Canad. Math. Bull. Vol. 17 (3), 1974

A NOTE ON CONTRACTIVE SELFMAPS OF l_∞

BY
T. F. McCABE

ABSTRACT. In this note it is shown that a contractive matrix selfmap T of l_∞ has T^2 a contraction; hence the techniques used in showing ITNC for $\ell_p (1 \le p < \infty)$, c_0, and c do not apply to ℓ_∞.

1. **Introduction.** A selfmap f of a metric space (X, d) is said to be *contractive* if $d(f(x), f(y)) < d(x, y)$ for distinct x, y in X. In [3], Nadler introduces and motivates the following definition: the *iterative test* is *conclusive* (ITC) for (X, d) provided that if f is a contractive selfmap of X with a fixed point then, for all $x \in X$, $\{f^n(x)\}$ converges to that fixed point. Nadler shows that the iterative test is conclusive for all finite dimensional Banach spaces, but that the iterative test is not conclusive (ITNC) for the spaces $\ell_p (1 \le p < \infty)$ and c_0. In [2] this author and Jack Bryant proved ITNC for c and extended this result to Banach spaces of the form $C(T)$ where T is a compact, Hausdorff space having a sequence of distinct points that converge. In demonstrating ITNC for each of $\ell_p (1 \le p < \infty)$, c_0, and c, examples of linear selfmaps were constructed which, of course, may be exhibited as infinite matrices.

Nadler questioned whether the iterative test is conclusive for ℓ_∞. It is the purpose of this paper to show that the techniques used in solving the problem for the other sequence spaces will not work for ℓ_∞.

2. **Results.** Considered as summability methods matrix selfmaps have been studied in detail [1]; we shall require only the following:

THEOREM 1. *Let $T = [a_{mn}]$ be a continuous matrix selfmap of ℓ_∞. Then*

$$\|T\| = \sup_n \left\{ \sum_{j=1}^{\infty} |a_{nj}| \right\}.$$

When first considering matrix maps, it is not obvious that there exist contractive matrix selfmaps of ℓ_∞ that are not contractions. The following example exhibits such a matrix.

EXAMPLE 2. Let T be defined by $T(\{x_n\}) = \{y_n\}$ where

$$y_1 = \tfrac{1}{2}x_1; \qquad y_n = \sum_{j=1}^{n-1} \frac{(-1)^j}{2^{j+1}} x_j + \frac{(-1)^{n+1}}{2} x_n \quad \text{for } n = 2, 3, \ldots.$$

https://doi.org/10.4153/CMB-1974-080-4 Published online by Cambridge University Press

T is represented by the matrix

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ -\frac{1}{4} & -\frac{1}{2} & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ -\frac{1}{4} & +\frac{1}{8} & \frac{1}{2} & 0 & \cdot & \cdot & \cdot & \cdot \\ -\frac{1}{4} & \frac{1}{8} & -\frac{1}{16} & -\frac{1}{2} & 0 & \cdot & \cdot & \cdot \\ -\frac{1}{4} & \frac{1}{8} & -\frac{1}{16} & \frac{1}{32} & \frac{1}{2} & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} = [a_{nm}].$$

Since $\sum_{m=1}^{\infty} |a_{nm}|<1$ for each n, and $\lim_n \sum_{m=1}^{\infty} |a_{nm}|=1$ then $\|T\|=1$; however T is also contractive. Indeed, upon considering the arrangement of positive and negative terms, if $\|x\|=1$ and $\|T(x)\|=1$ then $x=\lambda(-1, +1, -1, +1, \ldots)$ where $|\lambda|=1$; but if $T(x)=\{y_n\}$ then

$$\begin{aligned} y_n &= \lambda(-\tfrac{1}{4}(-1)+\tfrac{1}{8}(1)+(-1)(-\tfrac{1}{16})+\cdots \\ &\quad +(-1)^{n+1}(-1)^{n+1}(\tfrac{1}{2})^n+(-1)^{n+2}(\tfrac{1}{2})(-1)^{n+1}) \\ &= \lambda\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^n}-\frac{1}{2}\right) \end{aligned}$$

and $y_n \to 0$ as $n\to\infty$. Thus $\|T(x)\|<1$ and T is contractive.

The following proves that many matrix selfmaps of norm 1 cannot be contractive.

THEOREM 3. *If $T=[a_{nm}]$ has norm 1 and maps ℓ_∞ into c then there is an $x\in\ell_\infty$ such that $\|T(x)\|=\|x\|$ and hence T cannot be contractive.*

Proof. It is a consequence of the theorem of Silverman and Toeplitz [[1], p. 44] that any matrix $T=[a_{nm}]$ that maps ℓ_∞ into c has the property that the absolute row sums $\sum_{m=1}^{\infty} |a_{nm}|$ converge uniformly in n; that is, given $\varepsilon>0$ there is an N such that

$$\sum_{m=1}^{\infty} |a_{nm}| - \sum_{m=1}^{N} |a_{nm}| < \varepsilon \text{ for all } n.$$

If for some n, $\sum_{m=1}^{\infty} |a_{nm}|=1$, then select $x_m=\operatorname{sgn}(a_{nm})$ and let $x=(x_1, x_2, \ldots)$. Then obviously $\|T(x)\|=\|x\|=1$ and the proof would be complete.

If $\sum |a_{nm}|<1$ for all n we will construct $x\in\ell_\infty$ through the following process.

Choose $\{n_i\}$ such that $\lim_i \sum_{m=1}^{\infty} |a_{n_i m}|=1$. There is a number x_1 (either 1 or -1) such that $\operatorname{sng}(a_{n_i 1})=x_1$ for infinitely many $\{n_i\}$. We select this subsequence and denote it by $\{n_i^{(1)}\}$. Suppose $x_1, x_2, \ldots, x_m$ and $\{n_i^{(1)}\}, \ldots, \{n_i^{(m)}\}$ have been selected. Then there is a x_{m+1} (1 or -1) such that $x_{m+1}=\operatorname{sgn}(a_{n_i^{(m)} m+1})$ for infinitely many $\{n_i^{(m)}\}$. We select this subsequence and denote it by $\{n_i^{(m+1)}\}$. Continuing this process for all m, an element $x=(x_1, x_2, \ldots)\varepsilon\ell_\infty$ is constructed such that $\|T(x)\|=\|x\|$; for, let $\varepsilon>0$ be given and N be the integer as above. Then there is $n_i\in\{n_i^{(N)}\}$ such that $\sum_{m=1}^{N} |a_{n_i m}|>\sum_{m=1}^{\infty} |a_{n_i m}|-\varepsilon>1-2\varepsilon$ and thus $\sum_{m=1}^{N} a_{n_i m}x_m=\sum_{m=1}^{N} |a_{n_i m}|>1-2\varepsilon$. Therefore, $\|T(x)\|=\sup_n(|\sum_{m=1}^{\infty} a_{nm}x_m|)\geq 1-3\varepsilon=\|x\|-3\varepsilon$ for all $\varepsilon>0$ and hence $\|T(x)\|=\|x\|$.

https://doi.org/10.4153/CMB-1974-080-4 Published online by Cambridge University Press

We now show that no matrix map T will serve as an example to show that ITNC for ℓ_∞. Indeed, every such T which is contractive has T^2 a *contraction;* whence $T^n(x)\to 0$ for all $x\in\ell_\infty$.

THEOREM 4. *If $T=[a_{nm}]$ is a contractive selfmap of ℓ_∞, then T^2 must be a contraction.*

Proof. The proof is lengthy and detailed yet the idea behind the proof is simple. Partition the integers into disjoint sequences as follows:

Let j be any positive integer: the sequence $\{n_{ij}\}$ is defined by the following process. Let n_{1j} be the least integer l satisfying

$$1-\frac{1}{2^{j-1}}\leq\sum_{m=1}^{\infty}|a_{lm}|<1-\frac{1}{2^j}. \tag{1}$$

In general, let n_{ij} be the ith integer that is a solution l to (1).

Thus for each j we have a sequence (finite or infinite) $\{n_{ij}\}$. A technical lemma concerning these sequences must now be proven.

LEMMA 5. *There exists j_0 and $\varepsilon>0$ such that if $j>j_0$ then for all n_{ij},*

$$\sum_{m=1}^{j_0}|a_{n_{ij}m}|\geq\varepsilon.$$

Proof. Suppose the theorem is false. Then for each j_0 and $\varepsilon>0$ there is $j>j_0$ and n_{ij} such that $\sum_{m=1}^{\infty}|a_{n_{ij}m}|<\varepsilon$.

Let $\varepsilon=\frac{1}{8}$ and $j_0=3$. Then there is an $n_{ij}(j>3)$ such that $\sum_{m=1}^{3}|a_{n_{ij}m}|<\frac{1}{8}$. Let j_1 be an integer such that $\sum_{m=1}^{j_1}|a_{n_{ij}m}|>\frac{3}{4}$ and for $1\leq m\leq j_1$ define $x_m=\operatorname{sgn}(a_{n_{ij}m})$. In general, suppose $j_0, j_1,\ldots,j_n, x_1, x_2,\ldots,x_{j_n}$ have been defined; then choose $j^{n+1}>j_n$ such that for some $j>j_n$ and some n_{ij} we have

$$\sum_{m=1}^{j_n}|a_{n_{ij}m}|<\frac{1}{2^{j_n}}\quad\text{and}\quad\sum_{m=1}^{j_{n+1}}|a_{n_{ij}m}|>1-\frac{1}{2^{j_n}}.$$

Then define $x_m=\operatorname{sgn}(a_{n_{ij}m})$ for $j_n<m\leq j_{n+1}$. Thus an element $x=(x_1,x_2,\ldots)\in\ell_\infty$ is defined such that $\|x\|=1$. From the construction of T we must have $\|T(x)\|=\|x\|=1$, a contradiction, and the lemma must be true.

We return to the proof of the theorem. Let j_0 and $\varepsilon>0$ be the integer and number whose existence is guaranteed by the lemma and let $x=(x_1,x_2,\ldots)\in\ell_\infty$ such that $\|x\|=1$. Since T is contractive, $\sum_{m=1}^{\infty}|a_{nm}|<1$ for all n, and the first j_0 row sums must be bounded away from 1; that is, there is $0<d<1$ such that

$$\sum_{m=1}^{\infty}|a_{nm}|\leq d\quad\text{for}\quad n=1,2,\ldots,j_0$$

Hence letting $y=\{y_n\}=T(x)$ we have

$$\max\{|y_1|,|y_2|,\ldots,|y_{j_0}|\}\leq d<1$$

Thus

$$\left|\sum_{m=1}^{\infty} a_{nm}y_m\right| \le \sum_{m=1}^{j_0} |a_{nm}y_m| + \sum_{m=j_0+1}^{\infty} |a_{nm}|$$

$$\le d\sum_{m=1}^{j_0} |a_{nm}| + \sum_{m=j_0+1}^{\infty} |a_{nm}|$$

$$\le \max\left\{1-\frac{1}{2^{j_0}}, 1-(1-d)\varepsilon\right\} = \lambda < 1$$

So we have that if $x \in \ell_\infty$, then $\|T^2(x)\| \le \lambda \|x\|$ and T^2 is thus a contraction.

References

1. G. H. Hardy, *Divergent Series*, Oxford, Clarendon Press, 1956.
2. T. F. McCave and Jack Bryant, *A note on Edelstein's iterative test and spaces of continuous functions*, Pacific J. Math., (to appear).
3. S. B. Nadler, Jr., *A note on an iterative test of Edelstein*, Canad. Math. Bull., (to appear).

Texas A & M University,
College Station, Texas 77843
U.S.A.

Pan American University,
Edinburg, Texas 78539
U.S.A.

(1) *MR 42 Subject Classification:* Primary 5485, 4031.

Key Phrases: Iterative test conclusive, fixed point, contractive map, summability method.

https://doi.org/10.4153/CMB-1974-080-4 Published online by Cambridge University Press