# GENERALISED QUATERNION GROUPS AND DISTRIBUTIVELY GENERATED NEAR-RINGS

*by* J. J. MALONE

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In this paper two major questions concerning generalised quaternion groups and distributively generated (d.g.) near-rings are investigated. The d.g. near-rings generated, respectively, by the inner automorphisms, automorphisms, and endomorphisms of the group are described. It is also shown that these morphism near-rings are local near-rings and contain no non-trivial idempotents. Finally, it is demonstrated that exactly 16 d.g. near-rings can be defined on a given generalised quaternion group.

## 1. Properties of $Q_n$

The quaternion group of order $2^n$, $n \geqq 3$, will be designated by $Q_n$ and will be presented as $(a, b \mid a^{2^{n-1}}, bab^{-1}a, a^{2^{n-2}}b^2)$. Elements of $Q_n$ will be given in the form $a^x b^s$, $0 \leqq x \leqq 2^{n-1}-1$, $0 \leqq s \leqq 1$. Unless otherwise noted, it is assumed that $n > 3$.

**Lemma 1.** *The normal subgroups of $Q_n$ are the subgroups of $(a)$ and the normal subgroups generated by each of $b$ and $ab$. The latter two subgroups are each isomorphic to $Q_{n-1}$.*

**Proof.** Let $S(T)$ be the normal subgroup generated by $b(ab)$. Since an element of order 4 not in $(a)$ occurs in a set of $2^{n-2}$ conjugates (see p. 133 of (**1**)) it follows that $\{b, a^2b, a^4b, \ldots\} \subset S$ and $\{ab, a^3b, a^5b, \ldots\} \subset T$. Thus, $a^2 \in S$ and $a^2 \in T$ since

$$(a^2b)(a^{2^{n-2}}b) = a^2 = (a^3b)(a^{2^{n-2}+1}b).$$

From (**8**), pp. 191-192, it follows that each of $S$ and $T$ is isomorphic to $Q_{n-1}$ and also that the subgroups of $(a)$ are normal in $Q_n$.

**Theorem 2.** *$Q_n$ has $2^{n-1}$ inner automorphisms, $2^{2n-3}$ automorphisms, and $2^{2n-3}+4$ endomorphisms.*

**Proof.** The first statement follows since $Q_n$ has a centre of order 2 (see p. 192 of (**8**)), the second statement is given on page 133 of (**1**).

Since $Q_n$ modulo either $S$ or $T$ or $(a)$ is of order 2 and $Q_n$ contains a unique element of order 2, these normal subgroups each serve as the kernel for only one endomorphism. The subgroups of $(a)$, other than $(a)$ itself, lead to quotient groups which contain more than one element of order 2 and so cannot be the kernels of endomorphisms. The last endomorphism is the trivial one which has $Q_n$ as its kernel.