{aligned}
$$

Definition 2.4 ([19, Definition 2.11]) Let $\phi \in L^2(E; m)$ such that $0 < \phi \le 1$ *m-a.e.* and set $g := G_1\phi$. Then g is a 1-excessive function in $D(\mathcal{E})$ and strictly positive *m-a.e.* Define for $U \subset E$, U open,

$$\operatorname{cap}_\phi(U) := (g_U^1, \phi)$$

https://doi.org/10.4153/CJM-2011-055-5 Published online by Cambridge University Press

and for any $A \subset E$,

$$\operatorname{cap}_\phi(A) := \inf\{\operatorname{cap}_\phi(U) \mid A \subset U, U \text{ open}\}.$$

Theorem 2.5 *Let $A, A_1, A_2, \ldots$ be a sequence of subsets of E. If $\operatorname{cap}_\phi(A_n \triangle A) \to 0$ as $n \to \infty$, where $A_n \triangle A := (A_n - A) \cup (A - A_n)$, then for any $\alpha > 0$ and $h \in D(\mathcal{E})$, $h^\alpha_{A_n}$ converges to h^α_A in $D(\mathcal{E})$ as $n \to \infty$.*

Proof By Proposition 2.3(iii), it suffices to prove that if $\operatorname{cap}_\phi(A_n) \to 0$ as $n \to \infty$, then $h^\alpha_{A_n}$ converges to 0 in $D(\mathcal{E})$ as $n \to \infty$. Now we assume that

$$\lim_{n\to\infty} \operatorname{cap}_\phi(A_n) = 0.$$

Step 1 Assume that h is α-excessive.

Note that $\sup_{n\geq 1} \mathcal{E}_\alpha(h^\alpha_{A_n}, h^\alpha_{A_n}) \leq K^2_\alpha \mathcal{E}_\alpha(h, h) < \infty$ by (2.3). Then by [20, I. Lemma 2.12], for any subsequence $\{h^\alpha_{A_{n_k}}\}$ of $\{h^\alpha_{A_n}\}$, there exist a subsequence of $\{h^\alpha_{A_{n_k}}\}$ (we still denote it by $\{h^\alpha_{A_{n_k}}\}$ for simplicity of notation) and $h^* \in D(\mathcal{E})$ such that $h^\alpha_{A_{n_k}}$ converges weakly to h^* in $D(\mathcal{E})$ as $k \to \infty$. We will prove that $h^* = 0$. Once this is done, we obtain that $h^\alpha_{A_n}$ converges weakly to 0 in $D(\mathcal{E})$ as $n \to \infty$. Therefore, by Proposition 1.1(ii) and (iii), we get

$$\lim_{n\to\infty} \mathcal{E}_\alpha(h^\alpha_{A_n}, h^\alpha_{A_n}) = \lim_{n\to\infty} \mathcal{E}_\alpha(h^\alpha_{A_n}, h) = 0.$$

Since $\lim_{k\to\infty} \operatorname{cap}_\phi(A_{n_k}) = 0$, we can choose a subsequence of $\{A_{n_k}\}$, denoted by $\{A_{n_{k'}}\}$, such that

$$\sum_{k=1}^{\infty} \operatorname{cap}_\phi(A_{n_{k'}}) < \infty.$$

For $k = 1, 2, \ldots$, define $B_{n_{k'}} := \bigcup_{l=k}^{\infty} A_{n_{l'}}$. Then $\{B_{n_{k'}}\}$ is a decreasing sequence such that for any $k \geq 1, A_{n_{k'}} \subset B_{n_{k'}}$, and $\operatorname{cap}_\phi(B_{n_{k'}}) \leq \sum_{l=k}^{\infty} \operatorname{cap}_\phi(A_{n_{l'}}) \downarrow 0$. By Definition 2.4, there exists a decreasing sequence $\{C_{n_{k'}}\}$ of open subsets of E such that for any $k \geq 1, B_{n_{k'}} \subset C_{n_{k'}}$ and $\operatorname{cap}_\phi(C_{n_{k'}}) \downarrow 0$. For $k = 1, 2, \ldots$, define $F_{k'} := C^c_{n_{k'}} = E - C_{n_{k'}}$. Then $\{F_{k'}\}$ is an increasing sequence of closed subsets of E with $\operatorname{cap}_\phi(F^c_{k'}) \downarrow 0$. By [19, Theorem 2.14], we know that $\{F_{k'}\}$ is an $\mathcal{E}$-nest. Then following the proof of [19, Lemma 2.10(i)], we can show that $h^\alpha_{C_{n_{k'}}}$ converges to 0 in $D(\mathcal{E})$ as $k \to \infty$. Since $h^\alpha_{C_{n_{k'}}}$ is decreasing and $0 \leq h^\alpha_{A_{n_{k'}}} \leq h^\alpha_{B_{n_{k'}}} \leq h^\alpha_{C_{n_{k'}}}$ by Proposition 2.3(i), we get $h^* = 0$.

Step 2 Assume that $h = u - v$, where u and v are α-excessive functions in $D(\mathcal{E})$.

Since $h \leq u$, we have $h^\alpha_{A_n} \leq u^\alpha_{A_n}$ by Proposition 1.1(iv). Then following the proof of (2.3), we get

$$\mathcal{E}_\alpha(h^\alpha_{A_n}, h^\alpha_{A_n}) \leq K^2_\alpha \mathcal{E}_\alpha(u^\alpha_{A_n}, u^\alpha_{A_n}),$$

which together with Step 1 implies that $h^\alpha_{A_n}$ converges to 0 in $D(\mathcal{E})$ as $n \to \infty$.

Step 3 Assume that h is a general function in $D(\mathcal{E})$.

Note that $\mathcal{G} := \{u - v \mid u, v \text{ are } \alpha\text{-excessive functions in } D(\mathcal{E})\}$ is dense in $D(\mathcal{E})$. For any $\varepsilon > 0$, there exists $g \in \mathcal{G}$ such that

$$\mathcal{E}_\alpha(h-g, h-g)^{1/2} < \varepsilon. \tag{2.8}$$

By Step 2, there exists $N \in \mathbb{N}$ such that for $n \geq N$, we have

$$\mathcal{E}_\alpha(g^\alpha_{A_n}, g^\alpha_{A_n})^{1/2} < \varepsilon. \tag{2.9}$$

By the triangular inequality, (2.7)–(2.9), we obtain that for $n \geq N$

$$\begin{aligned}
\mathcal{E}_\alpha(h^\alpha_{A_n}, h^\alpha_{A_n})^{1/2} &\leq \mathcal{E}_\alpha(h^\alpha_{A_n} - g^\alpha_{A_n}, h^\alpha_{A_n} - g^\alpha_{A_n})^{1/2} + \mathcal{E}_\alpha(g^\alpha_{A_n}, g^\alpha_{A_n})^{1/2} \\
&\leq K_\alpha^{3/2}\big(\mathcal{E}_\alpha(h,h)^{1/2} + \mathcal{E}_\alpha(g,g)^{1/2}\big)^{1/2}\mathcal{E}_\alpha(h-g,h-g)^{1/4} + \varepsilon \\
&\leq K_\alpha^{3/2}\big(2\mathcal{E}_\alpha(h,h)^{1/2} + \varepsilon\big)^{1/2}\varepsilon^{1/2} + \varepsilon,
\end{aligned}$$

which implies that $h^\alpha_{A_n}$ converges to 0 in $D(\mathcal{E})$ as $n \to \infty$. ■

Proposition 2.6 *Let $\alpha > 0$, $h \in D(\mathcal{E})$, and $A_n \subset E$, $A_n \uparrow A$. Then:*

(i) $h^\alpha_{A_n} \uparrow h^\alpha_A$;

(ii) $\lim_{n\to\infty} \mathcal{E}_\alpha(h^\alpha_{A_n} - h^\alpha_A, h^\alpha_{A_n} - h^\alpha_A) = 0$.

Proof Part (ii) can be proved as in [19, Lemma 2.21]. By Proposition 2.3(i), $\{h^\alpha_{A_n}\}$ is an increasing sequence of nonnegative functions. Hence (ii) implies that (i) holds. ■

2.3 Operator $S(\cdot, A, h)$

In this subsection we fix $A \subset E$, $h \in D(\mathcal{E})$ and consider the operator $S(\cdot, A, h)$. It is well known that $(G_\alpha)_{\alpha>0}$ satisfies the following resolvent equations:

$$G_\alpha = G_\beta + (\beta - \alpha)G_\alpha G_\beta, \quad G_\alpha = G_\beta + (\beta-\alpha)G_\beta G_\alpha, \quad \forall \alpha, \beta > 0.$$

Theorem 2.7 *Let $A \subset E$ and $h \in D(\mathcal{E})$. Then for $\beta > \alpha > 0$, we have*

(i) $S^\beta_A h \leq S^\alpha_A h \leq S^\beta_A h + (\beta-\alpha)G_\alpha S^\beta_A h$;

(ii) $S^\beta_A h + (\beta-\alpha)G_\beta S^\alpha_A h \leq S^\beta_A h + (\beta-\alpha)G_\alpha S^\beta_A h$;

(iii) $G_\alpha S^\alpha_A h \leq G_\alpha[S^\beta_A h + (\beta-\alpha)G_\beta S^\alpha_A h]$.

Proof (i) The first inequality of Theorem 2.7(i) is a direct consequence of Proposition 1.1(iii) and Lemma 2.1(iv). We now prove the second inequality of Theorem 2.7(i). For any $w \in D(\mathcal{E})$ with $w \geq 0$, we obtain by Lemma 2.1(ii) that

$$\begin{aligned}
\mathcal{E}_\alpha(h^\beta_A + (\beta-\alpha)G_\alpha h^\beta_A, w) &= \mathcal{E}_\alpha(h^\beta_A, w) + (\beta-\alpha)\mathcal{E}_\alpha(G_\alpha h^\beta_A, w) \\
&= \big[\mathcal{E}_\beta(h^\beta_A, w) + (\alpha-\beta)(h^\beta_A, w)\big] + (\beta-\alpha)(h^\beta_A, w) \\
&= \mathcal{E}_\beta(h^\beta_A, w) \geq 0,
\end{aligned}$$

https://doi.org/10.4153/CJM-2011-055-5 Published online by Cambridge University Press

which implies that $h_A^\beta + (\beta - \alpha)G_\alpha h_A^\beta$ is α-excessive by Lemma 2.1(ii). Obviously,

$$(h_A^\beta + \widetilde{(\beta - \alpha)} G_\alpha h_A^\beta) \geq \widetilde{h} \quad \mathcal{E}\text{-}q.e. \text{ on } A.$$

Then, by Proposition 1.1(iii), we get $h_A^\alpha \leq h_A^\beta + (\beta-\alpha)G_\alpha h_A^\beta$, *i.e.*, , $S_A^\alpha h \leq S_A^\beta h + (\beta - \alpha)G_\alpha S_A^\beta h$.

(ii) By Theorem 2.7(i) and the resolvent equation, we get

$$\begin{aligned} G_\beta S_A^\alpha h &\leq G_\beta S_A^\beta h + (\beta - \alpha) G_\beta G_\alpha S_A^\beta h \\ &= G_\beta S_A^\beta h + G_\alpha S_A^\beta h - G_\beta S_A^\beta h = G_\alpha S_A^\beta h. \end{aligned} \tag{2.10}$$

Hence (ii) holds.

(iii) Let $f \in D(\mathcal{E})$ be an α-coexcessive function. Then, by the resolvent equation and (2.10), we get

$$\begin{aligned} (S_A^\alpha h - [S_A^\beta h + (\beta - \alpha) G_\beta S_A^\alpha h], f) &= \mathcal{E}_\alpha(G_\alpha\{S_A^\alpha h - [S_A^\beta h + (\beta - \alpha)G_\beta S_A^\alpha h]\}, f) \\ &= \mathcal{E}_\alpha(G_\beta S_A^\alpha h - G_\alpha S_A^\beta h, f) \leq 0. \end{aligned}$$

Taking $f = \widehat{G}_\alpha u$ with $u \in L_+^2(E;m)$, hereafter $L_+^2(E;m) := \{u \in L^2(E;m) \mid u \geq 0\}$, we obtain $G_\alpha S_A^\alpha h \leq G_\alpha[S_A^\beta h + (\beta - \alpha)G_\beta S_A^\alpha h]$. ∎

Remark 2.8 (i) In general, we do not have

$$S_A^\alpha h = S_A^\beta h + (\beta - \alpha) G_\alpha S_A^\beta h. \tag{2.11}$$

In fact, if $(\mathcal{E}, D(\mathcal{E}))$ is a symmetric Dirichlet form, then (2.11) holds if and only if $S_A^\beta h = 0$, which is proved as follows.

If $S_A^\beta h = 0$, then (2.11) holds by Theorem 2.7(i). Suppose that (2.11) holds, *i.e.*, $h_A^\alpha = h_A^\beta + (\beta - \alpha)G_\alpha h_A^\beta$. By Proposition 1.1(i), we have $\mathcal{E}_\alpha(h_A^\alpha, h_A^\beta) \geq \mathcal{E}_\alpha(h_A^\alpha, h_A^\alpha)$. Hence

$$\begin{aligned} &\mathcal{E}_\alpha\big(h_A^\beta + (\beta - \alpha)G_\alpha h_A^\beta, h_A^\beta\big) \geq \\ &\qquad \mathcal{E}_\alpha\big(h_A^\beta + (\beta - \alpha)G_\alpha h_A^\beta, h_A^\beta + (\beta - \alpha)G_\alpha h_A^\beta\big). \end{aligned} \tag{2.12}$$

Note that

$$\begin{aligned} &\mathcal{E}_\alpha(h_A^\beta + (\beta - \alpha)G_\alpha h_A^\beta, h_A^\beta) \\ &= \mathcal{E}_\alpha(h_A^\beta, h_A^\beta) + (\beta - \alpha)\mathcal{E}_\alpha(G_\alpha h_A^\beta, h_A^\beta) \\ &= \big[\mathcal{E}_\beta(h_A^\beta, h_A^\beta) + (\alpha - \beta)(h_A^\beta, h_A^\beta)\big] + (\beta - \alpha)(h_A^\beta, h_A^\beta) = \mathcal{E}_\beta(h_A^\beta, h_A^\beta) \end{aligned} \tag{2.13}$$

https://doi.org/10.4153/CJM-2011-055-5 Published online by Cambridge University Press

and

$$\begin{aligned}(2.14)\qquad &\mathcal{E}_\alpha(h_A^\beta + (\beta-\alpha)G_\alpha h_A^\beta, h_A^\beta + (\beta-\alpha)G_\alpha h_A^\beta)\\ &= \mathcal{E}_\alpha(h_A^\beta, h_A^\beta) + (\beta-\alpha)\mathcal{E}_\alpha(G_\alpha h_A^\beta, h_A^\beta)\\ &\quad + (\beta-\alpha)\mathcal{E}_\alpha(h_A^\beta, G_\alpha h_A^\beta) + (\beta-\alpha)^2\mathcal{E}_\alpha(G_\alpha h_A^\beta, G_\alpha h_A^\beta)\\ &= \big[\mathcal{E}_\alpha(h_A^\beta, h_A^\beta) + (\beta-\alpha)(h_A^\beta, h_A^\beta)\big]\\ &\quad + (\beta-\alpha)\big[\mathcal{E}_\alpha(h_A^\beta, G_\alpha h_A^\beta) + (\beta-\alpha)(h_A^\beta, G_\alpha h_A^\beta)\big]\\ &= \mathcal{E}_\beta(h_A^\beta, h_A^\beta) + (\beta-\alpha)\mathcal{E}_\beta(h_A^\beta, G_\alpha h_A^\beta).\end{aligned}$$

By (2.12)–(2.14), we get $\mathcal{E}_\beta(h_A^\beta, G_\alpha h_A^\beta) \le 0$. Since h_A^β is β-excessive and $G_\alpha h_A^\beta \ge 0$, we have $\mathcal{E}_\beta(h_A^\beta, G_\alpha h_A^\beta) \ge 0$. Thus

$$(2.15)\qquad \mathcal{E}_\beta(h_A^\beta, G_\alpha h_A^\beta) = 0.$$

Note that if $(\mathcal{E}, D(\mathcal{E}))$ is a symmetric Dirichlet form, then

$$\begin{aligned}(2.16)\qquad \mathcal{E}_\beta(h_A^\beta, G_\alpha h_A^\beta) &= \mathcal{E}_\alpha(h_A^\beta, G_\alpha h_A^\beta) + (\beta-\alpha)(h_A^\beta, G_\alpha h_A^\beta)\\ &\ge \mathcal{E}_\alpha(h_A^\beta, G_\alpha h_A^\beta) = (h_A^\beta, h_A^\beta).\end{aligned}$$

Equations (2.15) and (2.16) imply that $h_A^\beta = 0$, *i.e.*, $S_A^\beta h = 0$.

(ii) We do not know if the following inequality holds:

$$(2.17)\qquad S_A^\alpha h \le S_A^\beta h + (\beta-\alpha)G_\beta S_A^\alpha h.$$

Note that (2.17) is not a direct consequence of Theorem 2.7(iii). In fact, let $(\mathcal{E}, D(\mathcal{E}))$ be the Dirichlet form associated with the Brownian motion on $\mathbb{R}^1$. Define

$$f := 2x^2e^{-x^2},\quad g := 2e^{-x^2}.$$

Note that

$$(1-L)(-e^{-x^2}) = \Big(1-\frac{\triangle}{2}\Big)(-e^{-x^2}) = 2(x^2-1)e^{-x^2},$$

where $\triangle$ is the Laplacian operator. Hence

$$G_1f - G_1g = (1-L)^{-1}\big[2(x^2-1)e^{-x^2}\big] = -e^{-x^2} \le 0.$$

However, we do not have $f \le g$.

Proposition 2.9 *Let $A \subset E$ and $h \in D(\mathcal{E})$. Then for $\beta > \alpha > 0$, we have*

$$\begin{aligned}(2.18)\qquad h_A^\beta &\le (h_A^\alpha)_A^\beta \le h_A^\alpha \le (h_A^\beta)_A^\alpha \le (h_A^\alpha)_A^\beta + (\beta-\alpha)G_\alpha(h_A^\alpha)_A^\beta\\ &\le h_A^\beta + 2(\beta-\alpha)G_\alpha h_A^\beta + (\beta-\alpha)^2G_\alpha G_\alpha h_A^\beta.\end{aligned}$$

Proof Since $\widetilde{(h_A^\alpha)_A^\beta} \geq \widetilde{h_A^\alpha} \geq \widetilde{h}$ $\mathcal{E}$-*q.e.* on A and $(h_A^\alpha)_A^\beta$ is β-excessive (thus $(h_A^\alpha)_A^\beta \wedge h_A^\beta$ is β-excessive), by Proposition 1.1(iii), we have

$$h_A^\beta \leq (h_A^\alpha)_A^\beta. \tag{2.19}$$

Since h_A^α is α-excessive (and thus β-excessive by Proposition 2.1(iv)), we obtain by Proposition 1.1(iii) that

$$(h_A^\alpha)_A^\beta \leq h_A^\alpha. \tag{2.20}$$

Since $\widetilde{(h_A^\beta)_A^\alpha} \geq \widetilde{h_A^\beta} \geq \widetilde{h}$ $\mathcal{E}$-*q.e.* on A and $(h_A^\beta)_A^\alpha$ is α-excessive, by Proposition 1.1(iii), we have

$$h_A^\alpha \leq (h_A^\beta)_A^\alpha. \tag{2.21}$$

By Theorem 2.7(i), Proposition 1.1(iv), and the positivity preserving property of G_α, we get

$$(h_A^\beta)_U^\alpha \leq (h_A^\beta)_A^\beta + (\beta-\alpha)G_\alpha(h_A^\beta)_A^\beta \leq (h_A^\alpha)_A^\beta + (\beta-\alpha)G_\alpha(h_A^\alpha)_A^\beta. \tag{2.22}$$

By (2.20) and the positivity preserving of G_α, we get

$$(h_A^\alpha)_A^\beta + (\beta-\alpha)G_\alpha(h_A^\alpha)_A^\beta \leq h_A^\alpha + (\beta-\alpha)G_\alpha h_A^\alpha. \tag{2.23}$$

By Theorem 2.7(i) and the positivity preserving property of G_α, we get

$$\begin{aligned} &h_A^\alpha + (\beta-\alpha)G_\alpha h_A^\alpha \\ &\leq \left[h_A^\beta + (\beta-\alpha)G_\alpha h_A^\beta\right] + (\beta-\alpha)G_\alpha\left[h_A^\beta + (\beta-\alpha)G_\alpha h_A^\beta\right] \\ &= h_A^\beta + 2(\beta-\alpha)G_\alpha h_A^\beta + (\beta-\alpha)^2 G_\alpha G_\alpha h_A^\beta. \end{aligned} \tag{2.24}$$

Therefore, (2.18) holds by (2.19)–(2.24). ■

By Theorem 2.7, we can obtain the "continuity" of the operator $S(\,\cdot\,, A, h)$ on $(0,\infty)$.

Theorem 2.10 *Let $\alpha > 0$, $A \subset E$ and $h \in D(\mathcal{E})$. Then*

$$\lim_{\beta\to\alpha} \mathcal{E}_1(h_A^\beta - h_A^\alpha, h_A^\beta - h_A^\alpha) = 0. \tag{2.25}$$

Proof For $\beta > \alpha$, by Theorem 2.7(i) and the positivity preserving property of G_α, we get

$$0 \leq h_A^\alpha - h_A^\beta \leq (\beta-\alpha)G_\alpha h_A^\beta \leq (\beta-\alpha)G_\alpha h_A^\alpha,$$

which implies that $\lim_{\beta\downarrow\alpha} h_A^\beta = h_A^\alpha$. For $\beta \in (\alpha/2, \alpha)$, by Theorem 2.7(i) and the resolvent equation, we get

$$0 \leq h_A^\beta - h_A^\alpha \leq (\alpha-\beta)G_\beta h_A^\alpha \leq (\alpha-\beta)G_{\frac{\alpha}{2}} h_A^\alpha,$$

https://doi.org/10.4153/CJM-2011-055-5 Published online by Cambridge University Press

which implies that $\lim_{\beta\uparrow\alpha} h_A^\beta = h_A^\alpha$. Therefore,

$$\lim_{\beta\to\alpha} h_A^\beta = h_A^\alpha. \tag{2.26}$$

For $\beta > \alpha$, we obtain by Theorem 2.7(i), Lemma 2.1(ii), the positivity preserving property of G_α, and (2.26) that

$$\begin{aligned}
0 &\le \mathcal{E}_1(h_A^\alpha - h_A^\beta, h_A^\alpha - h_A^\beta)\\
&= \Big[\mathcal{E}_\alpha(h_A^\alpha, h_A^\alpha - h_A^\beta) + (1-\alpha)(h_A^\alpha, h_A^\alpha - h_A^\beta)\Big]\\
&\quad - \Big[\mathcal{E}_\beta(h_A^\beta, h_A^\alpha - h_A^\beta) + (1-\beta)(h_A^\beta, h_A^\alpha - h_A^\beta)\Big]\\
&\le \mathcal{E}_\alpha(h_A^\alpha, h_A^\alpha - h_A^\beta) + (1-\alpha)(h_A^\alpha, h_A^\alpha - h_A^\beta) - (1-\beta)(h_A^\beta, h_A^\alpha - h_A^\beta)\\
&\le \mathcal{E}_\alpha(h_A^\alpha, (\beta-\alpha)G_\alpha h_A^\beta) + (1-\alpha)(h_A^\alpha, h_A^\alpha - h_A^\beta) - (1-\beta)(h_A^\beta, h_A^\alpha - h_A^\beta)\\
&\le (\beta-\alpha)\mathcal{E}_\alpha(h_A^\alpha, G_\alpha h_A^\alpha) + (1-\alpha)(h_A^\alpha, h_A^\alpha - h_A^\beta) - (1-\beta)(h_A^\beta, h_A^\alpha - h_A^\beta)\\
&\to 0 \ \text{ as } \beta\to\alpha,
\end{aligned}$$

which implies that $\lim_{\beta\downarrow\alpha} \mathcal{E}_1(h_A^\beta - h_A^\alpha, h_A^\beta - h_A^\alpha) = 0$. For $\beta \in (\alpha/2, \alpha)$, by Theorem 2.7(i), Lemma 2.1(ii), the resolvent equation, (2.2), (2.4) and (2.26), we get

$$\begin{aligned}
0 &\le \mathcal{E}_1(h_A^\alpha - h_A^\beta, h_A^\alpha - h_A^\beta) = \mathcal{E}_1(h_A^\beta - h_A^\alpha, h_A^\beta - h_A^\alpha)\\
&= \Big[\mathcal{E}_\beta(h_A^\beta, h_A^\beta - h_A^\alpha) + (1-\beta)(h_A^\beta, h_A^\beta - h_A^\alpha)\Big]\\
&\quad - \Big[\mathcal{E}_\alpha(h_A^\alpha, h_A^\beta - h_A^\alpha) + (1-\alpha)(h_A^\alpha, h_A^\beta - h_A^\alpha)\Big]\\
&\le \mathcal{E}_\beta(h_A^\beta, h_A^\beta - h_A^\alpha) + (1+\alpha+\beta)(h_A^\alpha + h_A^\beta, h_A^\beta - h_A^\alpha)\\
&\le \mathcal{E}_\beta(h_A^\beta, (\alpha-\beta)G_\beta h_A^\alpha) + (1+\alpha+\beta)(h_A^\alpha + h_A^{\alpha/2}, h_A^\beta - h_A^\alpha)\\
&\le (\alpha-\beta)\mathcal{E}_\beta(h_A^\beta, G_{\frac{\alpha}{2}} h_A^\alpha) + (1+\alpha+\beta)(h_A^\alpha + h_A^{\alpha/2}, h_A^\beta - h_A^\alpha)\\
&\le (\alpha-\beta)(K_{\frac{\alpha}{2}}+1)\mathcal{E}_\beta(h_A^\beta, h_A^\beta)^{1/2}\mathcal{E}_\beta(G_{\frac{\alpha}{2}} h_A^\alpha, G_{\frac{\alpha}{2}} h_A^\alpha)^{1/2}\\
&\quad + (1+\alpha+\beta)(h_A^\alpha + h_A^{\alpha/2}, h_A^\beta - h_A^\alpha)\\
&\le (\alpha-\beta)(K_{\frac{\alpha}{2}}+1)^2\mathcal{E}_\beta(h, h)^{1/2}\mathcal{E}_\alpha(G_{\frac{\alpha}{2}} h_A^\alpha, G_{\frac{\alpha}{2}} h_A^\alpha)^{1/2}\\
&\quad + (1+\alpha+\beta)(h_A^\alpha + h_A^{\alpha/2}, h_A^\beta - h_A^\alpha)\\
&\le (\alpha-\beta)(K_{\frac{\alpha}{2}}+1)^2\mathcal{E}_\alpha(h, h)^{1/2}\mathcal{E}_\alpha(G_{\frac{\alpha}{2}} h_A^\alpha, G_{\frac{\alpha}{2}} h_A^\alpha)^{1/2}\\
&\quad + (1+\alpha+\beta)(h_A^\alpha + h_A^{\alpha/2}, h_A^\beta - h_A^\alpha)\\
&\to 0 \ \text{ as } \beta\to\alpha,
\end{aligned}$$

which implies that $\lim_{\beta\uparrow\alpha} \mathcal{E}_1(h_A^\beta - h_A^\alpha, h_A^\beta - h_A^\alpha) = 0$. Therefore (2.25) holds. ∎

https://doi.org/10.4153/CJM-2011-055-5 Published online by Cambridge University Press

By Theorem 2.7, there exists an $h_A^\infty \in L^2(E;m)$ such that

$$\lim_{\beta\to\infty} \|h_A^\beta - h_A^\infty\|_2 = 0. \tag{2.27}$$

Furthermore, we have the following proposition.

Proposition 2.11 *Let $\alpha > 0$, $A \subset E$ and $h \in D(\mathcal{E})$. If $h \le 0$ on A^c, then $h_A^\infty \in D(\mathcal{E})$ and h_A^β converges weakly to h_A^∞ in $D(\mathcal{E})$ (w.r.t. the $\mathcal{E}_1$-norm) as $\beta \to \infty$.*

Proof By (2.27) and [20, I. Lemma 2.12], it suffices to show that

$$\sup_{\beta\ge\alpha} \mathcal{E}_\alpha(h_A^\beta, h_A^\beta) < \infty. \tag{2.28}$$

For $\beta \ge \alpha$, by (1.1) and the sector condition, we get

$$\begin{aligned}
0 &\le \mathcal{E}_\alpha(h_A^\beta, h_A^\beta) \\
&= \mathcal{E}_\beta(h_A^\beta, h_A^\beta) - (\beta-\alpha)(h_A^\beta, h_A^\beta) \\
&\le \mathcal{E}_\beta(h_A^\beta, h) - (\beta-\alpha)(h_A^\beta, h_A^\beta) \\
&= \Big[\mathcal{E}_\alpha(h_A^\beta, h) + (\beta-\alpha)(h_A^\beta, h)\Big] - (\beta-\alpha)(h_A^\beta, h_A^\beta) \\
&\le K_\alpha \mathcal{E}_\alpha(h_A^\beta, h_A^\beta)^{1/2}\mathcal{E}_\alpha(h,h)^{1/2} + (\beta-\alpha)(h_A^\beta, h - h_A^\beta) \\
&\le K_\alpha \mathcal{E}_\alpha(h_A^\beta, h_A^\beta)^{1/2}\mathcal{E}_\alpha(h,h)^{1/2},
\end{aligned} \tag{2.29}$$

where in the last inequality we used the fact that $h_A^\beta \ge 0$ on E and $h_A^\beta \ge h$ on A, and the assumption that $h \le 0$ on A^c. It follows from (2.29) that

$$\mathcal{E}_\alpha(h_A^\beta, h_A^\beta) \le K_\alpha^2 \mathcal{E}_\alpha(h,h),$$

which implies (2.28). ■

Remark 2.12 (i) Equation (2.28) and thus Proposition 2.11 do not hold for general $h \in D(\mathcal{E})$. In fact, let A be a nearly Borel set of E and $h \in D(\mathcal{E})$ be a bounded τ-excessive function for some $\tau > 0$. Then, by Kuwae [16, Theorem 4.4] (cf. Proposition 3.1(ii)), we have

$$\widetilde{h_A^\beta}(x) = E_x[e^{-\beta\sigma_A}\widetilde{h}(X_{\sigma_A})], \ \ \forall \beta \ge \tau,$$

where X is the right process associated with $(\mathcal{E}, D(\mathcal{E}))$ and $\sigma_A := \inf\{t > 0 \mid X_t \in A\}$. Note that $\lim_{\beta\to\infty} \widetilde{h_A^\beta} = \widetilde{h}\cdot 1_{A^r}$, where A^r denotes the set of regular points of A, *i.e.*, $A^r = \{x \in E \mid P_x(\sigma_A > 0) = 0\}$. Hence $h_A^\infty = h \cdot 1_{A^r}$ (m-*a.e.* on E). However, in general, $h\cdot 1_{A^r}$ might not belong to $D(\mathcal{E})$. We give a concrete example as follows.

Let $(\mathcal{E}, D(\mathcal{E}))$ be the Dirichlet form associated with the Brownian motion on $\mathbb{R}^1$. Then $D(\mathcal{E}) = H^{1,2}(\mathbb{R}^1)$, the (1,2)-Sobolev space on $\mathbb{R}^1$, and

$$\mathcal{E}(u, v) = \frac{1}{2} \int_{\mathbb{R}^1} u' v' dx.$$

Define $h(x) := e^{-|x|}$ if $|x| > 1$ and $h(x) := (3 - x^2)e^{-1}/2$ otherwise. Then one can check that $h \in D(L)$ and $(1/2 - L)h(x) = (5 - x^2)1_{|x| \le 1}/4$. Hence h is a bounded 1/2-excessive function of $D(\mathcal{E})$. Let A be a finite closed subinterval of $\mathbb{R}^1$. Then we have $h \cdot 1_{A^r} = h \cdot 1_A \notin H^{1,2}(\mathbb{R}^1)$.

(ii) Let A be a nearly Borel set of E and $h \in D(\mathcal{E})$ (not necessarily excessive). Then for any $\alpha > 0$, we have

$$h_A^\infty = \lim_{\beta \to \infty} h_A^\beta \le \lim_{\beta \to \infty} (|h|_E^\alpha)_A^\beta = (|h|_E^\alpha) 1_{A^r}.$$

3 Balayage of Measures

In this section we discuss the balayage of measures. First, we make some preparation in Subsection 3.1.

3.1 Operators H_M^α and $\widehat{H}_M^\alpha$

Suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular semi-Dirichlet form on $L^2(E; m)$. Let $M \subset E$ and $D := M^c$. Define $\mathcal{F}_D := D(\mathcal{E})_D := \{f \in D(\mathcal{E}) \mid \widetilde{f} = 0\ \mathcal{E}\text{-}q.e. \text{ on } M\}$. Then, for any $\alpha > 0$, $\mathcal{F}_D$ is a closed subspace of the Hilbert space $D(\mathcal{E})$ (w.r.t. the $\mathcal{E}_\alpha$-norm). Let $u \in D(\mathcal{E})$. By applying [20, I.2.7, p. 18] to $J(w) = \mathcal{E}_\alpha(u, w), w \in D(\mathcal{E})$, and $\mathcal{C} = \mathcal{F}_D$, we obtain a unique function $\pi_{\mathcal{F}_D}^\alpha(u) \in \mathcal{F}_D$ such that

$$\mathcal{E}_\alpha(u - \pi_{\mathcal{F}_D}^\alpha(u), w) = 0, \quad \forall w \in \mathcal{F}_D.$$

For $\alpha > 0$ and $u \in D(\mathcal{E})$, define $H_M^\alpha u := u - \pi_{\mathcal{F}_D}^\alpha(u)$. Denote $\mathcal{H}_M^\alpha := \{H_M^\alpha u \mid u \in D(\mathcal{E})\}$. Then for any $u_1 \in \mathcal{H}_M^\alpha, u_2 \in \mathcal{F}_D$, we have $\mathcal{E}_\alpha(u_1, u_2) = 0$ and each $u \in D(\mathcal{E})$ can be uniquely decomposed into $u = u_1 + u_2, u_1 \in \mathcal{H}_M^\alpha, u_2 \in \mathcal{F}_D$. Therefore, we have the "orthogonal decomposition"

$$D(\mathcal{E}) = \mathcal{H}_M^\alpha \oplus \mathcal{F}_D. \tag{3.1}$$

Proposition 3.1 (i) *H_M^α is a continuous linear operator on $D(\mathcal{E})$ with respect to the $\mathcal{E}_\alpha$-norm.*

(ii) *If $u \in D(\mathcal{E})$ is α-excessive, then $H_M^\alpha u = u_M^\alpha$. In general, we have $H_M^\alpha f \le f_M^\alpha$ for any $f \in D(\mathcal{E})$.*

(iii) *H_M^α is sub-Markovian: if $u \in D(\mathcal{E})$ with $0 \le u \le 1$, then $0 \le H_M^\alpha u \le 1$.*

(iv) *Let M_1 and M_2 be two subsets of E with $M_1 \subset M_2$. Then, for any $\alpha > 0$ and $u \in D(\mathcal{E})$, $H_{M_1}^\alpha H_{M_2}^\alpha u = H_{M_2}^\alpha H_{M_1}^\alpha u = H_{M_1}^\alpha u$.*

Proof (i) It is easy to see that H_M^α is a linear operator. We now show that H_M^α is continuous. By (3.1) and the sector condition, we have

$$\mathcal{E}_\alpha(H_M^\alpha u, H_M^\alpha u) = \mathcal{E}_\alpha(H_M^\alpha u, u) \le K_\alpha \mathcal{E}_\alpha(H_M^\alpha u, H_M^\alpha u)^{1/2}\mathcal{E}_\alpha(u,u)^{1/2},$$

which implies that

$$\mathcal{E}_\alpha(H_M^\alpha u, H_M^\alpha u) \le K_\alpha^2 \mathcal{E}_\alpha(u,u). \tag{3.2}$$

It follows from (3.2) and the linearity of H_M^α that H_M^α is a continuous operator on $D(\mathcal{E})$.

(ii) If $u \in D(\mathcal{E})$ is α-excessive, then by Proposition 1.1(ii) and (iii), we have $\mathcal{E}_\alpha(u-(u-u_M^\alpha), w) = \mathcal{E}_\alpha(u_M^\alpha, w) = 0$ for all $w \in \mathcal{F}_D$ and $u - u_M^\alpha \in \mathcal{F}_D$. Hence $u - u_M^\alpha = \pi_{\mathcal{F}_D}^\alpha(u)$. Therefore, $H_M^\alpha u = u - \pi_{\mathcal{F}_D}^\alpha(u) = u_M^\alpha$.

Let $f \in D(\mathcal{E})$. We will show that $H_M^\alpha f \le f_M^\alpha = S_M^\alpha f$. Note that

$$\mathcal{G} := \{g_1 - g_2 \mid g_1, g_2 \text{ are } \alpha\text{-excessive functions in } D(\mathcal{E})\}$$

is dense in $D(\mathcal{E})$. By (i) and Theorem 2.2(iii), it suffices to show that $H_M^\alpha f \le S_M^\alpha f$ for any $f \in \mathcal{G}$. Suppose that $f = f_1 - f_2$ such that $f_1, f_2 \in D(\mathcal{E})$ are both α-excessive. Then, by the linearity of H_M^α and Theorem 2.2(ii), we get $H_M^\alpha f = H_M^\alpha f_1 - H_M^\alpha f_2 = S_M^\alpha f_1 - S_M^\alpha f_2 \le S_M^\alpha f$.

(iii) Let $u \in D(\mathcal{E})$ with $0 \le u \le 1$. By (ii) and Theorem 2.2(i), we get $H_M^\alpha u \le S_M^\alpha u \le 1$. For any $\varepsilon > 0$, we have $(-u)/\varepsilon \le 1$. Then $H_M^\alpha[(-u)/\varepsilon] \le S_M^\alpha[(-u)/\varepsilon] \le 1 \Rightarrow H_M^\alpha u \ge -\varepsilon$. Since ε is arbitrary, we get $H_M^\alpha u \ge 0$.

(iv) Since $M_1 \subset M_2$, we have $\widetilde{H_{M_1}^\alpha H_{M_2}^\alpha u} = \widetilde{H_{M_2}^\alpha u} = \widetilde{u}$ $\mathcal{E}$-*q.e.* on M_1. This together with the fact that

$$\mathcal{E}_\alpha(H_{M_1}^\alpha H_{M_2}^\alpha u, w) = 0, \quad \forall w \in \mathcal{F}_{M_1^c},$$

implies that $H_{M_1}^\alpha H_{M_2}^\alpha u = H_{M_1}^\alpha u$.

By the definition of $H_{M_2}^\alpha$, we have $\widetilde{H_{M_2}^\alpha H_{M_1}^\alpha u} = \widetilde{H_{M_1}^\alpha u}$ $\mathcal{E}$-*q.e.* on M_2. For any $w \in \mathcal{F}_{M_2^c} \subset \mathcal{F}_{M_1^c}$, we obtain from the definition of $H_{M_1}^\alpha$ that

$$\mathcal{E}_\alpha(H_{M_1}^\alpha u, w) = 0.$$

Hence $H_{M_2}^\alpha H_{M_1}^\alpha u = H_{M_1}^\alpha u$. ■

Let $(\widehat{\mathcal{E}}, D(\mathcal{E}))$ be the dual form of $(\mathcal{E}, D(\mathcal{E}))$. Then $(\widehat{\mathcal{E}}, D(\mathcal{E}))$ is a quasi-regular positive preserving form (cf. Ma–Röckner [21]). For any $\alpha > 0$ and $u \in D(\mathcal{E})$, there exists a unique function $\widehat{\pi}_{\mathcal{F}_D}^\alpha(u) \in \mathcal{F}_D$ such that

$$\mathcal{E}_\alpha(w, u - \widehat{\pi}_{\mathcal{F}_D}^\alpha(u)) = 0, \quad \forall w \in \mathcal{F}_D.$$

For $\alpha > 0$ and $u \in D(\mathcal{E})$, define $\widehat{H}_\alpha^M u := u - \widehat{\pi}_{\mathcal{F}_D}^\alpha(u)$. Then, for $f, g \in D(\mathcal{E})$, we have

$$\begin{aligned} \mathcal{E}_\alpha(H_M^\alpha f, g) &= \mathcal{E}_\alpha(H_M^\alpha f, H_M^\alpha g) = \mathcal{E}_\alpha(H_M^\alpha f, \widehat{H}_M^\alpha g) \\ &= \mathcal{E}_\alpha(\widehat{H}_M^\alpha f, \widehat{H}_M^\alpha g) = \mathcal{E}_\alpha(f, \widehat{H}_M^\alpha g). \end{aligned} \tag{3.3}$$

Proposition 3.2 (i) *$\widehat{H}^\alpha_M$ is a continuous linear operator on $D(\mathcal{E})$ with respect to the $\mathcal{E}_\alpha$-norm.*

(ii) *If $u \in D(\mathcal{E})$ is α-coexcessive, then $\widehat{H}^\alpha_M u = \widehat{u^\alpha_M}$. In general, we have $\widehat{H}^\alpha_M f \le \widehat{f^\alpha_M}$ for any $f \in D(\mathcal{E})$. Here $\widehat{h^\alpha_A}$ denotes the α-*cobalayaged *function of h on M.*

(iii) *$\widehat{H}^\alpha_M$ is positivity preserving: if $u \in D(\mathcal{E})$ with $u \ge 0$, then $\widehat{H}^\alpha_M u \ge 0$.*

(iv) *Let M_1 and M_2 be two subsets of E with $M_1 \subset M_2$. Then, for any $\alpha > 0$ and $u \in D(\mathcal{E})$, $\widehat{H}^\alpha_{M_1}\widehat{H}^\alpha_{M_2}u = \widehat{H}^\alpha_{M_2}\widehat{H}^\alpha_{M_1}u = \widehat{H}^\alpha_{M_1}u$.*

Proof The proofs of (i), (ii), and (iv) are similar to that of Proposition 3.1. We only prove (iii). Let $u \in D(\mathcal{E})$ with $u \ge 0$. Take $f = G_\alpha w$ with $w \in L^2_+(E;m)$. Then, by (3.3), Proposition 3.1(ii), and Lemma 2.1(ii), we get

$$(w, \widehat{H}^\alpha_M u) = \mathcal{E}_\alpha(f, \widehat{H}^\alpha_M u) = \mathcal{E}_\alpha(H^\alpha_M f, u) = \mathcal{E}_\alpha(f^\alpha_M, u) \ge 0.$$

Since $w \in L^2_+(E;m)$ is arbitrary, $\widehat{H}^\alpha_M u \ge 0$. ■

From now until the end of this subsection, we suppose that M is a nearly Borel set of E. Then one can check that $(\mathcal{E}, \mathcal{F}_D)$ is a semi-Dirichlet form on $L^2(D;m)$ in the wide sense, which means that $(\mathcal{E}, \mathcal{F}_D)$ satisfies all conditions of the semi-Dirichlet form on $L^2(D;m)$ except for the condition that $\mathcal{F}_D$ is dense in $L^2(D;m)$. Following the proof of [20, I. Theorem 2.8], there exist unique (not necessarily strongly continuous) contraction resolvents $(G^D_\alpha)_{\alpha>0}$ and $(\widehat{G}^D_\alpha)_{\alpha>0}$ on $L^2(D;m)$ such that $G^D_\alpha(L^2(D;m))$, $\widehat{G}^D_\alpha(L^2(D;m)) \subset \mathcal{F}_D$, and

$$\mathcal{E}_\alpha(G^D_\alpha f, u) = (f, u) = \mathcal{E}_\alpha(u, \widehat{G}^D_\alpha f), \quad \forall f \in L^2(D;m), u \in \mathcal{F}_D, \alpha > 0.$$

Lemma 3.3 (i) *For all $\alpha, \beta > 0$, $H^\alpha_M - H^\beta_M = (\beta-\alpha)G^D_\alpha H^\beta_M$ and $H^\alpha_M - H^\beta_M = (\beta-\alpha)G^D_\beta H^\alpha_M$.*

(ii) *For all $\alpha > 0$, $G_\alpha = G^D_\alpha + H^\alpha_M G_\alpha$, where $G^D_\alpha f := G^D_\alpha(f1_D)$ for $f \in L^2(E;m)$.*

The similar results hold for $(\widehat{G}^D_\alpha)_{\alpha>0}$.

Proof (i) Let $f \in D(\mathcal{E})$ and $g \in \mathcal{F}_D$. Then

$$\begin{aligned}\mathcal{E}_\alpha(H^\beta_M f + (\beta-\alpha)G^D_\alpha H^\beta_M f, g) &= \mathcal{E}_\alpha(H^\beta_M f, g) + (\beta-\alpha)(H^\beta_M f, g)\\ &= \mathcal{E}_\beta(H^\beta_M f, g) = 0\end{aligned}$$

and $\widetilde{H^\beta_M f + (\beta-\alpha)G^D_\alpha H^\beta_M f} = \widetilde{f}$ $\mathcal{E}$-*q.e.* on M. Hence $H^\alpha_M = H^\beta_M + (\beta-\alpha)G^D_\alpha H^\beta_M$. The second equality can be proved similarly.

(ii) Let $f \in L^2(E;m)$. Then $G_\alpha f - H^\alpha_M G_\alpha f \in \mathcal{F}_D$. For any $g \in \mathcal{F}_D$, we have

$$\mathcal{E}_\alpha(G_\alpha f - H^\alpha_M G_\alpha f, g) = (f,g) - \mathcal{E}_\alpha(H^\alpha_M G_\alpha f, g) = (f,g) = (f1_D, g) = \mathcal{E}_\alpha(G^D_\alpha f, g).$$

Hence $G_\alpha f - H^\alpha_M G_\alpha f = G^D_\alpha f$. ■

Corollary 3.4 *Let $\beta > \alpha > 0$ and $f \in D(\mathcal{E})$ with $f \ge 0$. Then $\widehat{H}^\beta_M f \le \widehat{H}^\alpha_M f$.*

3.2 Characterization of μ_B^α

From this point forward, we assume that $(\mathcal{E}, D(\mathcal{E}))$ is a regular semi-Dirichlet form on $L^2(E;m)$.

Lemma 3.5 *Each measure in S_0 charges no set of zero capacity.*

Proof Let $A \in \mathcal{B}(E)$ with $\mathrm{cap}_\phi(A) = 0$ and $\mu \in S_0$. We will show that $\mu(A) = 0$. Without loss of generality we assume that A is a compact subset of E. Then there exists a decreasing sequence of relatively compact open sets $\{U_n\}$ such that $A \subset U_n$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mathrm{cap}_\phi(U_n) = 0$. By [19, Definition 2.9 and Theorem 2.14], we know that $\{U_n^c\}$ is an $\mathcal{E}$-nest, *i.e.*, $\bigcup_{n\geq 1} D(\mathcal{E})_{U_n^c}$ is dense in $D(\mathcal{E})$.

We choose a $v \in D(\mathcal{E})$ satisfying $v \geq 1$ on U_1. Then $\sup_{n\geq 1} \mathcal{E}_1(v^1_{U_n}, v^1_{U_n}) \leq K_1^2 \mathcal{E}_1(v,v) < \infty$ by (2.3). Since $\{v^1_{U_n}\}$ is decreasing, we obtain by [20, I. Lemma 2.12] that $v^1_{U_n}$ converges weakly to some $f \in D(\mathcal{E})$ as $n \to \infty$. Let $w \in \bigcup_{n\geq 1} D(\mathcal{E})_{U_n^c}$. Then $\mathcal{E}_1(f,w) = \lim_{n\to\infty} \mathcal{E}_1(v^1_{U_n}, w) = 0$. By the density of $\bigcup_{n\geq 1} D(\mathcal{E})_{U_n^c}$ in $D(\mathcal{E})$, we get $f = 0$.

Set $g_n = n(U_1\mu - nG_{n+1}(U_1\mu))$, $n \in \mathbb{N}$. Then by Lemma 1.2 we know that $g_n \geq 0$. Note that $\lim_{n\to\infty}(g_n, w) = \mathcal{E}_1(U_1\mu, w)$ for any $w \in D(\mathcal{E})$. In particular, $\lim_{n\to\infty}(g_n, w) = \int_E w(x)\mu(dx)$ for any $w \in C_0(E) \cap D(\mathcal{E})$. Hence $g_n \cdot m$ converges vaguely to μ as $n \to \infty$. Therefore

$$\begin{aligned}\mu(A) &\leq \liminf_{n\to\infty} \mu(U_n) \leq \liminf_{n\to\infty} \liminf_{r\to\infty} \int_{U_n} g_r(x)m(dx)\\ &\leq \liminf_{n\to\infty} \liminf_{r\to\infty} (g_r, v^1_{U_n}) = \liminf_{n\to\infty} \mathcal{E}_1(U_1\mu, v^1_{U_n}) = 0. \qquad \blacksquare\end{aligned}$$

By Lemma 3.5, similar to [10, Theorem 2.2.2], we can show that for any $\mu \in S_0$ and any $v \in D(\mathcal{E})$, $\widetilde{v} \in L^1(E;\mu)$ and

$$\mathcal{E}_\alpha(U_\alpha\mu, v) = \int_E \widetilde{v}(x)\mu(dx), \quad \alpha > 0. \tag{3.4}$$

By (3.4), Proposition 3.1(ii) and (3.3), we obtain that for any $f \in D(\mathcal{E})$

$$\begin{aligned}\mu_B^\alpha(\widetilde{f}) &= \mathcal{E}_\alpha(U_\alpha\mu_B^\alpha, f) = \mathcal{E}_\alpha((U_\alpha\mu)_B^\alpha, f)\\ &= \mathcal{E}_\alpha(H_B^\alpha U_\alpha\mu, f) = \mathcal{E}_\alpha(U_\alpha\mu, \widehat{H}_B^\alpha f) = \int_E \widetilde{\widehat{H}_B^\alpha f}(x)\mu(dx).\end{aligned} \tag{3.5}$$

Proposition 3.6 *For any $\mu \in S_0$ and any $\alpha, \beta > 0$, we have*

$$U_\alpha\mu = U_\beta\mu + (\beta - \alpha)G_\alpha U_\beta\mu, \;\; U_\alpha\mu = U_\beta\mu + (\beta-\alpha)G_\beta U_\alpha\mu.$$

Proof Let $v \in C_0(E) \cap D(\mathcal{E})$. Then

$$\begin{aligned}&\mathcal{E}_\alpha(U_\beta\mu + (\beta-\alpha)G_\alpha U_\beta\mu, v)\\ &\quad = \mathcal{E}_\alpha(U_\beta\mu, v) + (\beta-\alpha)\mathcal{E}_\alpha(G_\alpha U_\beta\mu, v)\\ &\quad = [\mathcal{E}_\beta(U_\beta\mu, v) + (\alpha-\beta)(U_\beta\mu, v)] + (\beta-\alpha)(U_\beta\mu, v)\\ &\quad = \mathcal{E}_\beta(U_\beta\mu, v) = \int_E v(x)\mu(dx) = \mathcal{E}_\alpha(U_\alpha\mu, v),\end{aligned}$$

https://doi.org/10.4153/CJM-2011-055-5 Published online by Cambridge University Press

which implies the first equality. The second equality can be proved similarly. ■

Let μ be a measure on $(E, \mathcal{B}(E))$. We denote by $\operatorname{supp}_q[\mu]$ the *quasi-support* of μ, *i.e.*, the smallest quasi-closed set F such that $\mu(F^c) = 0$.

Proposition 3.7 *Let $u \in D(\mathcal{E})$ and F be a quasi-closed set of E. Then the following two conditions are equivalent:*

(i) $u = U_\alpha \mu$ *for some* $\mu \in S_0$ *with* $\operatorname{supp}_q[\mu] \subset F$*;*

(ii) $\mathcal{E}_\alpha(u, v) \geq 0, \forall v \in D(\mathcal{E}), \widetilde{v} \geq 0$ $\mathcal{E}$*-q.e. on* F*.*

Proof (i) $\Rightarrow$ (ii) is a direct consequence of (3.4). Suppose that (ii) holds. Then $u = U_\alpha \mu$ for some $\mu \in S_0$ by Lemma 1.2. By (3.4), we get

$$\int_{F^c} \widetilde{v}(x)\mu(dx) = 0, \ \ \forall v \in D(\mathcal{E})_{F^c}. \tag{3.6}$$

Let $\psi \in L^2(F^c; m)$ such that $0 < \psi \leq 1$ *m-a.e.* on F^c and set $w := G_1^{F^c}\psi$. Then $w \in D(\mathcal{E})_{F^c}$. By [19, Proposition 2.18(ii)], [16, Proposition 3.2], and considering the part semi-Dirichlet form $(\mathcal{E}, D(\mathcal{E})_{F^c})$, we know that $\widetilde{w} > 0$ $\mathcal{E}$*-q.e.* on F^c. Then by (3.6) and Lemma 3.5, we get $\mu(F^c) = 0$. Therefore $\operatorname{supp}_q[\mu] \subset F$, and the proof is complete. ■

Let $\mu \in S_0$, $B \subset E$ and $\alpha > 0$. We now consider the balayaged measure μ_B^α defined in Section 1. Note that

$$\mathcal{E}_\alpha(U_\alpha \mu_B^\alpha, v) = \mathcal{E}_\alpha((U_\alpha \mu)_B^\alpha, v) \geq 0, \ \forall v \in D(\mathcal{E}), \widetilde{v} \geq 0 \ \mathcal{E}\text{-}q.e. \text{ on } B.$$

By Proposition 3.7, we know that $\operatorname{supp}_q[\mu_B^\alpha] \subset \bar{B}^q$, the quasi-closure of B.

Until the end of this subsection, we suppose that $B \in \mathcal{B}(E)$.

Theorem 3.8 *For any $\mu \in S_0$, $B \in \mathcal{B}(E)$ and $\alpha > 0$, we have $\mu_B^\alpha \geq \mu$ when restricted on B.*

Proof Let $f \in D(\mathcal{E})$ with $f \geq 0$. By (3.5), the fact that $f - \widehat{H}_B^\alpha f = \widehat{\pi}^\alpha_{\mathcal{F}_{B^c}}(f) \in \mathcal{F}_{B^c}$ and the positivity preserving property of $\widehat{H}_B^\alpha$, we get

$$\int_E \widetilde{f} d\mu - \int_E \widetilde{f} d\mu_B^\alpha = \int_E \widetilde{f} d\mu - \int_E \widetilde{\widehat{H}_B^\alpha f} d\mu = \int_{B^c} (\widetilde{f} - \widetilde{\widehat{H}_B^\alpha f}) d\mu \leq \int_{B^c} \widetilde{f} d\mu,$$

which implies that

$$\int_E \widetilde{f} d\mu \leq \int_E \widetilde{f} d\mu_B^\alpha + \int_{B^c} \widetilde{f} d\mu. \tag{3.7}$$

For any compact set $F \subset B$ and $\varepsilon > 0$, there exists a relatively compact open set $G \supset F$ such that

$$\mu_B^\alpha(G) - \mu_B^\alpha(F) < \varepsilon, \ \mu(G) - \mu(F) < \varepsilon. \tag{3.8}$$

By the regularity of $(\mathcal{E}, D(\mathcal{E}))$, there exists $u \in C_0(E) \cap D(\mathcal{E})$ such that $0 \le u \le 1$, $\operatorname{supp}[u] \subset G$ and $u|_F \equiv 1$. Then, by (3.7) and (3.8), we get

$$\begin{aligned}\mu(F) &\le \int_E u d\mu \le \int_E u d\mu_B^\alpha + \int_{B^c} u d\mu \\ &\le \mu_B^\alpha(G) + \int_{G-B} u d\mu \le \mu_B^\alpha(G) + \int_{G-F} u d\mu \\ &\le \mu_B^\alpha(G) + \varepsilon \le \mu_B^\alpha(F) + 2\varepsilon.\end{aligned}$$

Since F and ε are arbitrary, $\mu_B^\alpha \ge \mu$ when restricted on B. ■

Theorem 3.9 *For any $\mu \in S_0$, $B \in \mathcal{B}(E)$ and $\alpha > 0$, μ_B^α is the measure in*

$$S_0(\alpha, B, \mu) := \{\nu \in S_0 \mid \nu \ge \mu \text{ when restricted on } B \text{ and } \widetilde{U_\alpha \nu} \ge \widetilde{U_\alpha \mu}\ \mathcal{E}\text{-q.e. on } B\}$$

with the smallest α-potential.

Proof By the definition of μ_B^α and Proposition 1.1(iii), we have

$$\widetilde{U_\alpha \mu_B^\alpha} = \widetilde{(U_\alpha \mu)_B^\alpha} \ge \widetilde{U_\alpha \mu}\ \mathcal{E}\text{-}q.e. \text{ on } B,$$

which together with Theorem 3.8 implies that $\mu_B^\alpha \in S_0(\alpha, B, \mu)$.

For any $\nu \in S_0(\alpha, B, \mu)$, we have $\widetilde{U_\alpha \nu} \ge \widetilde{U_\alpha \mu}$ $\mathcal{E}$-*q.e.* on B, and $U_\alpha \nu \wedge (U_\alpha \mu)_B^\alpha$ is α-excessive. Hence by Proposition 1.1(iii), we get $(U_\alpha \mu)_B^\alpha \le U_\alpha \nu$, *i.e.*, $U_\alpha \mu_B^\alpha \le U_\alpha \nu$. ■

Lemma 3.10 *Let $\mu, \nu \in S_0$ with $\mu \le \nu$. Then for any $\alpha > 0$, we have $U_\alpha \mu \le U_\alpha \nu$.*

Proof Since $\mu \le \nu$, $\nu - \mu \in S_0$. By (3.4), we have $U_\alpha(\nu - \mu) + U_\alpha \mu = U_\alpha \nu$. Furthermore, by Lemmas 1.2 and 2.1, we have $U_\alpha(\nu - \mu) \ge 0$. Hence $U_\alpha \mu \le U_\alpha \nu$. ■

Corollary 3.11 *Let $\mu \in S_0$ and $B \in \mathcal{B}(E)$. If $\operatorname{supp}_q[\mu] \subset B$, then for any $\alpha > 0$, we have*

(i) $\mu_B^\alpha = \mu$;
(ii) $U_\alpha \mu = (U_\alpha \mu)_B^\alpha$.

Proof (i) Note that $\mu \in S_0(\alpha, B, \mu)$. By the assumption that $\operatorname{supp}_q[\mu] \subset B$, we know $\mu \le \nu$ for any $\nu \in S_0(\alpha, B, \mu)$. Then, by Lemma 3.10, we get $U_\alpha \mu \le U_\alpha \nu$. Therefore $\mu_B^\alpha = \mu$ by Theorem 3.9.

(ii) By (i), we have $U_\alpha \mu = U_\alpha \mu_B^\alpha = (U_\alpha \mu)_B^\alpha$. ■

Corollary 3.12 *Let $\mu \in S_0$ and $B \in \mathcal{B}(E)$ be a quasi-closed set of E. Then for any $\alpha, \beta > 0$, we have $(\mu_B^\alpha)_B^\beta = \mu_B^\alpha$.*

Proof This is a direct consequence of Corollary 3.11 by noting that $\operatorname{supp}_q[\mu_B^\alpha] \subset B$. ■

Remark 3.13 Some results from this section may also be obtained by the technique of strongly supermedian functions and kernels developed in Feyel [6] and Beznea–Boboc [1]. We thank an anonymous referee for pointing this out to us.

https://doi.org/10.4153/CJM-2011-055-5 Published online by Cambridge University Press

3.3 Operator $T(\cdot,\cdot,\cdot)$

In this subsection we investigate some properties of the balayaged operator $T(\cdot,\cdot,\cdot)$ defined in (1.3). First, we fix $\alpha > 0$, $B \subset E$ and consider the operator $S(\alpha, B, \cdot)$ on S_0.

Proposition 3.14 *Let $\alpha > 0$ and $B \subset E$.*

(i) *If $\nu_1, \nu_2 \in S_0$, then $(\nu_1 + \nu_2)^\alpha_B = (\nu_1)^\alpha_B + (\nu_2)^\alpha_B$.*

(ii) *Let $\mu, \mu_1, \mu_2, \dots$ be a sequence of measures in S_0. If $\mu(\widetilde{f}) = \lim_{n\to\infty} \mu_n(\widetilde{f})$ for any $f \in D(\mathcal{E})$, then $(\mu_n)^\alpha_B$ converges vaguely to μ^α_B as $n \to \infty$.*

Proof (i) For any $v \in C_0(E) \cap D(\mathcal{E})$, by (3.5), we have

$$(\nu_1+\nu_2)^\alpha_B(v) = \int_E \widetilde{\widehat{H}^\alpha_B v}\, d(\nu_1+\nu_2) = \int_E \widetilde{\widehat{H}^\alpha_B v}\, d\nu_1 + \int_E \widetilde{\widehat{H}^\alpha_B v}\, d\nu_2 = \big((\nu_1)^\alpha_B + (\nu_2)^\alpha_B\big)(v),$$

which together with the regularity of $(\mathcal{E}, D(\mathcal{E}))$ implies that $(\nu_1+\nu_2)^\alpha_B = (\nu_1)^\alpha_B + (\nu_2)^\alpha_B$.

(ii) For any $v \in C_0(E) \cap D(\mathcal{E})$, by (3.5), we have

$$\begin{aligned}(\mu_n)^\alpha_B(f) - \mu^\alpha_B(f) &= \int_E \widetilde{\widehat{H}^\alpha_B f}\, d\mu_n - \int_E \widetilde{\widehat{H}^\alpha_B f}\, d\mu \\ &\to 0 \text{ as } n \to \infty,\end{aligned}$$

which together with the regularity of $(\mathcal{E}, D(\mathcal{E}))$ implies that $(\mu_n)^\alpha_B$ converges vaguely to μ^α_B as $n \to \infty$. ■

Second, we fix $\alpha > 0$, $\mu \in S_0$ and consider the operator $T(\alpha, \cdot, \mu)$ on 2^E.

Proposition 3.15 *Let $\alpha > 0$ and $\mu \in S_0$. Suppose that $B_1 \subset B_2 \subset E$. Then*

(i) *$(\mu^\alpha_{B_2})^\alpha_{B_1} = (\mu^\alpha_{B_1})^\alpha_{B_2} = \mu^\alpha_{B_1}$;*

(ii) *if $B_1 \in \mathcal{B}(E)$, then $\mu^\alpha_{B_2} \le \mu^\alpha_{B_1}$ when restricted on B_1;*

(iii) *if $B_1, B_2 \in \mathcal{B}(E)$, then $\mu^\alpha_{B_2} \le \mu^\alpha_{B_1} + \mu^\alpha_{B_2 - B_1}$ when restricted on B_2. Moreover, if B_2 is quasi-closed, then $\mu^\alpha_{B_2} \le \mu^\alpha_{B_1} + \mu^\alpha_{B_2 - B_1}$.*

Proof (i) By the definition of balayage of measures and Proposition 1.1(v), we have

$$U_\alpha\big((\mu^\alpha_{B_2})^\alpha_{B_1}\big) = (U_\alpha \mu^\alpha_{B_2})^\alpha_{B_1} = \big((U_\alpha\mu)^\alpha_{B_2}\big)^\alpha_{B_1} = (U_\alpha\mu)^\alpha_{B_1} = U_\alpha \mu^\alpha_{B_1},$$

which together with (3.4) and the regularity of $(\mathcal{E}, D(\mathcal{E}))$ implies that $(\mu^\alpha_{B_2})^\alpha_{B_1} = \mu^\alpha_{B_1}$. Similarly, we can prove that $(\mu^\alpha_{B_1})^\alpha_{B_2} = \mu^\alpha_{B_1}$.

Part (ii) holds by (i) and Theorem 3.8. Part (iii) is a direct consequence of (ii). ■

Theorem 3.16 *Let $\alpha > 0$ and $\mu \in S_0$. Suppose that one of the following two conditions holds:*

(i) *$\{B, B_n, n \ge 1\} \subset 2^E$ with $\lim_{n\to\infty} \operatorname{cap}_\phi(B_n \triangle B) = 0$;*

(ii) *$\{B_n\} \subset 2^E, B_n \uparrow B$.*

Then $\mu^\alpha_{B_n}$ converges vaguely to μ^α_B as $n \to \infty$.

https://doi.org/10.4153/CJM-2011-055-5 Published online by Cambridge University Press

Proof Let $f \in C_0(E) \cap D(\mathcal{E})$. Then, we obtain by (3.4) and Theorem 2.5 (resp. Proposition 2.6) that, as $n \to \infty$,

$$\begin{aligned}\mu^\alpha_{B_n}(f) &= \mathcal{E}_\alpha(U_\alpha\mu^\alpha_{B_n}, f) = \mathcal{E}_\alpha((U_\alpha\mu)^\alpha_{B_n}, f)\\ &\to \mathcal{E}_\alpha((U_\alpha\mu)^\alpha_B, f)\\ &= \mathcal{E}_\alpha(U_\alpha\mu^\alpha_B, f) = \mu^\alpha_B(f),\end{aligned}$$

which together with the regularity of $(\mathcal{E}, D(\mathcal{E}))$ implies that $\mu^\alpha_{B_n}$ converges vaguely to μ^α_B as $n \to \infty$. ∎

Finally, we fix $B \subset E$, $\mu \in S_0$ and consider the operator $T(\cdot, B, \mu)$ on $(0, \infty)$.

Theorem 3.17 *Let $B \in \mathcal{B}(E)$ and $\mu \in S_0$. Then for $\beta > \alpha > 0$, we have*

(i) $\mu^\beta_B \leq \mu^\alpha_B$;

(ii) $\mu^\alpha_B \leq \mu^\beta_B + (\beta - \alpha)(U_\alpha\mu \cdot m)^\beta_B$ *and* $\mu^\alpha_B \leq \mu^\beta_B + (\beta - \alpha)(U_\beta\mu \cdot m)^\alpha_B$.

Proof (i) Let $f \in C_0(E) \cap D(\mathcal{E})$ with $f \geq 0$. Then, by (3.5) and Corollary 3.4, we get

$$\mu^\beta_B(f) = \int_E \widetilde{\widehat{H}^\beta_B f} d\mu \leq \int_E \widetilde{\widehat{H}^\alpha_B f} d\mu = \mu^\alpha_B(f),$$

which together with the regularity of $(\mathcal{E}, D(\mathcal{E}))$ implies that $\mu^\beta_B \leq \mu^\alpha_B$.

(ii) Set $Q = B^c$. Let $f \in C_0(E) \cap D(\mathcal{E})$ with $f \geq 0$. Then, by (3.5) and Lemma 3.3, we get

$$\begin{aligned}\mu^\alpha_B(f) &= \int_E \widetilde{\widehat{H}^\alpha_B f} d\mu = \int_E \widetilde{\widehat{H}^\beta_B f} d\mu + (\beta - \alpha) \int_E \widehat{G}^Q_\alpha \widehat{H}^\beta_B f d\mu\\ &= \mu^\beta_B(f) + (\beta - \alpha)\mathcal{E}_\alpha(U_\alpha\mu, \widehat{G}^Q_\alpha \widehat{H}^\beta_B f)\\ &\leq \mu^\beta_B(f) + (\beta - \alpha)\mathcal{E}_\alpha(U_\alpha\mu, \widehat{G}_\alpha \widehat{H}^\beta_B f)\\ &= \mu^\beta_B(f) + (\beta - \alpha)\mathcal{E}_\alpha(G_\alpha U_\alpha\mu, \widehat{H}^\beta_B f)\\ &= \mu^\beta_B(f) + (\beta - \alpha) \int_E \widetilde{\widehat{H}^\beta_B f} d(U_\alpha\mu \cdot m)\\ &= \big(\mu^\beta_B + (\beta - \alpha)(U_\alpha\mu \cdot m)^\beta_B\big)(f),\end{aligned}$$

which together with the regularity of $(\mathcal{E}, D(\mathcal{E}))$ implies that $\mu^\alpha_B \leq \mu^\beta_B + (\beta - \alpha)(U_\alpha\mu \cdot m)^\beta_B$. The second equality can be proved similarly by using the formula $\widehat{H}^\alpha_B = \widehat{H}^\beta_B + (\beta - \alpha)\widehat{G}^Q_\beta \widehat{H}^\alpha_B$ (cf. Lemma 3.3). ∎

Lemma 3.18 *Let $\mu, \nu \in S_0$ with $\mu \leq \nu$. Then for any $\alpha > 0$ and $B \subset E$, we have $\mu^\alpha_B \leq \nu^\alpha_B$.*

Proof Let $f \in C_0(E) \cap D(\mathcal{E})$ with $f \geq 0$. Then, by (3.5), Proposition 3.2(iii) and the assumption that $\mu \leq \nu$, we get

$$\mu^\alpha_B(f) = \int_E \widetilde{\widehat{H}^\alpha_B f} d\mu \leq \int_E \widetilde{\widehat{H}^\alpha_B f} d\nu = \nu^\alpha_B(f),$$

https://doi.org/10.4153/CJM-2011-055-5 Published online by Cambridge University Press

which together with the regularity of $(\mathcal{E}, D(\mathcal{E}))$ implies that $\mu_B^\alpha \leq \nu_B^\alpha$. ■

Corollary 3.19 *Let $\mu \in S_0$, $B \in \mathcal{B}(E)$ and $\beta > \alpha > 0$. Then*

(i) $\mu_B^\beta \leq (\mu_B^\beta)_B^\alpha \leq \mu_B^\alpha$ *and* $\mu_B^\beta \leq (\mu_B^\alpha)_B^\beta \leq \mu_B^\alpha$;
(ii) $(\mu_B^\alpha)_B^\beta \leq (\mu_B^\beta)_B^\alpha + (\beta-\alpha)(U_\alpha\mu\cdot m)_B^\beta$ *and* $(\mu_B^\alpha)_B^\beta \leq (\mu_B^\beta)_B^\alpha + (\beta-\alpha)(U_\beta\mu\cdot m)_B^\alpha$;
(iii) *If B is a quasi-closed set of E, then* $(\mu_B^\beta)_B^\alpha \leq (\mu_B^\alpha)_B^\beta$.

Proof (i) By Proposition 3.15(i), Theorem 3.17(i), and Lemma 3.18, we get

$$\mu_B^\beta = (\mu_B^\beta)_B^\beta \leq (\mu_B^\beta)_B^\alpha \leq (\mu_B^\alpha)_B^\alpha = \mu_B^\alpha.$$

Similarly, we can prove $\mu_B^\beta \leq (\mu_B^\alpha)_B^\beta \leq \mu_B^\alpha$.

(ii) By Theorem 3.17, Lemma 3.18, Proposition 3.14(i), and Proposition 3.15(i), we get

$$\begin{aligned}(\mu_B^\alpha)_B^\beta &\leq \Big(\mu_B^\beta + (\beta-\alpha)(U_\alpha\mu\cdot m)_B^\beta\Big)_B^\beta = (\mu_B^\beta)_B^\beta + (\beta-\alpha)((U_\alpha\mu\cdot m)_B^\beta)_B^\beta \\ &\leq (\mu_B^\beta)_B^\alpha + (\beta-\alpha)(U_\alpha\mu\cdot m)_B^\beta.\end{aligned}$$

The second inequality can be proved similarly.

(iii) By Corollary 3.12 and Theorem 3.17(i), we get $(\mu_B^\beta)_B^\alpha = \mu_B^\beta \leq \mu_B^\alpha = (\mu_B^\alpha)_B^\beta$. ■

Theorem 3.20 *Let $\mu \in S_0$, $B \in \mathcal{B}(E)$ and $\alpha > 0$. Then μ_B^β converges vaguely to μ_B^α as $\beta \to \alpha$.*

Proof For any $\beta > \alpha$ and $f \in C_0(E) \cap D(\mathcal{E})$, by Theorem 3.17, we get

$$0 \leq \mu_B^\alpha(f) - \mu_B^\beta(f) \leq (\beta-\alpha)(U_\alpha\mu\cdot m)_B^\beta(f) \leq (\beta-\alpha)(U_\alpha\mu\cdot m)_B^\alpha(f),$$

which together with the regularity of $(\mathcal{E}, D(\mathcal{E}))$ implies that μ_B^β converges vaguely to μ_B^α as $\beta \downarrow \alpha$. For any $\beta \in (\alpha/2, \alpha)$ and $f \in C_0(E) \cap D(\mathcal{E})$, by Theorem 3.17, we get

$$0 \leq \mu_B^\beta(f) - \mu_B^\alpha(f) \leq (\alpha-\beta)(U_\alpha\mu\cdot m)_B^\beta(f) \leq (\alpha-\beta)(U_\alpha\mu\cdot m)_B^{\alpha/2}(f),$$

which together with the regularity of $(\mathcal{E}, D(\mathcal{E}))$ implies that μ_B^β converges vaguely to μ_B^α as $\beta \uparrow \alpha$. Therefore μ_B^β converges vaguely to μ_B^α as $\beta \to \alpha$. ■

Remark 3.21 Let $\mu \in S_0$ and $B \in \mathcal{B}(E)$. By Theorem 3.17(i), we know that μ_B^α is decreasing as α increases. For any $A \in \mathcal{B}(E)$, define

$$\mu_B^\infty(A) := \lim_{\alpha\to\infty} \mu_B^\alpha(A).$$

Then μ_B^∞ is a measure in S_0, and μ_B^α converges weakly to μ_B^∞ as $\alpha \to \infty$.

Acknowledgments We thank Professor Zhi-Ming Ma for bringing our attention to [2], Professor Jian-Gang Ying for discussions about the time change of Markov processes and providing us with the reference [18], and Xinfang Han and Li Ma for discussions that helped to give the concrete example in Remark 2.12(i) and the proof of Lemma 3.5. We also thank an anonymous referee for remarks that helped improve the presentation of this paper.

References

[1] L. Beznea and N. Boboc, *Potential theory and right processes.* Mathematics and its Applications, 572, Kluwer Academic Publishers, Dordrecht, 2004.

[2] J. Bliedtner and W. Hansen, *Potential theory, an analytic and probabilistic approach to balayage.* Universitext, Springer-Verlag, Berlin, 1986.

[3] Z.-Q. Chen, M. Fukushima, and J. Ying, *Traces of symmetric Markov processes and their characterizations.* Ann. Probab. **34**(2006), no. 3, 1052–1102. http://dx.doi.org/10.1214/009117905000000657

[4] ———, *Entrance law, exit system and Levy system of time changed processes.* Illinois J. Math. **50**(2006), no. 1–4, 269–312.

[5] J. L. Doob, *Classical potential theory and its probabilistic counterpart.* Grundlehren der Mathematischen Wissenschaften, 262, Springer-Verlag, New York, 1984.

[6] D. Feyel, *Sur la théorie fine du potentiel.* Bull. Soc. Math. France **111**(1983), no. 1, 41–57.

[7] P. J. Fitzsimmons, *On the quasi-regularity of semi-Dirichlet forms.* Potential Anal. **15**(2001), no. 3, 151–185. http://dx.doi.org/10.1023/A:1011249920221

[8] P. J. Fitzsimmons and R. K. Getoor, *Lévy systems and time changes.* In: Séminaire de Probabilités XLII, Lecture Notes in Math., 1979, Springer, Berlin, 2009, pp. 229–259.

[9] M. Fukushima, P. He, and J. Ying, *Time changes of symmetric diffusions and Feller measures.* Ann. Probab. **32**(2004), no. 4, 3138–3166. http://dx.doi.org/10.1214/009117904000000649

[10] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet forms and symmetric Markov processes.* de Gruyter Studies in Mathematics, 19, Walter de Gruyter, Berlin, 1994.

[11] M. Fukushima and T. Uemura, *Jump-type processes generated by lower bounded semi-Dirichlet forms.* Ann. Probab., to appear. http://www.imstat.org/aop/future_papers.htm

[12] Z.-C. Hu and Z.-M. Ma, *Beurling-Deny formula of semi-Dirichlet forms.* C. R. Math. Acad. Sci. Paris **338**(2004), no. 7, 521–526.

[13] Z.-C. Hu, Z.-M. Ma, and W. Sun, *Extensions of Lévy-Khintchine formula and Beurling-Deny formula in semi-Dirichlet forms setting.* J. Funct. Anal. **239**(2006), no. 1, 179–213. http://dx.doi.org/10.1016/j.jfa.2005.12.015

[14] ———, *Some remarks on representations of non-symmetric local Dirichlet forms.* In: Potential Theory and Stochastics in Albac, Theta Ser. Adv. Math., 11, Theta, Bucharest, 2009, pp. 145–156.

[15] ———, *On representations of non-symmetric Dirichlet forms.* Potential Anal. **32**(2010), no. 2, 101–131. http://dx.doi.org/10.1007/s11118-009-9145-5

[16] K. Kuwae, *Maximum principles for subharmonic functions via local semi-Dirichlet forms.* Canad. J. Math. **60**(2008), no. 4, 822–874. http://dx.doi.org/10.4153/CJM-2008-036-8

[17] Y. LeJan, *Balayage et formes de Dirichlet.* Z. Warsch. Verw. Gebiete **37**(1976/77), no. 4, 297–319.

[18] ———, *Mesures associées à une forme de Dirichlet. Applications.* Bull. Soc. Math. France **106**(1978), no. 1, 61–112.

[19] Z. M. Ma, L. Overbeck, and M. Röckner, *Markov processes associated with semi-Dirichlet forms.* Osaka J. Math. **32**(1995), no. 1, 97–119.

[20] Z. M. Ma and M. Röckner, *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms.* Springer-Verlag, Berlin-Heidelberg-New York, 1992.

[21] ———, *Markov processes associated with positivity preserving coercive forms.* Canad. J. Math. **47**(1995), no. 4, 817–840. http://dx.doi.org/10.4153/CJM-1995-042-6

[22] Z. M. Ma and W. Sun, *Some topics on Dirichlet forms.* In: New trends in stochastic analysis and related topics: a volume in honour of Professor K. D. Elworthy. World Scientific Publishing Company, 2011.

[23] L. Overbeck, M. Röckner, and B. Schmuland, *An analytic approach to Fleming-Viot processes with interactive selection.* Ann. Probab. **23**(1995), no. 1, 1–36. http://dx.doi.org/10.1214/aop/1176988374

[24] M. Röckner M and B. Schmuland, *Quasi-regular Dirichlet forms: examples and counterexamples.* Canad. J. Math. **47**(1995), no. 1, 165–200. http://dx.doi.org/10.4153/CJM-1995-009-3

[25] M. Silverstein, *Dirichlet spaces and random time change.* Illinois J. Math. **17**(1973), 1–72.

[26] ———, *The reflected Dirichlet space.* Illinois J. Math. **18**(1974), 310–355.

[27] ———, *Symmetric Markov processes.* Lecture Notes in Math., 426, Springer-Verlag, Berlin-New York, 1974.

Department of Mathematics, Nanjing University, Nanjing, 210093, China
e-mail: huzc@nju.edu.cn

Department of Mathematics and Statistics, Concordia University, Montreal, H3G 1M8
e-mail: wsun@mathstat.concordia.ca