TOPOLOGY OF SOME KÄHLER MANIFOLDS II

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Topology of positively curved compact Kähler manifolds had been studied by several authors (cf. [6; 2]); these manifolds are simply connected and their second Betti number is one [1]. We will restrict ourselves to the study of some compact homogeneous Kähler manifolds. The aim of this paper is to supplement some results in [9]. We prove, among other results, that a compact, simply connected homogeneous complex manifold whose Euler number is a prime $p \geq 2$ is isomorphic to the complex projective space $P_{p-1}(C)$; in the case of surfaces, we prove that a compact, simply connected, homogeneous almost complex surface with Euler-Poincaré characteristic positive, is hermitian symmetric. We first recall the following result (cf. [3]) which has some interesting consequences:

PROPOSITION 1. *Let V be a compact Kähler manifold of constant scalar curvature; if the second Betti number of V is one, then V is a Kähler-Einstein space.*

Proof. Since V has constant scalar curvature, the Ricci form Φ of the Kähler metric of V is co-closed [3]. But Φ is always closed and hence harmonic. Since the second Betti number $b_2(V)$ is equal to one we see that Φ is proportional to the Kähler form and consequently V is Kähler-Einstein.

Evidently the Ricci curvature of such a Kähler manifold V is everywhere positive, zero or negative; if V is, moreover, homogeneous, then the Ricci curvature is non negative.

COROLLARY 1 *Let V be a compact homogeneous (assumed effective in what follows) Kähler manifold whose second Betti number is one; if the Ricci curvature is non-zero, then V is simply connected* (cf. [9, Proposition 1]). *If* $\dim_C V = 2$, *then V is isomorphic to the complex projective plane* (cf. Theorem 1, below; note that the Euler-Poincaré characteristic of such a surface is equal to three).

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We may point out in this connection the following interesting result which is essentially due to Andreotti (cf. [6, page 172]): Let V be a complete Kähler surface of positive Ricci curvature; if its second Betti number is one and if V has no exceptional curves of the first kind, then V is birationally equivalent, without exceptions, to the complex projective plane.

The remark in [9, page 168] is false in general. However, we have the following result which is implicitly contained in [5].

THEOREM 1. Let V be a compact, simply connected, homogeneous complex manifold whose Euler-Poincaré characteristic is a prime number $p \geq 2$; then V is isomorphic to the complex projective space $P_{p-1}(C)$.

Proof. Since the Euler-Poincaré characteristic is a prime number, we may assume that $V = K/L$ where K is compact simple with center reduced to $\{e\}$ and L is a maximal connected subgroup of maximal rank [4]. Now the classification of [4] shows that V is isomorphic to the complex projective space $P_{p-1}(C)$ or to one of the following spaces: the sphere S^{2n} $(n > 1)$, S^6, the Cayley plane $F_{4/spin(9)}$, the Cayley line $G_{2/SO(4)}$ and the quaternionic projective space $P_{p-1}(K)$. But none of the latter spaces is homogeneous complex since their second Betti number is zero (cf. [5, page 500]).

In the case of a compact complex surface $(n = 2)$, we have the following more precise result:

THEOREM 2. Let V be a compact simply connected homogeneous complex surface whose Euler-Poincaré characteristic is positive; then V is hermitian symmetric.

Proof. We may assume that $V = K/L$ where K is a compact semi-simple Lie group and L is a closed subgroup of maximal rank; moreover, L has a non-discrete center. If the linear isotropy group $\tilde{L}$ of L is irreducible, then V is hermitian symmetric. If $\tilde{L}$ is reducible, then $\tilde{L}$ is of 1 or 2 parameters; if $\tilde{L}$ is of one parameter, then K is of five real parameters which is impossible since K is semi-simple. Consequently, $\tilde{L}$ is of two parameters and hence V is again symmetric.

In fact, using some results of Hermann [7], we can obtain the following result.

COROLLARY 2. *Let* V *be a compact homogeneous almost complex surface with a compact effective transformation group and whose Euler-Poincaré characteristic is different from zero; then* V *is homogeneous complex. If* V *is simply connected, then* V *is hermitian symmetric.*

The first part is proved in [7] where the author gives a classification of such surfaces. As a consequence, we have the following interesting result (cf. [10])

COROLLARY 3. *Any homogeneous almost complex structure on the complex projective plane* $P_2(C)$ *is isomorphic to the usual complex structure (or its conjugate).*

Note that second Betti number of a compact homogeneous almost complex surface whose Euler-Poincaré characteristic is positive, is different from zero; in fact, this is true for any compact almost complex surface whose Euler-Poincaré characteristic is different from zero (cf. [11, Lemma]). On the other hand, there exist such compact almost complex surfaces which are not complex analytic; these examples are due to Van de Ven. Such surfaces are not *homogeneous* almost complex by Corollary 2. Similar examples exist in higher dimensions; we will discuss them in a future publication.

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