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Abstract

For the p-adic Galois representation associated to a Hilbert modular form, Carayol has shown that, under a certain assumption, its restriction to the local Galois group at a finite place not dividing p is compatible with the local Langlands correspondence. Under the same assumption, we show that the same is true for the places dividing p, in the sense of p-adic Hodge theory, as is shown for an elliptic modular form. We also prove that the monodromy-weight conjecture holds for such representations.

1. Introduction

We consider the p-adic Galois representation associated to a Hilbert modular form. Carayol has shown that, under assumption (C) of Theorem 2.1, its restriction to the local Galois group at a finite place not dividing p is compatible with the local Langlands correspondence, see [Car86b]. In this paper, under the same assumption (C), we show that the same is true for the places dividing p, in the sense of p-adic Hodge theory [Fon94], as is shown for an elliptic modular form in [Sai97] complemented in [Sai00]. We also prove that the monodromy-weight conjecture holds for such representations.

We prove the compatibility by comparing the p-adic and ℓ-adic representations, for it is already established for ℓ-adic representations [Car86b]. More precisely, we prove it by comparing the traces of Galois actions and proving the monodromy-weight conjecture. The first task is to construct the Galois representation in a purely geometric way in terms of étale cohomology of an analogue of the Kuga–Sato variety and algebraic correspondences acting on it. Then we apply the comparison theorem of p-adic Hodge theory [Tsu99] and the weight spectral sequences [Mok93, RZ82] to compute the traces and the monodromy operators in terms of the reduction modulo p. We obtain the required equality between traces by applying the Lefschetz trace formula which has the same form for ℓ-adic and for crystalline cohomologies. We deduce the monodromy-weight conjecture from the Weil conjecture and a certain vanishing of global sections. The last vanishing result is an analogue of the vanishing of the fixed part $(\mathrm{Sym}^{k-2}T_\ell E)^{SL_2(\mathbb{Z}_\ell)}$ for $k > 2$ for the universal elliptic curve E over a modular curve in positive characteristic.

We state the main compatibility result, Theorem 2.2, and the monodromy-weight conjecture, Theorem 2.4, in § 2 after briefly recalling the basic terminology on the ℓ-adic representation associated to a Hilbert modular form. We recall a cohomological construction of the ℓ-adic representation in § 3. After introducing Shimura curves in § 4 and recalling their modular interpretation in § 5, we give a geometric construction of the ℓ-adic representation in § 6. We extend the geometric construction to semi-stable models in § 7 and prove Theorems 2.2 and 2.4

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in § 8, admitting Proposition 8.3 on the vanishing. The last § 9 is devoted to the proof of Proposition 8.3.

The strategy of the proof is the same as in the previous work in [Sai97] complemented in [Sai00]. An essential part of the work consists of understanding the papers [Car86a, Car86b] of Carayol.

2. The ℓ-adic representation associated to a Hilbert modular form: main results

Let F be a totally real number field of degree $g > 1$ and $I = \{\sigma_1, \ldots, \sigma_g\}$ be the set of real embeddings $F \hookrightarrow \mathbb{R}$. We fix a multiweight

$$k = (k_1, \ldots, k_g, w) \in \mathbb{N}^{g+1} \tag{2.1}$$

satisfying the conditions $k_i \geqslant 2$ and $k_i \equiv w \bmod 2$.

We recall some terminology on the ℓ-adic representation associated to a Hilbert modular form. Let $\pi = \bigotimes_v \pi_v$ be a cuspidal automorphic representation of the adele group $GL_2(\mathbb{A}_F)$ such that, for the infinite places, the σ_i-component π_{σ_i} is a holomorphic discrete series representation D_{k_i}. The finite part $\pi^\infty = \bigotimes_{\mathfrak{p}\nmid\infty} \pi_\mathfrak{p}$ is an admissible representation of the finite adeles $GL_2(\mathbb{A}_F^\infty)$, where $\mathbb{A}_F^\infty = F \otimes_\mathbb{Z} \hat{\mathbb{Z}}$. Let $\mathfrak{n} \subset O_F$ be the level of π.

Let L be a sufficiently large number field of finite degree over $\mathbb{Q}$ such that π^∞ admits an L-structure π_L^∞. The fixed part $(\pi_L^\infty)^{K_1(\mathfrak{n})}$ is of dimension 1 and generated by an eigen newform f. Let $\pi_L^\infty = \bigotimes_\mathfrak{p} \pi_{\mathfrak{p},L}$ be the factorization into the tensor product of irreducible admissible representations $\pi_{\mathfrak{p},L}^\infty$ of $GL_2(F_\mathfrak{p})$ over L. To attach an L-rational representation of the Weil–Deligne group to the L-representation $\pi_{\mathfrak{p},L}$ of $GL_2(F_\mathfrak{p})$, we briefly recall the local Langlands correspondence.

To an irreducible admissible representation π of $GL_2(F_\mathfrak{p})$ defined over L, the local Langlands correspondence associates an L-rational F-semi-simple representation $\sigma(\pi)$ of the Weil–Deligne group $'W(\bar{F}_\mathfrak{p}/F_\mathfrak{p})$ of degree two. An F-semi-simple representation of the Weil–Deligne group is a pair of a semi-simple representations (ρ, V) of the Weil group $W(\bar{F}_\mathfrak{p}/F_\mathfrak{p})$ with open kernel and a nilpotent endomorphism N of V satisfying $\rho(\sigma)N\rho(\sigma)^{-1} = (q_\mathfrak{p})^{n(\sigma)}N$. Here $q_\mathfrak{p}$ is the norm of $\mathfrak{p}$ and $n : W(\bar{F}_\mathfrak{p}/F_\mathfrak{p}) \to \mathbb{Z}$ is the canonical surjection sending a geometric Frobenius in $W(\bar{F}_\mathfrak{p}/F_\mathfrak{p})$ to 1. A representation (ρ', N) of the Weil–Deligne group is called unramified if ρ' is unramified and $N = 0$. Among several ways to normalize the local Langlands correspondence, we consider the so-called Hecke correspondence: $\pi \mapsto \sigma_h(\pi)$ (see [Del73]).

We apply the construction $\pi \mapsto \sigma_h(\pi)$ to the local component $\pi_{\mathfrak{p},L}$ of a cuspidal automorphic representation and further take the dual representation $\check{\sigma}_h(\pi)$. Thus, we obtain an F-semi-simple L-rational representation $\check{\sigma}_h(\pi_\mathfrak{p})$ of the Weil–Deligne group $'W(\bar{F}_\mathfrak{p}/F_\mathfrak{p})$. For a finite place $\mathfrak{p} \nmid \mathfrak{n}$, the representation $\pi_\mathfrak{p}$ is an unramified principal series and hence the L-factor $L_\mathfrak{p}(\pi, T) \in L[T]$ is equal to the characteristic polynomial $\det(1 - Fr_\mathfrak{p}T : \check{\sigma}_h(\pi_\mathfrak{p}))$ of the geometric Frobenius $Fr_\mathfrak{p}$ and is of degree two.

Let λ be a finite place of L and $\rho : \mathrm{Gal}(\bar{F}/F) \to GL_2(L_\lambda)$ be a continuous representation of degree two. We say that ρ is attached to π if, at almost all finite place $\mathfrak{p} \nmid \mathfrak{n}$, the representation ρ is unramified and we have an equality

$$\det(1 - \rho(Fr_\mathfrak{p})T) = L_\mathfrak{p}(\pi, T). \tag{2.2}$$

The finite subset of places to be omitted actually consists of those dividing the product $\mathfrak{n}\ell$ of the level $\mathfrak{n}$ of f and the prime ℓ below λ.

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The existence is established by an accumulation of works of many people: [BR93, Car86b, Oht82, RT83, Tay89]. Since it is known to be irreducible by [Tay95, Proposition 3.1], Chebotarev density implies the uniqueness. In the following, we recall a theorem of Carayol (see [Car86b]) which asserts not only the existence but also gives a precise description of the restriction to the decomposition group $\mathrm{Gal}(\bar{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$ at finite places $\mathfrak{p} \nmid \ell$ including those dividing the level $\mathfrak{n}$.

To an ℓ-adic representation of the local Galois group $G_{F_{\mathfrak{p}}} = \mathrm{Gal}(\bar{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$, we attach a representation of the Weil–Deligne group $'W(\bar{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$. First we consider the case where $\mathfrak{p} \nmid \ell$. Let L_λ be a finite extension of $\mathbb{Q}_\ell$. Let $\rho : G_{F_{\mathfrak{p}}} \to GL_{L_\lambda}(V)$ be a continuous ℓ-adic representation. Take a lifting $\tilde{F} \in W(\bar{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$ of the geometric Frobenius and an isomorphism $\mathbb{Z}_\ell(1) \to \mathbb{Z}_\ell$ and identify them. Let $t_\ell : I_{\mathfrak{p}} \to \mathbb{Z}_\ell(1) \to \mathbb{Z}_\ell$ be the canonical surjection. Then, by the monodromy theorem of Grothendieck, there is a representation $'\rho = (\rho', N)$ of the Weil–Deligne group $'W(\bar{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$ characterized by the condition

$$\rho(\tilde{F}^n \sigma) = \rho'(\tilde{F}^n \sigma) \exp(t_\ell(\sigma) N)$$

for $n \in \mathbb{Z}$ and $\sigma \in I_{\mathfrak{p}}$. The isomorphism class of the representation (ρ', N) of the Weil–Deligne group is independent of the choice of the lifting $\tilde{F}$ or the isomorphism $\mathbb{Z}_\ell(1) \to \mathbb{Z}_\ell$ and is determined by ρ.

For an ℓ-adic representation ρ of $\mathrm{Gal}(\bar{F}/F)$, let $\rho_{\mathfrak{p}}$ denote the restriction to $\mathrm{Gal}(\bar{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$. Let $'\rho_{\mathfrak{p}}$ denote the representation of the Weil–Deligne group attached to $\rho_{\mathfrak{p}}$ and let $'\rho_{\mathfrak{p}}^{F\text{-ss}}$ denote its F-semi-simplification.

THEOREM 2.1 [Car86b]. *Let f be an eigen newform of multiweight k and $\lambda|\ell$ be a finite place of the number field L. We assume the following condition is satisfied.*

(C) *If the degree $g = [F : \mathbb{Q}]$ is even, there exists a finite place $\mathfrak{v}$ such that the $\mathfrak{v}$-factor $\pi_{f,\mathfrak{v}}$ lies in the discrete series.*

Then there exists an ℓ-adic representation

$$\rho = \rho_{f,\lambda} : \mathrm{Gal}(\bar{F}/F) \longrightarrow GL_{L_\lambda}(V_{f,\lambda}) \tag{2.3}$$

satisfying the following property.

For a finite place $\mathfrak{p} \nmid \ell$, there is an isomorphism

$$'\rho_{f,\lambda,\mathfrak{p}}^{F\text{-ss}} \simeq \check{\sigma}_h(\pi_{f,\mathfrak{p}}) \tag{2.4}$$

of representations of the Weil–Deligne group $'W(\bar{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$.

Remark. Since the right-hand side is L-rational, Theorem 2.1 implies that so is the left-hand side. For $\mathfrak{p} \nmid \mathfrak{n}\ell$, the isomorphism means that we have an equality

$$\det(1 - Fr_{\mathfrak{p}} T : V_{f,\lambda}) = \det(1 - Fr_{\mathfrak{p}} T : \check{\sigma}_h(\pi)) = L_{\mathfrak{p}}(f, T). \tag{2.5}$$

Hence $V_{f,\lambda}$ in Theorem 2.1 is the ℓ-adic representation associated to f.

In this paper, we study the case where $\mathfrak{p}$ divides ℓ. Let p be the characteristic of a finite place $\mathfrak{p}$ of F. Let $F_{\mathfrak{p},0}$ denote the maximal absolutely unramified subfield in $F_{\mathfrak{p}}$.

We describe the construction attaching a representation of the Weil–Deligne group to a p-adic representation of the local Galois group due to Fontaine [Fon94]. Let B_{st} be the ring defined by Fontaine. It is an $\widehat{F_{\mathfrak{p},0}^{\mathrm{nr}}}$-algebra and it admits a natural action of the absolute Galois group $G_{F_{\mathfrak{p}}}$, a semi-linear action of the Frobenius φ and an action of the monodromy operator N. For an open subgroup $J \subset I$ of the inertia, the fixed part B_{st}^J is the completion $\widehat{F_{\mathfrak{p},0}^{\mathrm{nr}}}$ of a maximal

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unramified extension of $F_{\mathfrak{p},0}$. In this paper, we neglect the filtration. Let L_μ be a finite extension of $\mathbb{Q}_p$ and consider a continuous p-adic representation $\mathrm{Gal}(\bar{F}_\mathfrak{p}/F_\mathfrak{p}) \to GL_{L_\mu}(V)$ of finite degree. Let $\widehat{L_\mu^{\mathrm{nr}}}$ denote the completion of the maximum unramified extension of L_μ. We choose an arbitrary factor of $\widehat{F_{\mathfrak{p},0}^{\mathrm{nr}}} \otimes_{\mathbb{Q}_p} L_\mu$. This is the same thing as fixing an embedding $\widehat{F_{\mathfrak{p},0}^{\mathrm{nr}}} \to \widehat{L_\mu^{\mathrm{nr}}}$. For an L_μ-representation $G_{F_\mathfrak{p}} \to GL_{L_\mu}(V)$ of finite degree, we put

$$D(V) = D_{\mathrm{pst}}(V) = \bigcup_{J \subset I} (B_{\mathrm{st}} \otimes V)^J \otimes_{(\widehat{F_{\mathfrak{p},0}^{\mathrm{nr}}} \otimes_{\mathbb{Q}_p} L_\mu)} \widehat{L_\mu^{\mathrm{nr}}}. \tag{2.6}$$

Here J runs through the open subgroups of the inertia subgroup $I = I_\mathfrak{p}$ and $-^J$ denotes the J-fixed part. The union $\bigcup_{J \subset I}(B_{\mathrm{st}} \otimes V)^J$ is an $\widehat{F_{\mathfrak{p},0}^{\mathrm{nr}}} \otimes_{\mathbb{Q}_p} L_\mu$-module since $B_{\mathrm{st}}^J = \widehat{F_{\mathfrak{p},0}^{\mathrm{nr}}}$. It is known that $D(V)$ is an $\widehat{L_\mu^{\mathrm{nr}}}$-vector space of finite dimension and $\dim_{\widehat{L_\mu^{\mathrm{nr}}}} D(V) \leqslant \dim_{L_\mu} V$. We say V is potentially semi-stable (pst for short) if we have the equality $\dim_{\widehat{L_\mu^{\mathrm{nr}}}} D(V) = \dim_{L_\mu} V$.

For a pst-representation V, Fontaine defines a natural representation on $D(V)$ of the Weil–Deligne group $'W(\bar{F}_\mathfrak{p}/F_\mathfrak{p})$ as follows (see [Fon94]). By the Galois actions on B_{st} and on V, the quotient $G_{F_\mathfrak{p}}/J$ acts on the J-fixed part $(B_{\mathrm{st}} \otimes V)^J$ for a normal subgroup $J \subset G_{F_\mathfrak{p}}$. Passing to the limit, we obtain an action of $\mathrm{Gal}(\bar{F}/F)$ on the $\widehat{F_{\mathfrak{p},0}^{\mathrm{nr}}} \otimes_{\mathbb{Q}_p} L_\mu$-module $\bigcup_{J \subset I}(B_{\mathrm{st}} \otimes V)^J$. The kernel is open in the inertia $I_\mathfrak{p}$. This Galois action is semi-linear with respect to its natural action on $\widehat{F_{\mathfrak{p},0}^{\mathrm{nr}}}$ and the trivial action on L_μ. We modify it by using the Frobenius φ to get a $\widehat{F_{\mathfrak{p},0}^{\mathrm{nr}}} \otimes_{\mathbb{Q}_p} L_\mu$-linear action of the Weil group $W(\bar{F}_\mathfrak{p}/F_\mathfrak{p})$ as follows.

Let $\mathbb{F}_\mathfrak{p}$ denote the residue field of $\mathfrak{p}$. Recall that the Weil group $W(\bar{F}_\mathfrak{p}/F_\mathfrak{p})$ is the inverse image of the inclusion $\mathbb{Z} \to \mathrm{Gal}(\overline{\mathbb{F}_\mathfrak{p}}/\mathbb{F}_\mathfrak{p})$ sending 1 to the geometric Frobenius $Fr_\mathfrak{p}$ by the canonical map $\mathrm{Gal}(\bar{F}_\mathfrak{p}/F_\mathfrak{p}) \to \mathrm{Gal}(\overline{\mathbb{F}_\mathfrak{p}}/\mathbb{F}_\mathfrak{p})$. Let $n : W(\bar{F}_\mathfrak{p}/F_\mathfrak{p}) \to \mathbb{Z}$ be the canonical map and $q_\mathfrak{p} = p^f$ be the norm of $\mathfrak{p}$. Then by letting $\sigma \in W(\bar{F}_\mathfrak{p}/F_\mathfrak{p})$ act on $D(V)$ by $(\varphi^{f \cdot n(\sigma)} \otimes 1) \circ \sigma \otimes \sigma$, we get a $\widehat{F_{\mathfrak{p},0}^{\mathrm{nr}}} \otimes_{\mathbb{Q}_p} L_\mu$-linear action. Taking the $\widehat{L_\mu^{\mathrm{nr}}}$-component, we obtain an $\widehat{L_\mu^{\mathrm{nr}}}$-linear representation $D(V)$ of the Weil group $W(\bar{F}_\mathfrak{p}/F_\mathfrak{p})$. The monodromy operator N on B_{st} induces an $\widehat{L_\mu^{\mathrm{nr}}}$-linear nilpotent operator on $D(V)$ satisfying $\sigma N = (q_\mathfrak{p})^{n(\sigma)} N \sigma$ since $\varphi N = pN\varphi$. Thus an $\widehat{L_\mu^{\mathrm{nr}}}$-linear action $'\rho_{\mu,\pi,v}$ of the Weil–Deligne group on $D(V)$ is defined.

We apply the construction $V \mapsto D(V)$ (2.6) to the restriction $\rho_{f,\mu,\mathfrak{p}}$ of the p-adic representation associated to π_f, to the decomposition group $\mathrm{Gal}(\bar{F}_\mathfrak{p}/F_\mathfrak{p})$ for a place $\mathfrak{p}|p$. Thus we obtain an $\widehat{L_\mu^{\mathrm{nr}}}$-representation $'\rho_{f,\mu,\mathfrak{p}}$ of the Weil–Deligne group $'W(\bar{F}_\mathfrak{p}/F_\mathfrak{p})$. Our main result is the following.

Theorem 2.2. *Let the assumptions including* (C) *be the same as in Theorem 2.1 and let μ be a place of L dividing the characteristic of a prime $\mathfrak{p}$ of F. Then, the representation $\rho_{f,\mu,\mathfrak{p}}$ of* $\mathrm{Gal}(\bar{F}_\mathfrak{p}/F_\mathfrak{p})$ *is potentially semi-stable and there is an isomorphism*

$$'\rho_{f,\mu,\mathfrak{p}}^{F\text{-ss}} \simeq \check{\sigma}_h(\pi_{f,\mathfrak{p}}) \tag{2.7}$$

of representations of the Weil–Deligne group $'W(\bar{F}_\mathfrak{p}/F_\mathfrak{p})$.

Remark. By the semi-stability of $\rho_{f,\mu,\mathfrak{p}}$, the representation $'\rho_{f,\mu,\mathfrak{p}}$ is of degree two. Similarly as in the ℓ-adic case, Theorem 2.2 implies that the left-hand side $'\rho_{f,\mu,\mathfrak{p}}^{F\text{-ss}}$ is L-rational.

Since the D_{pst}-functor of Fontaine does not preserve the integral structure, the author does not know how to remove the assumption (C) in Theorem 2.2 and in Theorem 2.4 below, by trying to apply for example the congruence argument as in [Tay89]. A partial result in this direction was recently obtained by Kisin (see [Kis08]). More recently, Liu has announced a proof in the general case [Liu09].

By the argument using a quadratic base change as in [Car86b], we may assume that the finite place $\mathfrak{v}$ in the condition (C) is different from $\mathfrak{p}$ in the case where $g=[F:\mathbb{Q}]$ is even.

We will prove Theorem 2.2 by comparing p-adic cohomology with ℓ-adic cohomology. Let λ be a place of L dividing a prime $\ell \neq p$. By Theorem 2.1 applied to $\rho_{f,\lambda,\mathfrak{p}}$, it is enough to compare $'\rho_{f,\lambda,\mathfrak{p}}$ with $'\rho_{f,\mu,\mathfrak{p}}$. More precisely, we prove the following.

CLAIM 2.3. Let the notation be as in Theorem 2.2. Let $\mathfrak{p}|p$ be a finite place of F and let $\lambda|\ell \neq p$ and $\mu|p$ be places of L. Then the following hold.

(i) The representation $\rho_{f,\mu,\mathfrak{p}}$ is potentially semi-stable.

(ii) For $\sigma \in W^+=\{\sigma \in W(\bar{F}_\mathfrak{p}/F_\mathfrak{p}) \mid n(\sigma) \geqslant 0\}$, we have an equality in some finite extension of L,

$$\mathrm{Tr}\ {}'\rho_{f,\lambda,\mathfrak{p}}(\sigma)=\mathrm{Tr}\ {}'\rho_{f,\mu,\mathfrak{p}}(\sigma). \tag{2.8}$$

(iii) Let N_λ and N_μ be the nilpotent monodromy operators for $'\rho_{\lambda,\pi,\mathfrak{p}}$ and $'\rho_{\mu,\pi,\mathfrak{p}}$ respectively. Then $N_\lambda=0$ if and only if $N_\mu=0$.

By [Sai97, Lemma 1], Theorem 2.2 follows from Claim 2.3. In assertion (ii), we may allow a finite extension since we already know that the left-hand side is in L.

The assertion (i) is a special case of assertion (ii) where $\sigma=1$. We deduce the assertion (iii) from assertion (ii) together with the monodromy-weight conjecture, Theorem 2.4 below, asserting that the monodromy filtration coincides with the weight filtration up to a shift.

Let V be a representation of the Weil–Deligne group $'W_\mathfrak{p}$. We assume $N^2=0$. Then $0 \subset W_{-1}V=\mathrm{Image}\ N \subset W_0V=\mathrm{Ker}\ N \subset W_1V=V$ is a filtration by subrepresentations of V. It is called the monodromy filtration. We put $\mathrm{Gr}_1^W(V)=V/\mathrm{Ker}\ N$, $\mathrm{Gr}_0^W(V)=\mathrm{Ker}\ N/\mathrm{Image}\ N$ and $\mathrm{Gr}_{-1}^W(V)=\mathrm{Image}\ N$. Then each graded piece is a representation of the Weil group. The monodromy operator N induces an isomorphism $\mathrm{Gr}_1^W(V)(1)\to \mathrm{Gr}_{-1}^W(V)$. For a lifting $\widetilde{F}$ of the geometric Frobenius Fr, the eigenvalues, up to roots of unity, are independent of the choice of the lifting. We say an algebraic number is pure of weight n if the complex absolute value of its conjugates are $(q_\mathfrak{p})^{n/2}$ where $q_\mathfrak{p}$ denotes the norm of $\mathfrak{p}$. Then, for an integer $n\in\mathbb{Z}$, we say that the monodromy filtration of V is pure of weight n, if the eigenvalues of a lifting $\widetilde{F}$ of Fr acting on Gr_i^W for each i are algebraic numbers of weight $n+i$.

THEOREM 2.4. *Let the notation be as in Claim 2.3. Then the monodromy filtration of the representations $'\rho_{f,\lambda,\mathfrak{p}}$ and $'\rho_{f,\mu,\mathfrak{p}}$ of the Weil–Deligne group are pure of weight $w-1$. In other words, the eigenvalues α of $'\rho_{f,\lambda,\mathfrak{p}}(\widetilde{F})$ and of $'\rho_{f,\mu,\mathfrak{p}}(\widetilde{F})$ for an arbitrary lifting $\widetilde{F}\in W(\bar{F}_\mathfrak{p}/F_\mathfrak{p})$ of the geometric Frobenius is of weight n, where*

$$n=\begin{cases} w-1 & \text{if } N=0,\\ w-2 & \text{if } N\neq 0 \text{ and } \alpha \text{ is the eigenvalue on } \mathrm{Ker}\ N,\\ w & \text{if } N\neq 0 \text{ and } \alpha \text{ is the eigenvalue on } \mathrm{Coker}\ N.\end{cases} \tag{2.9}$$

Remark. The assertion for the case $N\neq 0$ is easy since we know the determinant and $N:\mathrm{Gr}_1^W(V)(1)\to \mathrm{Gr}_{-1}^W(V)$ is an isomorphism.

We show that Theorem 2.4 and the assertion Claim 2.3(ii) imply Claim 2.3(iii). In fact, by assertion (ii), the eigenvalues of a lifting $\widetilde{F}$ of Frobenius are the same for λ and μ. By Theorem 2.4, we distinguish the two cases $N = 0$ and $N \neq 0$ by their absolute values. Thus assertion (iii) follows from assertion (ii) and Theorem 2.4. Thus Theorem 2.2 is reduced to the assertion Claim 2.3(ii) and Theorem 2.4.

3. Cohomological construction of the ℓ-adic representation

Carayol constructs an ℓ-adic representation associated to a Hilbert modular form by decomposing the étale cohomology $H^1(M_{K,\bar{F}}, \mathcal{F}_\lambda)$ of a Shimura curve with coefficient sheaf $\mathcal{F}_\lambda$. Here, we briefly recall the construction with a slight modification. Using that construction, we state Claim 3.2, which implies the main results.

First, we recall the definition of the Shimura curve. We fix a real place τ_1 of the totally real field F and regard F as a subfield of $\mathbb{R} \subset \mathbb{C}$ by τ_1. If the degree $g = [F : \mathbb{Q}]$ is even, we also fix a finite place $\mathfrak{v}$. Let B be a quaternion algebra over F ramifying exactly at the other real places $\{\tau_2, \ldots, \tau_g\}$ if $g = [F : \mathbb{Q}]$ is odd and at $\{\tau_2, \ldots, \tau_g, \mathfrak{v}\}$ if g is even.

Let $G = \mathrm{Res}_{F/\mathbb{Q}} B^\times$ denote the Weil restriction to $\mathbb{Q}$ of the algebraic group $B^\times$ over F. Here and in the following, we identify algebraic groups over $\mathbb{Q}$ and their $\mathbb{Q}$-valued points. Let X be the $G(\mathbb{R})$-conjugacy class of the map

$$\begin{array}{rccl} h: & \mathbb{C}^\times & \to & G(\mathbb{R}) = (B \otimes_{\mathbb{Q}} \mathbb{R})^\times \simeq \quad GL_2(\mathbb{R}) \times \mathbb{H}^\times \cdots \times \mathbb{H}^\times, \\ & a + b\sqrt{-1} & \mapsto & \left(\begin{pmatrix} a & b \\ -b & a \end{pmatrix}^{-1}, 1, \ldots, 1 \right). \end{array} \tag{3.1}$$

The conjugacy class X is naturally identified with the union $\mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$ of the upper and lower half planes. Let $M = M(G, X) = (M_K)_K$ be the canonical model of the Shimura variety defined for G and X. Here and in what follows, we call a projective system of varieties simply a variety, using a standard abuse of terminology. The Shimura variety $M = (M_K)_K$ is defined over the reflex field F. Here K runs through the open compact subgroups of $G(\mathbb{A}^\infty) = (B \otimes_F \mathbb{A}_F^\infty)^\times$. Each M_K is a proper and smooth, but not necessarily geometrically connected, curve over F. Since the reciprocity map $F^\times \to G^{\mathrm{ab}} = F^\times$ is the identity, the constant field F_K of M_K is the abelian extension of F corresponding to the compact open subgroup $\mathrm{Nrd}_{B/F} K \subset \mathbb{A}_F^{\infty\,\times}$. The projective system $(M_K)_K$ has a natural right action of the finite adeles $G(\mathbb{A}^\infty)$. For $g \in G(\mathbb{A}^\infty)$ and open compact subgroups $K, K' \subset G(\mathbb{A}^\infty)$ such that $g^{-1}Kg \subset K'$, we have $g : M_K \to M_{K'}$. The set of $\mathbb{C}$-valued points $M_K(\mathbb{C})$ are identified with the set of double cosets $G(\mathbb{Q}) \backslash X \times G(\mathbb{A}^\infty)/K$. The action of $G(\mathbb{Q}) = B^\times$ on X is induced by $B^\times \to (B \otimes_{F,\tau_1} \mathbb{R})^\times \simeq GL_2(\mathbb{R})$. For g, K, K' as above, the map $g : M_K(\mathbb{C}) \to M_{K'}(\mathbb{C})$ is induced by $(x, g_1) \mapsto (x, g_1 g)$.

We will define a smooth L_λ-sheaf $\mathcal{F}_\lambda^{(k)}$ on the Shimura curve M. It is the dual of the sheaf denoted $\mathcal{F}_\lambda$ in [Car86b]. We prefer the dual because it is related directly to a direct summand of a cohomology sheaf as we will see in later sections. Let $k = (k_1, \ldots, k_g, w) \in \mathbb{N}^{g+1}$ be a multiweight as in (2.1) and put $n = n(k) = \prod_i (k_i - 1)$. The algebraic group denoted G^c in [Mil90, ch. III] for our group $G = B^\times$ is the quotient of G by $\mathrm{Ker}(N_{F/\mathbb{Q}} : F^\times \to \mathbb{Q}^\times)$. Here we identify algebraic groups over $\mathbb{Q}$ and their $\mathbb{Q}$-valued points, and $F^\times \subset B^\times$ denotes the center of G.

In order to define the sheaf $\mathcal{F}_\lambda^{(k)}$, we take a number field $L \subset \mathbb{C}$ splitting F and B and we fix an isomorphism $B \otimes_{\mathbb{Q}} L \simeq M_2(L)^I$. We identify $\{\tau_i : F \to L\} = \{\tau_i : F \to \mathbb{C}\}$ by the inclusion $L \to \mathbb{C}$. We define a representation $\rho = \rho^{(k)} : G \to GL_n$ defined over L. We have $B \otimes_{\mathbb{Q}} \mathbb{C} \simeq M_2(\mathbb{C})^I$ where $I = \{\tau_1, \ldots, \tau_g\}$ is the set of embeddings $F \to \mathbb{C}$. It induces an isomorphism $G_{\mathbb{C}} \xrightarrow{\simeq} GL_{2,\mathbb{C}}^I$. We define the morphism $\rho = \rho^{(k)} : G \to GL_n$ to be the composite of this isomorphism with the tensor

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product $\bigotimes_{i\in I}((\mathrm{Sym}^{k_i-2}\otimes \det{}^{(w-k_i)/2})\circ \check{pr}_i)$. Here $\check{pr}_i$ denotes the contragradient representation of the ith projection $pr_i : GL_{2,\mathbb{C}}^I \to GL_{2,\mathbb{C}}$. Since the restriction to the center $F^\times$ is the multiplication by $N_{F/\mathbb{Q}}^{-(w-2)}$, it factors through the quotient $\rho^{(k)} : G^c \to GL_n$. The representation $\rho^{(k)} : G \to GL_n$ is defined over L.

We define the smooth L_λ-sheaf $\mathcal{F}_\lambda^{(k)}$ on M to be the L_λ-component of the smooth $L\otimes\mathbb{Q}_\ell$-sheaf $V_\ell(\rho^{(k)})$ attached to the representation $\rho^{(k)}$ [Mil90, ch. III, § 7]. We consider the inductive limit

$$H^1(M_{\bar F},\mathcal{F}_\lambda^{(k)}) = \varinjlim_K H^1(M_{K,\bar F},\mathcal{F}_{K,\lambda}^{(k)}). \tag{3.2}$$

By the natural action of $G(\mathbb{A}^\infty)$ on the projective system $(M_K,\mathcal{F}_{K,\lambda}^{(k)})_K$, it is a representation of $G(\mathbb{A}^\infty)\times \mathrm{Gal}(\bar F/F)$. The structure as a birepresentation is described as follows.

LEMMA 3.1. *Let k be a multiweight as in (2.1) and let $L\subset\mathbb{C}$ be a number field splitting F and B. If the degree $g=[F:\mathbb{Q}]$ is even, let $\mathfrak{v}$ be a finite place of F.*

Then, we have the following.

(i) *Let π be a cuspidal automorphic representation of $GL_2(\mathbb{A}_F)$ of multiweight k such that the finite part π^∞ is defined over a number field L. Assume, if the degree $g=[F:\mathbb{Q}]$ is even, that the $\mathfrak{v}$-factor $\pi_{f,\mathfrak{v}}$ lies in the discrete series. Then the finite part π'^∞ of the representation π' of $G(\mathbb{A})$ corresponding to π by the Jacquet–Langlands correspondence has an L-structure π'^∞_L.*

(ii) *There exists an isomorphism*

$$H^1(M_{\bar F},\mathcal{F}_\lambda^{(k)}) \simeq \bigoplus_{f'}\left(\pi'^\infty_{f',L(f')}\otimes_{L(f')}\bigoplus_{\lambda'|\lambda} V_{\lambda',f'}\right) \tag{3.3}$$

of representations of $G(\mathbb{A}^\infty)\times\mathrm{Gal}(\bar F/F)$ over L_λ. Here f' runs through the conjugacy classes over L, up to scalar multiplication, of eigen newforms of multiweight k, such that, if $g=[F:\mathbb{Q}]$ is even, the $\mathfrak{v}$-component $\pi_{f',\mathfrak{v}}$ lies in the discrete series. The extension of L generated by the Hecke eigenvalues acting on f' is denoted by $L(f')$ and λ' runs finite places of $L(f')$ above λ.

Although the proof of Lemma 3.1 is well-known to specialists, we include it here for the sake of completeness.

First we define an admissible representation S_L of $G(\mathbb{A}^\infty)$ over L. We define the automorphic vector bundle (see [Mil90, ch. III]) $\mathcal{V}(\mathcal{J})$ associated to a G^c-equivariant vector bundle $\mathcal{J}=\mathcal{J}^{(k)}$ on the compact dual $\check X$ and its canonical model $\mathcal{V}(\mathcal{J})_L$. Then S_L is defined as the limit of the spaces of global sections

$$S_L=\Gamma(M\otimes_F L,\Omega^1_M\otimes\mathcal{V}(\mathcal{J})_L)=\varinjlim_K \Gamma(M_K\otimes_F L,\Omega^1_M\otimes\mathcal{V}(\mathcal{J})_L). \tag{3.4}$$

We use the notation of [Mil90, ch. III] The compact dual $\check X$ is $\mathbb{P}^1_{\mathbb{C}}$ in our case. We define a G^c-equivariant vector bundle $\mathcal{J}=\mathcal{J}^{(k)}$ on $\check X$ in the following way. Let ω be the dual of the tautological quotient bundle on $\check X=\mathbb{P}^1_{\mathbb{C}}$. We put $\mathcal{J}^{(k)}=\omega^{\otimes k_1-2}\otimes\bigotimes_{i=2}^g \mathrm{Sym}^{k_i-2}(\mathbb{C}^{\oplus 2})$. We define the action of $G_{\mathbb{C}}={GL_{2,\mathbb{C}}}^I$ on $\mathcal{J}^{(k)}$ by giving the action of each factor in the following way. The first factor $GL_{2,\mathbb{C}}$ acts on $\check X$ in the natural way. On $\omega^{\otimes k_1-2}$, we consider $\det^{-(w-k_1)/2}$ times the natural action. For $i\neq 1$, the ith factor $GL_{2,\mathbb{C}}$ acts on $\check X$ trivially. On $\mathrm{Sym}^{k_i-2}(\mathbb{C}^{\oplus 2})$, we consider $\det^{-((w-k_i)/2)}$ times the action induced by the contragradient action of GL_2. By taking the tensor product, we obtain a $G_{\mathbb{C}}$-equivariant bundle $\mathcal{J}=\mathcal{J}^{(k)}$. Since the center $\mathbb{G}_m^I$ acts by the inverse of $(w-2)$nd power of the product character, it defines a $G_{\mathbb{C}}^c$-equivariant bundle. It is clearly defined over the number field $L\supset F$. Hence by [Mil90, Theorem 5.1(a), ch. III], we obtain a $G(\mathbb{A}^\infty)$-equivariant vector bundle $\mathcal{V}(\mathcal{J})_L$ on M_L. Thus the representation $S_L=\Gamma(M\otimes L,\Omega^1_M\otimes\mathcal{V}(\mathcal{J})_L)$ is defined.

By the Jacquet–Langlands correspondence (see [JL70]), we have an isomorphism

$$S_L\otimes_L\mathbb{C}\simeq\bigoplus_{f'}\pi'_{f'} \tag{3.5}$$

as a representation of $G(\mathbb{A}^\infty)$ over $\mathbb{C}$ where f' runs cuspidal automorphic representation of $GL_2(\mathbb{A}_F^\infty)$ of multiweight k such that, if $[F:\mathbb{Q}]$ is even, the $\mathfrak{v}$-component $\pi_{f',\mathfrak{v}}$ is in the discrete series.

Attached to the representation $\rho^{(k)}:G^c\to GL_n$ defined over L, we have a local system $\mathcal{F}^{(k)}=V(\rho^{(k)})$ of L-vector spaces on $M(\mathbb{C})$. We construct a Hodge decomposition, which is a generalization of the Eichler–Shimura isomorphism. Let $\mathcal{F}^{(k)}_{\mathbb{C}}=\mathcal{F}^{(k)}\otimes_L\mathbb{C}$. We regard it as a local system of $\mathbb{R}$-vector spaces endowed with a ring homomorphism $\mathbb{C}\to\operatorname{End}(\mathcal{F}^{(k)}_{\mathbb{C}})$. We consider the filtration on $\mathcal{F}^{(k)}_{\mathbb{C}}\otimes_{\mathbb{R}}\mathcal{O}_{M(\mathbb{C})}$ defined by $\rho\circ h_x$. It defines on $\mathcal{F}^{(k)}_{\mathbb{C}}$ a structure of variation of polarizable $\mathbb{R}$-Hodge structures of weight $w-2$.

We put $\mathcal{F}^{(k)}_{\mathbb{C}}\otimes_{\mathbb{C}}\mathcal{O}_{M(\mathbb{C})}=\mathcal{V}^{(k)}(=\mathcal{V}(\rho^{(k)}))$ and let $\sigma:M(\mathbb{C})\to M(\mathbb{C})$ denote the complex conjugate. We identify $\mathcal{F}^{(k)}_{\mathbb{C}}\otimes_{\mathbb{R}}\mathcal{O}_{M(\mathbb{C})}=\mathcal{V}^{(k)}\oplus\sigma^*\mathcal{V}^{(k)}$. The Hodge filtration $F^{w-2}(\mathcal{V}^{(k)}\oplus\sigma^*\mathcal{V}^{(k)})$ is given by $\mathcal{V}(\mathcal{J}^{(k)})\oplus\sigma^*\mathcal{V}(\mathcal{J}^{(k)})$. Hence the Hodge decomposition gives a $G(\mathbb{A}^\infty)$-equivariant isomorphism

$$\begin{aligned}H^1(M(\mathbb{C}),\mathcal{F}^{(k)}_{\mathbb{C}})&\simeq H^1(M(\mathbb{C}),\Omega^\bullet_M\otimes\mathcal{V}^{(k)})\\&\simeq H^0(M(\mathbb{C}),\Omega^1_M\otimes\mathcal{V}(\mathcal{J}^{(k)}))\oplus\sigma^*H^0(M(\mathbb{C}),\Omega^1_M\otimes\mathcal{V}(\mathcal{J}^{(k)})).\end{aligned} \tag{3.6}$$

We have $S_{\mathbb{C}}=H^0(M(\mathbb{C}),\Omega^1_M\otimes\mathcal{V}(\mathcal{J}^{(k)}))$ by definition and its complex conjugate $\sigma^*H^0(M(\mathbb{C}),\Omega^1_M\otimes\mathcal{V}(\mathcal{J}^{(k)}))$ is identified with $H^0(M(\mathbb{C}),\Omega^1_M\otimes\mathcal{V}(\sigma^*\mathcal{J}^{(k)}))$. The $G(\mathbb{A}^\infty)$-equivariant bundle $\mathcal{J}^{(k)}$ on $\check{X}$ is isomorphic to its complex conjugate $\sigma^*\mathcal{J}^{(k)}$ since the GL_2-action on the tautological quotient bundle on $\mathbb{P}^1$ is defined over $\mathbb{R}$, and the standard representation $\mathbb{H}^\times\to GL_2$ defined over $\mathbb{C}$ is $GL_2(\mathbb{C})$-conjugate to its complex conjugate. Thus, we obtain an isomorphism

$$H^1(M_{\bar F},\mathcal{F}^{(k)})\otimes_L\mathbb{C}\simeq S_L^{\oplus 2}\otimes_L\mathbb{C} \tag{3.7}$$

as a representation of $G(\mathbb{A}^\infty)$ over $\mathbb{C}$.

Proof of Lemma 3.1. (i) We show $\pi'_{f'}$ is defined over $L(f')$. If $g=[F:\mathbb{Q}]$ is odd, we have $G(\mathbb{A}^\infty)\simeq GL_2(\mathbb{A}_F^\infty)$ and $\pi_{f'}=\pi'_{f'}$, and there is nothing to prove. We show the case where g is even. It is enough to show that each factor $\pi'_{f',\mathfrak{r}}$ of $\pi'_{f'}=\bigotimes_v\pi'_{f',\mathfrak{r}}$ is defined over $L(f')$. Let $\mathfrak{n}$ be the level of f' and $K_1(\mathfrak{n})=K_1(\mathfrak{n})_{\mathfrak{r}}\cdot K_1(\mathfrak{n})^{\mathfrak{r}}\subset GL_2(\mathbb{A}_F^\infty)$. Then the representation $\pi_{f',\mathfrak{r}}$ is given as the fixed subspace $\pi_{f',\mathfrak{r}}=\pi_{f'}^{K_1(\mathfrak{n})^{\mathfrak{r}}}$ and is defined over $L(f')$. For $\mathfrak{r}\neq\mathfrak{v}$, we have $\pi_{f',\mathfrak{r}}=\pi'_{f',\mathfrak{r}}$ and it is defined over $L(f')$. Finally we consider the case $\mathfrak{r}=\mathfrak{v}$. Then by the isomorphism (3.5), we see that the intertwining space $\operatorname{Hom}_{G(\mathbb{A}^{\infty\mathfrak{v}})}(\bigotimes_{\mathfrak{r}\neq\mathfrak{v}}\pi'_{f',\mathfrak{r},L(f')},S_L\otimes_L L(f'))$ is an $L(f')$-structure of $\pi'_{f',\mathfrak{v}}$.

(ii) In [Car86b], it is shown that for $\mathcal{F}_\lambda$, we have a direct sum decomposition of the form $H^1(M_{\bar F},\mathcal{F}_\lambda)\simeq\bigoplus_{\pi'}\pi'\otimes\check{\sigma}_h(\pi')$ over $\bar L_\lambda$. Since $\mathcal{F}^{(k)}_\lambda$ here is the dual of $\mathcal{F}_\lambda$ there and since $\check{\sigma}_h(\check{\pi}') \simeq \sigma_h(\pi')(-1)$, we have an isomorphism $H^1(M_{\bar{F}}, \mathcal{F}_\lambda^{(k)}) \simeq \bigoplus_{\pi'} \pi' \otimes \sigma_h(\pi')$ over $\bar{L}_\lambda$ by Poincaré duality. Since the $\mathrm{Gal}(\bar{F}/F)$-representation

$$V_{f',\lambda'} = \mathrm{Hom}_{G(\mathbb{A}^\infty),L(f')_{\lambda'}}(\pi'_{f',L(f')} \otimes_{L(f')} L(f')_{\lambda'}, H^1(M_{\bar{F}}, \mathcal{F}_\lambda^{(k)})) \tag{3.8}$$

gives an $L(f')_{\lambda'}$-structure, the assertion follows. □

Using Lemma 3.1, we reduce Theorems 2.2 and 2.4 to a statement below, Claim 3.2, on the cohomology. Let $K \subset G(\mathbb{A}^\infty)$ be a sufficiently small open compact subgroup. We take an integral ideal $\mathfrak{n}$ of O_F, divisible by $\mathfrak{p}$ and by $\mathfrak{v}$ if g is even. We assume K to be of the form $K = K_\mathfrak{n} K^\mathfrak{n}$. Here $K_\mathfrak{n} \subset \prod_{\mathfrak{r}|\mathfrak{n}} B_\mathfrak{r}^\times$ is an open compact subgroup and $K^\mathfrak{n} = \prod_{\mathfrak{r}\nmid\mathfrak{n}} GL_2(O_{F_\mathfrak{r}})$ for some isomorphism $\prod'_{\mathfrak{r}\nmid\mathfrak{n}} B_\mathfrak{r} \simeq \prod'_{\mathfrak{r}\nmid\mathfrak{n}} M_2(F_\mathfrak{r})$. Let $T^\mathfrak{n} = L[T_\mathfrak{r}; \mathfrak{r}\nmid\mathfrak{n}]$ be the free L-algebra generated by the Hecke operators $T_\mathfrak{r}$ for $\mathfrak{r}\nmid\mathfrak{n}$. We consider $H^1(M_{K,\bar{F}}, \mathcal{F}_\lambda^{(k)})$ as a $T^\mathfrak{n}$-module. In the following statement, the letter D denotes Fontaine's D_{pst}-functor (2.6).

CLAIM 3.2. Let $K \subset G(\mathbb{A}^\infty)$ be a sufficiently small open compact subgroup and let $\mathfrak{n} \subset O_F$ be an integral ideal. We assume $K = K_\mathfrak{n} K^\mathfrak{n}$ as above. Then the following hold.

(i) The representation $H^q(M_{K,\bar{F}}, \mathcal{F}_\mu^{(k)})$ of $G_{F_\mathfrak{p}}$ for $q = 0, 1, 2$ is potentially semi-stable.

(ii) For $\sigma \in W^+$ and $T \in T^\mathfrak{n}$, we have equalities in a finite extension of L,

$$\sum_{q=0}^{2} (-1)^q \,\mathrm{Tr}(\sigma \circ T | H^q(M_{K,\bar{F}}, \mathcal{F}_\lambda^{(k)})) = \sum_{q=0}^{2} (-1)^q \,\mathrm{Tr}(\sigma \circ T | D(H^q(M_{K,\bar{F}}, \mathcal{F}_\mu^{(k)}))). \tag{3.9}$$

(iii) For the representations $H^1(M_{K,\bar{F}}, \mathcal{F}_\lambda^{(k)})$ and $D(H^1(M_{K,\bar{F}}, \mathcal{F}_\mu^{(k)}))$ of the Weil–Deligne group ${}'W_{F_\mathfrak{p}}$, their monodromy filtrations are pure of weight $w-1$.

We prove that the assertions in Claim 3.2 imply the corresponding assertions (i) and (ii) in Claim 2.3 and Theorem 2.4. Let f be a normalized eigen new cuspform of multiweight k. By Lemma 3.1, the finite part $\pi_f'^\infty$ is defined over $L(f)$. Replacing L by $L(f)$ if necessary, we may assume $L = L(f)$. Let K be a sufficiently small open compact subgroup satisfying $\pi_f'^K \neq 0$ and Claim 3.2. The representations $V_{f,\lambda}$ and $V_{f,\mu}$ are direct summands of $H^1(M_{K,\bar{F}}, \mathcal{F}_\lambda^{(k)})$ and $H^1(M_{K,\bar{F}}, \mathcal{F}_\mu^{(k)})$ by Lemma 3.1 respectively. Hence the assertion Claim 3.2(i) implies the assertion Claim 2.3(i) and the assertion Claim 3.2(iii) implies Theorem 2.4.

We show that the equality (3.9) of the traces implies the equality (2.8). First we show that the equality (3.9) for the alternating sum implies the equality for each piece

$$\mathrm{Tr}(\sigma \circ T | H^q(M_{K,\bar{E}}, \mathcal{F}_\lambda^{(k)})) = \mathrm{Tr}(\sigma \circ T | D(H^q(M_{K,\bar{E}}, \mathcal{F}_\mu^{(k)}))) \tag{3.10}$$

for $q = 0, 1, 2$. In fact, it is sufficient to show the equality (3.10) for $q = 0, 2$.

We show that $H^0 = H^2 = 0$ if $k \neq (2, \ldots, 2, w)$. The fundamental group $\pi_1(M_{K,\bar{F}})$ of the geometric fiber is isomorphic to $\mathrm{Ker}(\mathrm{Nrd}_{B/F} : K \to \hat{O}_F^\times)$. Hence its Lie algebra generates $B^0 = \mathrm{Ker}(\mathrm{Trd}_{B/F} : B \to F)$ over F. The Lie algebra $B^0 \otimes_\mathbb{Q} L \simeq \mathfrak{sl}_2(L_\lambda)^g$ is also generated by the Lie algebra of $\pi_1(M_{K,\bar{F}})$ over L. It follows easily from this that the representation of $\pi_1(M_{K,\bar{F}})$ corresponding to the sheaf $\mathcal{F}_\lambda$, hence the sheaf itself, is irreducible. Hence its largest geometrically constant subsheaf and quotient sheaves are zero, unless $k = (2, \ldots, 2, w)$.

We assume $k = (2, \ldots, 2, w)$ and we show the equality (3.10) for $q = 0, 2$. Then the sheaf $\mathcal{F}_\lambda^{(k)}$ is defined by the character $N_{F/\mathbb{Q}}^{-(w-2)/2} \circ \mathrm{Nrd}_{B/F} : G \to \mathbb{G}_m$ and is isomorphic to the Tate twist $L_\lambda(-(w-2)/2)$. It is sufficient to show the assertion for H^0 since $H^2 \simeq H^0(-1)$. Let $F_K = \Gamma(M_K, \mathcal{O}_{M_K})$ be the constant field of M_K. Then there is an isomorphism

$$H^0(M_{\bar{F}}, \mathcal{F}_\lambda^{(k)}) \simeq \varprojlim_K H^0\left(F_K \otimes_F \bar{F}, L_\lambda\left(-\frac{w-2}{2}\right)\right) \tag{3.11}$$

of $\mathrm{Gal}(\bar{F}/F) \times G(\mathbb{A}^\infty)$-module. On the right-hand side, the Galois action is the natural one. The action of $G(\mathbb{A}^\infty)$ is defined by that induced by its action on $\varprojlim_K \mathrm{Spec}\, F_K$ multiplied by the character

$$G(\mathbb{A}^\infty) \xrightarrow{N_{F/\mathbb{Q}}^{(w-2)/2} \circ \mathrm{Nrd}_{B/F}} \mathbb{A}^{\infty\,\times}/\mathbb{Q}^{+\times} \xrightarrow{\sim} \hat{\mathbb{Z}}^\times \longrightarrow \mathbb{Z}_\ell^\times \subset L_\lambda^\times.$$

From this, we easily deduce the equality (3.10) for $q = 0$.

We deduce the equality (3.9) from the equality (3.10) for $q = 1$. By the strong multiplicity one theorem, the image of the Hecke algebra $T^{\mathfrak{n}}$ in $\mathrm{End}_L(S_L^K)$, where S_L^K denotes the K-fixed part, is $\prod_{f'} L(f')$ where f' runs the conjugacy class of eigen newforms f' as in Lemma 3.1 such that $\pi'^K_{f'} \neq 0$. Let $e \in T^{\mathfrak{n}}$ be an element whose image is the idempotent corresponding to $f' = f$. Then if we put $d = \dim \pi'^K_f$, we see that $e \cdot H^1(M_{K,\bar{F}}, \mathcal{F}_\lambda^{(k)})$ is isomorphic to the direct sum $\check{\sigma}_h(\pi)^{\oplus d}$ by Lemma 3.1(ii). Hence we have

$$d \cdot \mathrm{Tr}\ {}'\rho_{\lambda,f,\mathfrak{p}}(\sigma) = \mathrm{Tr}(\sigma \circ e | H^1(M_{K,\bar{F}}, \mathcal{F}_\lambda^{(k)})),$$
$$d \cdot \mathrm{Tr}\ {}'\rho_{\mu,f,\mathfrak{p}}(\sigma) = \mathrm{Tr}(\sigma \circ e | D(H^1(M_{K,\bar{F}}, \mathcal{F}_\mu^{(k)}))).$$

Thus the equality (2.8) follows from (3.9). It is clear that the assertion Claim 3.2(iii) implies Theorem 2.4. Therefore Theorems 2.2 and 2.4 are reduced to Claim 3.2.

4. Shimura curves and sheaves on them

The construction of the sheaf $\mathcal{F}_\lambda^{(k)}$ in the last section is geometric in the sense that it is defined by using a Barsotti–Tate group on M. However, it is not geometric enough in a stricter sense that it is not a part of a higher direct image of a proper, smooth family of varieties parametrized by M. This is due to the fact that M is a so-called exotic model and is not a Shimura variety of PEL-type (polarization, endomorphism and level structure). However, the argument, by Carayol in [Car86a], showing that the Barsotti–Tate group extends to the integral model of M, enables a geometric construction, in the stricter sense.

To give this geometric construction, we introduce two more Shimura curves M' and M'' in what follows that are related to each other in the diagram (4.4) below. We show in § 5 that the Shimura variety M' has a modular interpretation and we construct the required sheaf using the universal family of abelian varieties in § 6.1. After extending the family to M'' in § 6.2, we complete the construction on M in § 6.3.

First, we recall the definition of several Shimura varieties introduced in [Car86a]. We take an imaginary quadratic field $E_0 = \mathbb{Q}(\sqrt{-a})$. We fix an embedding $E_0 \subset \mathbb{C}$. We assume that the prime p splits in E_0 and we also fix an embedding $E_0 \subset \mathbb{Q}_p$ defined by a prime ideal $\mathfrak{q}_0$ of O_{E_0} above p. We put $E = FE_0 = F \otimes_{\mathbb{Q}} E_0$ and $D = B \otimes_F E = B \otimes_{\mathbb{Q}} E_0$. We consider the reductive group $G'' = B^\times \times_{F^\times} E^\times \simeq B^\times \cdot E^\times \subset D^\times$. We keep the abuse of notation to use its $\mathbb{Q}$-valued points $G(\mathbb{Q})$ to denote an algebraic group G over $\mathbb{Q}$. As in [Car86a], the notation $B^\times \times_{F^\times} E^\times$ does not mean the fiber product but the amalgamate sum. Let G' be the inverse image of $\mathbb{Q}^\times \subset F^\times$ by the map $\nu = \mathrm{Nrd}_{B/F} \times N_{E/F} : G'' \to F^\times$. We also consider tori $T = E^\times$

and $T_0 = E_0^\times$. We consider the $G'(\mathbb{R})$-conjugacy class X' (respectively $G''(\mathbb{R})$-conjugacy class X'') of the morphism

$$\begin{array}{rccl} h': & \mathbb{C}^\times & \to & G'(\mathbb{R}) \subset \\ & & & G''(\mathbb{R}) = \quad GL_2(\mathbb{R})\cdot\mathbb{C}^\times \times \mathbb{H}^\times\cdot\mathbb{C}^\times \times\cdots\times \mathbb{H}^\times\cdot\mathbb{C}^\times, \\ & z = x+\sqrt{-1}y & \mapsto & \left(\begin{pmatrix} x & y \\ -y & x\end{pmatrix}^{-1}\otimes 1, 1\otimes z^{-1},\ldots,1\otimes z^{-1}\right). \end{array} \tag{4.1}$$

We also consider morphisms

$$\begin{array}{rll} h_E: & \mathbb{C}^\times \to T(\mathbb{R}) = \mathbb{C}^\times\times\mathbb{C}^\times\times\cdots\times\mathbb{C}^\times, & z\mapsto (z^{-1},1,\ldots,1), \\ h_0: & \mathbb{C}^\times \to T_0(\mathbb{R}) = \mathbb{C}^\times, & z\mapsto z^{-1}. \end{array} \tag{4.2}$$

The conjugacy classes X', X'' have natural structures of complex manifolds and are isomorphic to the upper half plane X^+ and to the union of upper and lower half planes X, respectively. Let $M' = M(G', X')$, $M'' = M(G'', X'')$, $N = M(T, h_E)$ and $N_0 = M(T_0, h_0)$ be the canonical models of the Shimura varieties defined over the reflex fields E, E, E and E_0, respectively. The reciprocity map $E^\times \to E^\times$ is the identity for (T, h_E). For an open compact subgroup $K\subset \mathbb{A}_E^{\infty\,\times}$, the canonical model N_K is the spectrum of the abelian extension E_K corresponding to K by class field theory. The same thing holds for the canonical model of N_0.

We define morphisms between Shimura curves. We consider the morphism $\alpha: G\times T\to G''$ of algebraic groups inducing

$$B^\times\times E^\times \to G''(\mathbb{Q}) \subset (B\otimes E)^\times : (b,e)\to b\otimes N_{E/E_0}(e)\cdot e^{-1} \tag{4.3}$$

on $\mathbb{Q}$-valued points. Since $h' = \alpha\circ(h\times h_E)$, it induces a homomorphism of Shimura varieties $M\times N\to M''$ defined over E. We let α also denote the morphism $M\times N\to M''$. The inclusion $G'\to G''$ induces a natural map $M'\to M''$ of Shimura varieties over E. Let $\beta: G\times T\to T_0$ be the morphism inducing $N_{E/E_0}\circ \mathrm{pr}_2 : B^\times\times E^\times\to E_0^\times$ on $\mathbb{Q}$-valued points. Since $h_0 = N_{E/E_0}\circ h$, a homomorphism of Shimura varieties $M\times N\to N_0$ defined over E is thus induced. We also let the map $M\times N\to N_0$ be denoted by β. We consider the diagram

$$\begin{array}{ccccccc} M & \xleftarrow{pr_1} & M\times N & \xrightarrow{\alpha} & M'' & \longleftarrow & M' \\ & & \downarrow{\scriptstyle\beta} & & & & \\ & & N_0 & & & & \end{array} \tag{4.4}$$

of (weakly) canonical models of Shimura varieties over E.

We define an L_λ-sheaf $\mathcal{F}_\lambda''^{(k)}$ on M'' analogous to $\mathcal{F}_\lambda^{(k)}$. Let $k = (k_1,\ldots,k_g,w)$ be the multiweight and put $n = n(k) = \prod_i (k_i - 1)$. The algebraic group denoted G''^c in [Mil90, ch. III] for the group G'' is the quotient of G'' by $\mathrm{Ker}(N_{F/\mathbb{Q}}: F^\times\to\mathbb{Q}^\times)$. Here $F^\times$ is regarded as a subgroup of the center $Z(G'') = E^\times$. We define a representation of algebraic group $\rho = \rho''^{(k)}: G''\to GL_n$ factoring the quotient G''^c as follows. Recall that we have an isomorphism $B\otimes_{\mathbb{Q}}\mathbb{C}\simeq M_2(\mathbb{C})^I$. It induces an injection $G''_{\mathbb{C}}\to (GL_{2,\mathbb{C}}\times GL_{2,\mathbb{C}})^I$. For each $i\in I$, the first component corresponds to the inclusion $E_0\to\mathbb{C}$ and the second one corresponds to its complex conjugate. We define the morphism $\rho'' = \rho''^{(k)}: G\to GL_n$ to be the composite of the injection with the tensor product $\bigotimes_{i\in I}((\mathrm{Sym}^{k_i-2}\otimes\det{}^{(w-k_i)/2})\circ\check{pr}_{2,i})$. Here $\check{pr}_{2,i}$ denotes the contragradient representation of the $(2,i)$th projection $(GL_{2,\mathbb{C}}\times GL_{2,\mathbb{C}})^I\to GL_{2,\mathbb{C}}$. Since the restriction to the subgroup $F^\times\subset G''$ is the scalar multiplication by $N_{F/\mathbb{Q}}^{-(w-2)}$, it factors through the quotient

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$\rho''^{(k)} : G''^c \to GL_n$. The morphism $\rho'' = \rho''^{(k)} : G \to GL_n$ is defined over the composite field LE_0. Replacing L by LE_0 if necessary, we assume it is defined over L.

We may also define it as follows. Let $p_2 : G'' \to G$ be the map defined over E_0 induced by the second projection on $(D \otimes_{\mathbb{Q}} E_0)^\times = D^\times \times D^\times$ corresponding to the conjugate $E_0 \to E_0$. Then we have $\rho''^{(k)} = \rho^{(k)} \circ p_2$.

We define the smooth L_λ-sheaf $\mathcal{F}''^{(k)}_\lambda$ on M'' to be the L_λ-component of the smooth $L \otimes \mathbb{Q}_\ell$-sheaf $V_\ell(\rho''^{(k)})$ attached to the representation $\rho''^{(k)}$ [Mil90, ch. III, § 6]. By restriction, we obtain a smooth L_λ-sheaf $\mathcal{F}'^{(k)}_\lambda$ on M' attached to the representation $\rho'^{(k)} = \rho''^{(k)}|_{G'}$.

We also define a sheaf $\mathcal{F}(\chi)_\lambda$ on N_0. The algebraic group T_0^c in [Mil90, ch. III] is T_0 itself. We define a character $\chi : T_0 \to \mathbb{G}_m$. Over $\mathbb{C}$, we have $T_{0,\mathbb{C}} \simeq \mathbb{G}_m \times \mathbb{G}_m$. Here the first component corresponds to the inclusion $E_0 \to \mathbb{C}$ and the second one corresponds to its complex conjugate. We define the morphism $\chi : T_0 \to \mathbb{G}_m$ to be the inverse of the first projection. We also define the morphism $\bar{\chi}$ to be the inverse of the second projection. Their product $\chi_0 = \chi\bar{\chi}$ is the inverse of the norm map $\chi_0 = N_{E/\mathbb{Q}}^{-1} : T_0 \to \mathbb{G}_m$. They are defined over $E_0 \subset L$. We define the smooth L_λ-sheaf $\mathcal{F}(\chi)$ on N_0 to be the L_λ-component of the smooth $L \otimes \mathbb{Q}_\ell$-sheaf $V_\ell(\chi)$ attached to the representation χ. The sheaf $\mathcal{F}(\chi_0)$ is defined similarly.

We have $\rho''^{(k)} \circ \alpha = (\rho^{(k)} \circ pr_1) \times (\bar{\chi}^{(w-2)(g-1)} \circ N_{E/E_0} \circ pr_2)$. In other words, we have a commutative diagram

$$\begin{array}{ccc} G \times T & \xrightarrow{\alpha\times\beta} & G'' \times T_0 \\ {\scriptstyle \rho^{(k)}\circ pr_1}\downarrow & & \downarrow{\scriptstyle \rho''^{(k)}\times\chi^{(g-1)(w-2)}\chi_0^{-(g-1)(w-2)}} \\ GL_n & \xleftarrow[\text{product}]{} & GL_n \times \mathbb{G}_m \end{array} \tag{4.5}$$

of homomorphisms defined over L. By the commutativity of the diagram, we obtain an isomorphism of smooth L_λ-sheaves

$$pr_1^*\mathcal{F}^{(k)} \otimes \beta^*\mathcal{F}(\chi_0)^{\otimes(g-1)(w-2)} \simeq \alpha^*\mathcal{F}''^{(k)} \otimes \beta^*\mathcal{F}(\chi)^{\otimes(g-1)(w-2)} \tag{4.6}$$

on $M \times N$. The isomorphism is equivariant with respect to the action of $G(\mathbb{A}^\infty) \times T(\mathbb{A}^\infty)$.

The sheaf $\beta^*\mathcal{F}(\chi_0)$ together with the action of $T(\mathbb{A}^\infty)$ on it is identified as follows. Let $\beta_1 : N \to N_0$ denote the map induced by N_{E/E_0}. It is sufficient to describe $\beta_1^*\mathcal{F}(\chi_0)$. If we forget the action, it is just the Tate twist $L_\lambda(-1)$. The action of $T(\mathbb{A}^\infty)$ is that induced by the natural action of $T(\mathbb{A}^\infty)$ on N multiplied by the character

$$T(\mathbb{A}^\infty) \xrightarrow{N_{E/\mathbb{Q}}} \mathbb{A}^{\infty\,\times}/\mathbb{Q}^{+\times} \xleftarrow{\sim} \hat{\mathbb{Z}}^\times \longrightarrow \mathbb{Z}_\ell^\times \subset L_\lambda^\times. \tag{4.7}$$

Thus the geometric construction of $pr_1^*\mathcal{F}^{(k)}$ is reduced to that of $\mathcal{F}''^{(k)}$ and that of $\mathcal{F}(\chi)$.

Before constructing $\mathcal{F}''^{(k)}$ geometrically in the next section, we will study its restriction $\mathcal{F}'^{(k)}$ to M'. We prepare some notations. We consider the representation

$$\rho' : G' \subset G'' \subset D^\times \xrightarrow{b\mapsto\bar{b}^{-1}} D^\times \subset GL(D) \tag{4.8}$$

defined over $\mathbb{Q}$. Since the algebraic group G'^c for G' is equal to G' itself, the representation ρ' gives rise to a smooth ℓ-adic sheaf $\mathcal{F}'_\ell$ on M' for each prime ℓ. It is a smooth sheaf of $D \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$-modules of rank one.

Recall that we have an isomorphism

$$D \otimes_{\mathbb{Q}} L \simeq (M_2(L) \times M_2(L))^I. \tag{4.9}$$

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For each $i \in I$, the first component corresponds to the embedding $E_0 \subset L \subset \mathbb{C}$ and the second to its conjugate. For each $i \in I$, let $e_i \in D \otimes_{\mathbb{Q}} L$ denote the idempotent whose $(2, i)$th component is $\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$ and the other components are zero under the isomorphism (4.9). For each finite place $\lambda | \ell$, we regard the L_λ-sheaf $\mathcal{F}' \otimes_{\mathbb{Q}_\ell} L_\lambda$ as a $D \otimes_{\mathbb{Q}} L \simeq (M_2(L) \times M_2(L))^I$-module. For each $i \in I$, let $\mathcal{F}'_i$ denote the e_i-part $e_i(\mathcal{F}' \otimes_{\mathbb{Q}_\ell} L_\lambda)$. It is easy to see that

$$\mathcal{F}'^{(k)} = \bigotimes_{i \in I} (\mathrm{Sym}^{k_i - 2} \mathcal{F}'_i \otimes (\det \mathcal{F}'_i)^{\otimes (w - k_i)/2}) \tag{4.10}$$

as a smooth L_λ-sheaf on M' with an action of $G'(\mathbb{A}^\infty)$.

In § 6.1, we will construct the sheaf $\mathcal{F}'$ and the idempotents e_i after recalling a modular interpretation of M' in § 5. We will also construct $\mathcal{F}(\chi)$ on N_0 in a similar way. After that, we study the relation between M' and M'' and extend $\mathcal{F}'$ to M'' in § 6.2.

5. Modular interpretations of M' and N_0

We recall the modular interpretation of the Shimura curve M' on the category of schemes over E (see [Car86a, 2.3]). In the notation of [Del71, (4.9) and (4.13)], we put $L = V = D$. Let the involution $*$ on $D = B \otimes_F E$ be the tensor product of the main involution of B and the conjugate of E and let ψ be the non-generate alternating form on D defined by

$$\psi(x, y) = \mathrm{Tr}_{E/\mathbb{Q}}(\sqrt{-a}\,\mathrm{Trd}_{D/E}\, xy^*). \tag{5.1}$$

Then the group G in [Del71, (4.9)] is G' here and G_1 in [Del71, (4.13)] is G'' here.

We prepare some terminology to formulate a moduli problem for M'. Let O_D be a maximal order in D stable under the involution $*$. An abelian scheme A over a scheme S is called an O_D-abelian scheme over S when a ring homomorphism $m : O_D \to \mathrm{End}_S(A)$ is given. Let Lie A denote the locally free $\mathcal{O}_S$-module $\mathrm{Lie}(A/S) = \mathrm{Hom}_{\mathcal{O}_S}(0^*\Omega^1_{A/S}, \mathcal{O}_S)$, where $0 : S \to A$ denotes the 0-section. When S is a scheme over Spec E, for an O_D-abelian scheme A on S, we define direct summands $\mathrm{Lie}^2 A \supset \mathrm{Lie}^{1,2} A$ of the $O_D \otimes_{\mathbb{Z}} \mathcal{O}_S = D \otimes_{\mathbb{Q}} \mathcal{O}_S$-module Lie A as follows. The submodule $\mathrm{Lie}^2 A$ is defined to be the submodule on which the action of $E_0 \subset D$ and that of $E_0 \subset \mathcal{O}_S$ are conjugate to each other over $\mathbb{Q}$. Similarly $\mathrm{Lie}^{1,2} A$ is the submodule where the action of $E \subset D$ and that of $E \subset \mathcal{O}_S$ are the conjugate to each other over F. They are the same as the tensor products $\mathrm{Lie}^2 A = \mathrm{Lie}\ A \otimes_{E_0 \otimes E_0} E_0$, $\mathrm{Lie}^{1,2} A = \mathrm{Lie}\ A \otimes_{E \otimes E} E$ and hence are direct summands. If A is an O_D-abelian scheme, the dual A^* is considered as an O_D-abelian scheme by the composite map

$$m^* : O_D \xrightarrow{\ *\ } O_D^{\mathrm{opp}} \xrightarrow{\ m\ } \mathrm{End}(A)^{\mathrm{opp}} \xrightarrow{\ *\ } \mathrm{End}(A^*)$$

where opp denotes the opposite ring. A polarization $\theta \in \mathrm{Hom}(A, A^*)^{\mathrm{sym}}$ of an O_D-abelian scheme A is called an O_D-polarization if it is O_D-linear.

Let $K \subset \hat{O}_D^\times \subset G'(\mathbb{A}^\infty)$ be a sufficiently small compact open subgroup. Take a maximal order O_D of D and let $\hat{O}_D = O_D \otimes \hat{\mathbb{Z}} \subset D \otimes \mathbb{A}^\infty$ be the corresponding maximal order. We assume $K \subset \hat{O}_D^\times$. Let $\hat{T} \subset D \otimes \mathbb{A}^\infty$ be an $\hat{O}_D$-lattice satisfying $\psi(\hat{T}, \hat{T}) \subset \hat{\mathbb{Z}}$. We define a functor $M'_{K', \hat{T}}$ on the category of schemes over E as follows. For a scheme S over E, let $M'_{K', \hat{T}}(S)$ be the set of isomorphism classes of the triples $(A, \theta, \bar{k})$ consisting of the following data.

(i) An O_D-abelian scheme A on S of dimension 4g such that $\mathrm{Lie}^2 A = \mathrm{Lie}^{1,2} A$ and that it is a locally free $\mathcal{O}_S$-module of rank two.

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(ii) An O_D-polarization $\theta \in \mathrm{Hom}(A, A^*)^{\mathrm{sym}}$ of A.

(iii) A K-equivalent class $\bar{k}$ of an $O_D \otimes \hat{\mathbb{Z}}$-linear isomorphism $k : \hat{T}(A) \to \hat{T}$ such that there exists a $\hat{\mathbb{Z}}$-linear isomorphism k' making the diagram

$$\begin{array}{ccccc} \hat{T}(A) \times \hat{T}(A) & \xrightarrow{(1,\theta_*)} & \hat{T}(A) \times \hat{T}(A^*) & \longrightarrow & \hat{\mathbb{Z}}(1) \\ {\scriptstyle k\times k}\downarrow & & & & \downarrow{\scriptstyle k'} \\ \hat{T} \times \hat{T} & & \xrightarrow[\mathrm{Tr}\,\psi]{} & & \hat{\mathbb{Z}} \end{array} \tag{5.2}$$

commutative.

Only in this section, changing the notation from previous sections, k will denote a level structure $k : \hat{T}(A) \to \hat{T}$ as in property (iii) above. It is shown in [Car86a, (2.3), (2.6.2)], that the scheme $M'_{K'}$ represents the functor $M'_{K',\hat{T}}$. It is easily checked that the functor is independent of the choice of $\hat{T}$ up to unique canonical isomorphism. Let $A_{K',\hat{T}}$ denote the universal abelian scheme over $M_{K'}$. They form a projective system $A = (A_{K',\hat{T}})_{K',\hat{T}}$.

We give a modular interpretation of the action of $G'(\mathbb{A}^\infty)$ on M' and on A. Let $g \in G'(\mathbb{A}^\infty)$ and $K, K' \subset G'(\mathbb{A}^\infty)$ be sufficiently small open subgroups satisfying $g^{-1}Kg \subset K'$. We take a maximal order O_D and let $\hat{T}$ and $\hat{T}'$ be a K-stable $O_D \otimes \hat{\mathbb{Z}}$-lattice and a K'-stable $O_D^g \otimes \hat{\mathbb{Z}}$-lattice of $V \otimes \mathbb{A}^\infty$ satisfying $g^{-1}\hat{T} \subset \hat{T}'$ and $\psi(\hat{T}, \hat{T}), \psi(\hat{T}', \hat{T}') \subset \hat{\mathbb{Z}}$. The functor

$$g_* : \mathcal{M}_K \to \mathcal{M}_{K'}, \quad [(A, \theta, k)] \mapsto [(A', \theta', k')] \tag{5.3}$$

is described as follows. (Ind-)étale locally on S, we take an isomorphism $\tilde{k} : \hat{T} \to \hat{T}(A)$ in the K-equivalent class k and identify $\hat{T}(A)$ with $\hat{T}$ by $\tilde{k}$. Let $g_* : A \to A'$ be the isogeny of O_D-abelian schemes such that $\hat{T}(A') = g\hat{T}' \supset \hat{T} = \hat{T}(A)$. The K'-equivalent class k' is the class of the isomorphism $g : \hat{T}' \to g\hat{T}' = \hat{T}(A')$. The pair (A', k') is independent of the choice of $\tilde{k}$. The polarization θ' on A' is the map making the diagram

$$\begin{array}{ccc} A & \xrightarrow{\nu^+(g)\theta} & A^* \\ {\scriptstyle g_*}\downarrow & & \uparrow{\scriptstyle {}^t g_*} \\ A' & \xrightarrow[\theta']{} & A'^* \end{array} \tag{5.4}$$

commutative. Here $\nu^+ : G'(\mathbb{A}^\infty) \to \mathbb{Q}^{\times +}$ is the composite

$$G'(\mathbb{A}^\infty) \xrightarrow{\nu} \mathbb{A}^{\infty\times} \to \mathbb{A}^{\infty\times}/\hat{\mathbb{Z}}^\times \xleftarrow{\sim} \mathbb{Q}^{\times +}. \tag{5.5}$$

We have the universal O_D-isogeny $g : A_{K,\hat{T}} \to g^* A_{K',\hat{T}'}$ and a commutative diagram.

$$\begin{array}{ccc} A_{K,\hat{T}} & \xrightarrow{g_*} & A_{K',\hat{T}'} \\ \downarrow & & \downarrow \\ M'_K & \xrightarrow[g]{} & M'_{K'} \end{array} \tag{5.6}$$

For later use, we will extend the action of $G'(\mathbb{A}^\infty)$ on M' and on A to a larger subgroup $\tilde{G} \subset G''(\mathbb{A}^\infty)$. Let $G''(\mathbb{R})_+$ be the inverse image of $GL_2(\mathbb{R})^+\mathbb{C}^\times \subset GL_2(\mathbb{R})\mathbb{C}^\times$ by the first projection $G''(\mathbb{R}) \to GL_2(\mathbb{R}) \cdot \mathbb{C}^\times$ and let $G''(\mathbb{Q})_+ = G''(\mathbb{Q}) \cap G''(\mathbb{R})_+ = \{\gamma \in G''(\mathbb{Q}) \mid \nu(\gamma)$ is totally positive$\}$. We put

$$\tilde{G} = G''(\mathbb{Q})_+ \cdot G'(\mathbb{A}^\infty) \subset G''(\mathbb{A}^\infty). \tag{5.7}$$

We extend the action of $G'(\mathbb{A}^\infty)$ on M' to an action of $\tilde{G}$. For $g \in G''(\mathbb{A}^\infty)$ and open compact subgroups $K' \subset G'(\mathbb{A}^\infty)$ and $K'' \subset G''(\mathbb{A}^\infty)$ such that $g^{-1}K'g \subset K''$, let $g : M'_{K'} \to M''_{K''}$ denote the composite $M'_{K'} \to M''_{gK''g^{-1}} \xrightarrow{g} M''_{K''}$. For $g \in \tilde{G}$ and open compact subgroups $K'_1, K'_2 \subset G'(\mathbb{A}^\infty)$ such that $g^{-1}K'_1 g \subset K'_2$, the map $g : M'_{K'_1} \to M'_{K'_2}$ is defined as follows. We may take an open compact subgroup $K'' \supset K'_2$ of $G''(\mathbb{A}^\infty)$ such that the canonical map $M'_{K'_2} \to M''_{K''}$ is an open immersion (see Lemma 6.1 in § 6.2). Then since $M'_{K'}(\mathbb{C}) = G'(\mathbb{Q}) \backslash G'(\mathbb{A}^\infty) \times X'/K' = G''(\mathbb{Q})_+ \backslash \tilde{G} \times X'/K'$, the image of $g : M'_{K'_1} \to M''_{K''}$ is contained in $M'_{K'_2}$. Hence the required map $M'_{K'_1} \to M'_{K'_2}$ is thus induced. The modular interpretation of the action of $\tilde{G}$ on M' is described in the same way as above. The only modification is that ν^+ is extended to $\tilde{G}$ as the composite.

$$\tilde{G} \xrightarrow{\ \nu\ } F^{\times +}\mathbb{A}_{\mathbb{Q}}^{\infty\times} \to F^{\times +}\mathbb{A}_{\mathbb{Q}}^{\infty\times}/\hat{\mathbb{Z}}^\times \xleftarrow{\ \sim\ } F^{\times +} \tag{5.8}$$

Similarly, we have a modular interpretation for N_0 in terms of elliptic curves with complex multiplication by O_{E_0}. Let $H \subset \hat{O}_{E_0}^\times$ be a sufficiently small open subgroup. We take a fractional ideal $R \subset E_0$ satisfying $\mathrm{Tr}_{E_0/\mathbb{Q}}(\sqrt{-a}R\bar{R}) \subset \mathbb{Z}$. Let $\hat{R} = R \otimes \hat{O}_E$ be the corresponding ideal. We define a functor $N_{0,H,\hat{R}}$ on the category of schemes over E_0 as follows. For a scheme S over E_0, let $N_{0,H,\hat{R}}(S)$ be the set of isomorphism classes of the pairs $(A, \bar{k})$ of the following data.

(i) An elliptic curve A endowed with a ring homomorphism $O_{E_0} \to \mathrm{End}_S(A)$ such that the induced homomorphism $O_{E_0} \to \mathrm{End}_{\mathcal{O}_S}(\mathrm{Lie}\ A) = \mathcal{O}_S$ is the same as that defined by the structure morphism $S \to \mathrm{Spec}\ E_0$.

(ii) An H-equivalent class $\bar{k}$ of an $\hat{O}_{E_0}$-isomorphism $k : T(A) \to T$ such that there exists a $\hat{\mathbb{Z}}$-isomorphism k' making the diagram

$$\begin{array}{ccc} \hat{T}(A) \times \hat{T}(A) & \longrightarrow & \hat{\mathbb{Z}}(1) \\ \downarrow{\scriptstyle k \times k} & & \downarrow{\scriptstyle k'} \\ \hat{R} \times \hat{R} & \xrightarrow[(x,y) \mapsto \mathrm{Tr}_{E_0/\mathbb{Q}}(\sqrt{-a}x\bar{y})]{} & \hat{\mathbb{Z}} \end{array} \tag{5.9}$$

commutative.

It is easily checked that the functor $N_{0,H,\hat{R}}$ is independent of the choice of R up to unique canonical isomorphism.

By the theory of complex multiplication, for a sufficiently small H, the functor $N_{0,H,\hat{R}}$ is represented by $N_H = \mathrm{Spec}\ E_{0,H}$ where $E_{0,H}$ is the abelian extension corresponding to the open subgroup $H \subset \mathbb{A}_{E_0}^{\infty\times}$ by the isomorphism $\mathbb{A}_{E_0}^{\infty\times}/E^\times \simeq \mathrm{Gal}(E_0^{\mathrm{ab}}/E_0)$ of class field theory. Similarly as above, a natural action of $T_0(\mathbb{A}^\infty) = \mathbb{A}_{E_0}^{\infty\times}$ on the projective systems $N = (N_K)_K$ and on the universal CM elliptic curve $b : A_0 = (A_{0,\hat{T},K})_{\hat{T},K} \to N$ is defined.

6. Geometric constructions

In this section, we construct the sheaves $\mathcal{F}', \mathcal{F}''$ etc. geometrically, using the modular interpretation. In §§ 6.1, 6.2 and 6.3, we study M', M'' and M, respectively.

6.1 Geometric construction on M', N_0

We show that the direct image $R^1a_*\mathbb{Q}_\ell$ of the universal abelian scheme $a: A \to M'$ gives the sheaf $\mathcal{F}'$. Using it, we construct the sheaf $\mathcal{F}'^{(k)}$ on M' in a purely geometric way. We will also define geometrically $\mathcal{F}(\chi)$ on N_0.

Let $K' \subset \hat{O}_D^\times$, $\hat{T}$ and the universal O_D-abelian scheme $a_{K'}: A_{K',\hat{T}} \to M'_{K'}$ be as in the modular interpretation in § 5. By the ring homomorphism $O_D \to \mathrm{End}_{M'_{K'}}(A_{K',\hat{T}})$, we regard the direct image $R^1a_{K'*}\mathbb{Q}_\ell$ as a sheaf of $D \otimes \mathbb{Q}_\ell$-modules for every ℓ. It is independent of the choice of lattice $\hat{T}$. A canonical action of $G'(\mathbb{A}^\infty)$ is defined on the system of sheaves $R^1a_*\mathbb{Q}_\ell = (R^1a_{K'*}\mathbb{Q}_\ell)_{K'}$. By the modular interpretation, it is easy to see that the sheaf $R^1a_*\mathbb{Q}_\ell$ is isomorphic to the sheaf $\mathcal{F}'$ with the action of $G'(\mathbb{A}^\infty)$ defined at the end of § 5. We will identify them in the following.

For each $i \in I$, let $e_i \in D \otimes_\mathbb{Q} L$ be the idempotent defined at the end of § 5. We regard $R^1a_*L_\lambda$ as a sheaf of $D \otimes_\mathbb{Q} L$-modules. Then $e_i \in D \otimes_\mathbb{Q} L$ acts on it as a projector and the e_i-part $e_i \cdot R^1a_*L_\lambda$ is isomorphic to $\mathcal{F}'_i$. Since D is generated by $1 + pO_D$, we may write each e_i as an L-linear combination of elements in $1 + pO_D$. Therefore e_i is an L-linear combination of endomorphisms of A over M' whose degrees are prime to p.

One finds easily an idempotent $e^{(k_i)} \in \mathbb{Q}[\mathfrak{S}_{w-2}]$ of the group algebra of a symmetric group such that the $e^{(k_i)}$-part $e^{(k_i)} \cdot \mathcal{F}_i'^{\otimes w-2}$ is equal to $\mathrm{Sym}^{k_i-2}\mathcal{F}'_i \otimes (\det \mathcal{F}'_i)^{\otimes(w-k_i)/2}$. The action of the symmetric group $\mathfrak{S}_{w-2}$ on $\mathcal{F}_i'^{\otimes w-2}$ is induced by its action on the fiber product $a^{w-2}: A^{w-2} \to M'$ over M' as permutations. One can also find easily a $\mathbb{Q}$-linear combination e^1 of the multiplications by prime-to-p integers on A such that $e^1R^1a_*\mathbb{Q}_\ell = R^1a_*\mathbb{Q}_\ell$ and $e^1R^qa_*\mathbb{Q}_\ell = 0$ for $q \neq 1$.

Taking their product, we obtain an algebraic correspondence e' on the $(w-2)g$-fold self-fiber product $A^{(w-2)g}$ of $A \to M'$ with coefficients in L satisfying the following conditions.

(i) It is an L-linear combination of permutations in $\mathfrak{S}_{g(w-2)}$ and endomorphisms of $A^{(w-2)g}$ as an abelian scheme over M' whose degrees are prime to p.

(ii) It acts as an idempotent on the cohomology sheaf $R^qa_*^{(w-2)g}L_\lambda$ where $a^{(w-2)g}$ denotes the map $A^{(w-2)g} \to M'$. We have $e'R^qa_*^{(w-2)g}L_\lambda = \mathcal{F}'^{(k)}$ for $q = (w-2)g$ and $e'R^qa_*^{(w-2)g}L_\lambda = 0$ otherwise.

Similarly, we construct $\mathcal{F}(\chi)$. Let $H \subset \hat{O}_{E_0}^\times$, $\hat{R}$ and the universal O_{E_0}-elliptic curve $b_H: A_{0,H,\hat{R}} \to N_{0,H}$ be as in the modular interpretation in § 5. We regard the direct image $R^1b_{H*}\mathbb{Q}_\ell$ as a sheaf of $E_0 \otimes \mathbb{Q}_\ell$-modules by the ring homomorphism $O_{E_0} \to \mathrm{End}_{N_{0,H}}(A_{0,H,\hat{R}})$, for every ℓ. It is independent of the choice of lattice $\hat{R}$. A canonical action of $\mathbb{A}_{E_0}^{\infty\times}$ is defined on the system of sheaves $R^1b_*\mathbb{Q}_\ell = (R^1b_{H*}\mathbb{Q}_\ell)_H$. By the modular interpretation, it is easy to see that the sheaf $R^1b_*\mathbb{Q}_\ell$ is isomorphic to the sheaf on N_0 associated to the inverse of the tautological representation $E^\times \to GL_\mathbb{Q}(E): t \mapsto t^{-1} \times -$. We will identify them in the following.

Let $e_0 \in E_0 \otimes_\mathbb{Q} L$ be the idempotent corresponding to the inclusion $E_0 \to L$. We regard $R^1a_{0,*}L_\lambda$ as a sheaf of $E_0 \otimes_\mathbb{Q} L$-modules. Then $e_0 \in E_0 \otimes_\mathbb{Q} L$ acts on it as a projector and the e_0-part $e_0 \cdot R^1a_{0,*}L_\lambda$ is isomorphic to $\mathcal{F}(\chi)$. Similarly as above, we may write each e_0 as an L-linear combination of elements in $1 + pO_D$. Therefore e_0 is an L-linear combination of endomorphisms of A_0 over N_0 whose degrees are prime to p. Similarly as above, after modifying e_0 if necessary, we also have $e_0 \cdot R^qb_*L_\lambda = 0$ for $q \neq 1$.

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6.2 Geometric construction on M''

We extend the geometric construction on M' to M''. We first study the relation between them. Recall that $G'' = B^\times \times_{F^\times} E^\times$ and G' is the inverse image of $\mathbb{Q}^\times \subset F^\times$ by

$$\nu = \mathrm{Nrd}_{B/F} \times N_{E/F} : G'' \to F^\times.$$

For an open compact subgroup $K'' \subset G''(\mathbb{A}^\infty)$ and for $g \in G''(\mathbb{A}^\infty)$, we put $K'^g = G'(\mathbb{A}^\infty) \cap gK''g^{-1}$. Recall that $g : M'_{K'^g} \to M''_{K''}$ denotes the composition $M'_{K'^g} \to M''_{gK''g^{-1}} \xrightarrow{g} M''_{K''}$. The double coset $\tilde{G}\backslash G''(\mathbb{A}^\infty)/K''_1 = F^{\times +}\backslash \mathbb{A}_F^{\infty\times}/\mathbb{A}_{\mathbb{A}}^{\infty\times}\nu(K''_1)$ is finite where the subgroup $\tilde{G} \subset G''(\mathbb{A}^\infty)$ is defined in (5.7). If $\Sigma \subset G''(\mathbb{A}^\infty)$ is a complete set of representatives, we have a finite étale surjection $\amalg\, g : \coprod_{g\in\Sigma} M'_{K'^g} \to M''_{K''}$.

Lemma 6.1. *Let $K'' \subset G''(\mathbb{A}^\infty)$ be a compact open subgroup and put $K' = K'' \cap G'(\mathbb{A}^\infty)$. Then for a sufficiently small open subgroup $K''_1 \subset K''$ containing K' and for a complete set Σ of representatives of the finite set $\tilde{G}\backslash G''(\mathbb{A}^\infty)/K''_1$, the map*

$$\amalg\ g : \coprod_{g\in\Sigma} M'_{K'^g} \to M''_{K''_1} \tag{6.1}$$

is an isomorphism.

Proof. Since it is an étale surjection, it is enough to show the map is injective on the $\mathbb{C}$-valued points. Since Σ is a complete set of representatives, it is enough to consider each map g. Let $\bar\nu : G''(\mathbb{A}^\infty) \to \mathbb{A}_F^{\infty\times}/\mathbb{A}_{\mathbb{Q}}^{\infty\times}$ denote the map induced by ν. We show the following.

Sublemma 6.2. *The equality $\bar\nu(K'') \cap (\overline{O_F^\times}/\mathbb{Z}^\times) = \bar\nu(K'' \cap \overline{O_E^\times})$ implies the injectivity of the map $g : M'_{K'^g}(\mathbb{C}) \to M''_{K''}(\mathbb{C})$.*

We prove Lemma 6.1, admitting Sublemma 6.2. Namely, we prove that for a sufficiently small open subgroup $K''_1 \supset K'$ of K'', we have an equality $\bar\nu(K''_1) \cap (\overline{O_F^\times}/\mathbb{Z}^\times) = \bar\nu(K''_1 \cap \overline{O_E^\times})$. Since $N_{E/F}(O_E^\times)$ is of finite index in $O_F^\times$, the right-hand side $\bar\nu(K'' \cap \overline{O_E^\times})$ is an open subgroup of the left-hand side $\bar\nu(K'') \cap (\overline{O_F^\times}/\mathbb{Z}^\times)$. Hence, for a sufficiently small open subgroup $\overline{K_1}$ of $K''/K' \simeq \bar\nu(K'')$ we have $\overline{K_1} \cap (\overline{O_F^\times}/\mathbb{Z}^\times) = \overline{K_1} \cap \bar\nu(K'' \cap \overline{O_E^\times})$. For the corresponding open subgroup $K''_1 = K'' \cap \bar\nu^{-1}(\overline{K_1})$, this is nothing but the required equality $\bar\nu(K''_1) \cap (\overline{O_F^\times}/\mathbb{Z}^\times) = \bar\nu(K''_1 \cap \overline{O_E^\times})$.

We prove Sublemma 6.2. Namely, we assume $\bar\nu(K'') \cap (\overline{O_F^\times}/\mathbb{Z}^\times) = \bar\nu(K'' \cap \overline{O_E^\times})$ and prove the map $g : M'_{K'^g}(\mathbb{C}) \to M''_{K''}(\mathbb{C})$ is injective. Replacing K'' by $gK''g^{-1}$, it is enough to show that the map $M'_{K'}(\mathbb{C}) \to M''_{K''}(\mathbb{C})$ is injective for $K' = K'' \cap G'(\mathbb{A}^\infty) = \mathrm{Ker}(\bar\nu : K'' \to \hat{O}_F^\times/\hat{\mathbb{Z}}^\times)$. We consider the commutative diagram of exact sequences.

$$\begin{array}{ccccccccc} & & K'' \cap \overline{O_E^\times} & \longrightarrow & K''/K' & \longrightarrow & K''/(K'' \cap \overline{O_E^\times})K' & \longrightarrow & 1 \\ & & \downarrow \bar\nu & & \downarrow \bar\nu\ \cap & & \downarrow & & \\ 1 & \longrightarrow & \overline{O_F^\times}/\mathbb{Z}^\times & \longrightarrow & \hat{O}_F^\times/\hat{\mathbb{Z}}^\times & \longrightarrow & \hat{O}_F^\times/\hat{\mathbb{Z}}^\times\overline{O_F^\times} & & \end{array}$$

The middle vertical arrow is injective by the definition of K'. By the snake lemma, the equality $\bar\nu(K'') \cap (\overline{O_F^\times}/\mathbb{Z}^\times) = \bar\nu(K'' \cap \overline{O_E^\times})$ is equivalent to the injectivity of the right vertical arrow. Since $\hat{O}_F^\times/\hat{\mathbb{Z}}^\times\overline{O_F^\times}$ is a subgroup of $\mathbb{A}_F^{\infty\times}/\mathbb{A}_{\mathbb{Q}}^{\infty\times}\overline{F^\times}$, we get an exact sequence.

$$K'/(K' \cap \overline{O_E^\times}) \longrightarrow K''/(K'' \cap \overline{O_E^\times}) \longrightarrow \mathbb{A}_F^{\infty\times}/\mathbb{A}_{\mathbb{Q}}^{\infty\times}\overline{F^\times} \tag{6.2}$$

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We consider the following commutative diagram.

$$\begin{array}{ccc} M'(\mathbb{C}) = \varprojlim_{K'} M'_{K'}(\mathbb{C}) & \longrightarrow & M''(\mathbb{C}) = \varprojlim_{K''} M''_{K''}(\mathbb{C}) \\ \downarrow & & \downarrow \\ \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^{\infty\times} & \longrightarrow & \overline{F^\times} \backslash \mathbb{A}_F^{\infty\times} \end{array}$$

The horizontal arrows are injective by [Del71, Variante 1.15.1 and Lemme 1.15.3]. We have $M'_{K'}(\mathbb{C}) = M'(\mathbb{C})/K'$ and $M''_{K''}(\mathbb{C}) = M''(\mathbb{C})/K''$. From these facts, it is straightforward to show that the exactness (6.2) implies the injectivity of the canonical map $M'_{K'}(\mathbb{C}) \to M''_{K''}(\mathbb{C})$. □

We extend the universal O_D-abelian scheme A on M' to an O_D-abelian scheme also denoted by A on M''. Let $K'' \subset G''(\mathbb{A}^\infty)$ be a sufficiently small open subgroup. We assume that the map $\amalg\, g : \coprod_{g\in\Sigma} M'_{K'^g} \to M''_{K''_1}$ (6.1) is an isomorphism. We take an $\hat{O}_D$-lattice $\hat{T}$ in $D \otimes \mathbb{A}^\infty$. For each $g \in \Sigma$, we have a gO_Dg^{-1}-abelian scheme $A_{K'_g, g\hat{T}}$ on $M'_{K'_g}$ since $g\hat{T}$ is K'_g-stable. We define an abelian scheme $A_{K'',\hat{T}}$ on $M''_{K''}$ to be $A_{K'_g, g\hat{T}}$ on the image of $M'_{K'_g}$. We define an O_D-multiplication on $A_{K'',\hat{T}}$ as $O_D \xrightarrow{a\mapsto gag^{-1}} gO_Dg^{-1} \to \mathrm{End}_{M'_{K'_g}}(A_{K'_g,g\hat{T}})$ on $M'_{K'_g}$. By the action of $\tilde{G}$ described in § 5, we see that the abelian scheme $A_{K'_g,g\hat{T}}$ is independent of the choice of representatives Σ. We also see by the action of $\tilde{G}$ that, for $g \in G''(\mathbb{A}^\infty)$, compact open subgroups $K''_1, K''_2 \subset G''(\mathbb{A}^\infty)$ and K''_i-stable $\hat{O}_D$-lattices $\hat{T}_i$ satisfying $g^{-1}K''_1 g \subset K''_2$, we have an isogeny $A_{K''_1,\hat{T}_1} \to g^* A_{K''_2,\hat{T}_2}$. Thus we obtain an action of $G''(\mathbb{A}^\infty)$ on the projective system $A = (A_{K'',\hat{T}})_{K'',\hat{T}}$ over $M'' = (M''_{K''})_{K''}$.

On the $(w-2)g$-fold self-fiber product $A^{(w-2)g}$ of $A \to M''$, we define an algebraic correspondence e' with coefficients in L exactly in the same way as in the case of M'. Then, it is an L-linear combinations of permutations in $\mathfrak{S}_{g(w-2)}$ and endomorphisms of $A^{(w-2)g}$ as an abelian scheme over M' whose degrees are prime to p. Further, it acts as an idempotent on the cohomology sheaf $R^q a_*^{(w-2)g} L_\lambda$ where $a^{(w-2)g}$ denotes the map $A^{(w-2)g} \to M'$. We have

$$e' R^q a_*^{(w-2)g} L_\lambda = \bigotimes_i \mathrm{Sym}^{k_i-2}(e_i \cdot R^1 a_* L_\lambda) \otimes (\det e_i \cdot R^1 a_* L_\lambda)^{\otimes (w-k_i)/2}$$

for $q = (w-2)g$, and $e' R^q a_*^{(w-2)g} L_\lambda = 0$ otherwise. By the modular interpretation of M', we see that the K''-equivalent class of the isomorphism $\hat{T} \to T(A_{K'',\hat{T}})$ is well-defined. Passing to the limit, we obtain an isomorphism $D \otimes \mathbb{A}^\infty \to R^1 a_* \mathbb{Q}_\ell$ on $\varprojlim_{K''} M_{K''}$. The isomorphism is compatible with the action of $G''(\mathbb{A}^\infty)$. On the left-hand side $D \otimes \mathbb{A}^\infty$, the group $G''(\mathbb{A}^\infty) \subset (D \otimes \mathbb{A}^\infty)^\times$ acts by the multiplication by the inverse of the main involution: $t \mapsto \bar{t}^{-1} \times -$. Thus similarly as on M', we have

$$e' R^{(w-2)g} a_*^{(w-2)g} L_\lambda = \mathcal{F}''^{(k)}. \tag{6.3}$$

6.3 Geometric construction on M

We will define an analogue $c : X \to M \times N$ of the Kuga–Sato variety and an algebraic correspondence $e = e^{(k)}$ on X with coefficient in L satisfying the following property: it is an L-linear combination of endomorphisms of X as an abelian scheme over $M \times N$, whose degrees are prime to p. The algebraic correspondence e acts as an idempotent on the higher direct image $R^q c_* \mathbb{Q}_\ell \otimes L = \prod_{\lambda|\ell} R^q c_* L_\lambda$. The image of the projector $e \cdot R^q c_* L_\lambda$ is a smooth L_λ-sheaf

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isomorphic to

$$\alpha^*\mathcal{F}^{(k)} \otimes \beta^* F(\chi)^{\otimes(w-2)(g-1)} = pr_1^*\mathcal{F}^{(k)} \otimes \beta^* F(\chi_0)^{\otimes(w-2)(g-1)} \tag{6.4}$$

for $q = q_0 = (2g-1)(w-2)$ and is zero otherwise.

We define X to be the fiber product

$$X = \alpha^* A^{g(w-2)} \times_{M_E \times N} \beta^* A_0^{(g-1)(w-2)}. \tag{6.5}$$

Here $\alpha^* A^{g(w-2)}$ denotes the base change by $\alpha : M \times N \to M''$ of the $g(w-2)$-fold self fiber product of $A \to M''$. Similarly $\beta^* A_0^{g(w-2)}$ denotes the base change by $\beta : M \times N \to N_0$ of the $(g-1)(w-2)$-fold self fiber product of $A_0 \to N_0$. The symbol X denotes the projective system $X = (X_{K,H,\hat{T},\hat{R}})_{K,H,\hat{T},\hat{R}}$ of abelian schemes over $M \times N = (M_K \times N_H)_{K,H}$.

Next we define an algebraic correspondence $e = e^{(k)}$ on X. We have defined algebraic correspondences e' on $A^{g(w-2)}$ over M'' and e_0 on A_0 over N_0 at the end of §§6.2 and 6.1 respectively. Let $e_0^{\otimes(g-1)(w-2)} = \prod_{i=1}^{(g-1)(w-2)} pr_i e_0$ be the algebraic correspondence on the $(g-1)(w-2)$nd self fiber product $A_0^{(g-1)(w-2)}$ defined as the product of the pull-back of the algebraic correspondence e_0 on A_0 by projections. We define an algebraic correspondence e on X as the product of the pull-back of e' by α with the pull-back of $e_0^{\otimes(g-1)(w-2)}$ by β. Namely we put $e = \alpha^* e' \times \beta^* e_0^{\otimes(g-1)(w-2)}$. Then it satisfies the required property stated at (6.4).

Let $H \subset \mathbb{A}_E^{\infty\times}$ be a sufficiently small open compact subgroup. Let $\mathfrak{m} = \mathfrak{n} O_E$ be a sufficiently divisible integral ideal of O_E. We assume $H = H^{\mathfrak{m}} \cdot H_{\mathfrak{m}}$ is the product of the prime-to-$\mathfrak{m}$ component $H^{\mathfrak{m}} = \prod_{\mathfrak{s} \nmid \mathfrak{m}} O_{E_{\mathfrak{s}}}^\times$ with the $\mathfrak{m}$-primary component $H_{\mathfrak{m}}$. Let $T_0^{\mathfrak{m}} = L[P_{\mathfrak{s}};\ \mathfrak{s} \nmid \mathfrak{m}]$ be the free L-algebra generated by the class $P_{\mathfrak{s}}$ of the inverse of prime element for $\mathfrak{s} \nmid \mathfrak{m}$. We consider $H^q(X_{K,H,\hat{T},\hat{R}} \otimes_E \bar{E}, L_\lambda)$ as a $T^{\mathfrak{n}} \times T_0^{\mathfrak{m}}$-module and $H^0(N_{H,\bar{E}}, \mathcal{F}(\chi_0))$ as a $T_0^{\mathfrak{m}}$-module.

Applying the Leray spectral sequence to $c : X_{K,H,\hat{T},\hat{R}} \to M_K \times_F N_H$, we obtain the following lemma.

LEMMA 6.3. *Let $K \subset G(\mathbb{A}^\infty)$ and $H \subset \mathbb{A}_E^{\infty\times}$ be sufficiently small open compact subgroups and let $\hat{T} \subset V \otimes \mathbb{A}^\infty$ and $\hat{R} \subset E_0 \otimes \mathbb{A}^\infty$ be an $\hat{O}_D$-lattice and an $\hat{O}_{E_0}$-lattice respectively. Let $X = X_{K,H,\hat{T},\hat{R}}$ be the analogue of the Kuga–Sato variety (6.5). Then there is an algebraic correspondence e on X with coefficients in L satisfying the following properties.*

(i) *There exists elements $a_i \in L$, permutations $\tau_i \in \mathfrak{S}_{g(w-2)}$ of the first $g(w-2)$-factors in X and endomorphisms $\varphi_i \in \mathrm{End}_M X$ of degrees prime to p such that*

$$e = \sum_i a_i \tau_i \varphi_i. \tag{6.6}$$

(ii) *For each finite place λ of L, the action of e on $H^q(X_{K,H,\hat{T},\hat{R}} \otimes_E \bar{E}, L_\lambda)$ is a projector. Put $q_0 = (2g-1)(w-2)$. Then, there is an isomorphism*

$$\begin{aligned} &e \cdot H^q(X_{K,H,\hat{T},\hat{R}} \otimes_E \bar{E}, L_\lambda) \\ &\quad \simeq H^{q-q_0}(M_K \otimes_F \bar{F}, \mathcal{F}_\lambda^{(k)}) \otimes_{\mathbb{Q}_\ell} H^0(N_H \otimes_E \bar{E}, \mathcal{F}(\chi_0^{(g-1)(w-2)})). \end{aligned} \tag{6.7}$$

The isomorphism is compatible with the actions of the absolute Galois group $G_E = \mathrm{Gal}(\bar{E}/E)$ and of the Hecke algebra $T^{\mathfrak{n}} \otimes T_0^{\mathfrak{m}}$.

Using Lemma 6.3, we state Claim 6.4, in terms of X and e, that implies Claim 3.2 and hence Theorems 2.2 and 2.4. Recall that we fixed an isomorphism $E_{0,\mathfrak{q}_0} \to \mathbb{Q}_p$. Let $\mathfrak{q}$ be the place of E dividing $\mathfrak{p}$ and $\mathfrak{q}_0$. The local field $E_{\mathfrak{q}}$ is canonically isomorphic to $F_{\mathfrak{p}}$. We identify $F_{\mathfrak{p}} = E_{\mathfrak{q}}$ by

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the canonical isomorphism. Since we want to prove the assertions on the action of Galois group $\mathrm{Gal}(\bar{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$, it is enough to consider the action of $\mathrm{Gal}(\bar{E}_{\mathfrak{q}}/E_{\mathfrak{q}})$, induced by the isomorphism.

Claim 6.4. We keep the notation in Claim 3.2. Let $K \subset G(\mathbb{A}^\infty)$ and $H \subset \mathbb{A}_E^{\infty\times}$ be sufficiently small open compact subgroups. Let $X = X_{K,H,\hat{T},\hat{R}}$ denote the analogue of the Kuga–Sato variety (6.5). Then, the following hold.

(i) The p-adic representation $H^q(X \otimes_E \bar{E}_{\mathfrak{q}}, \mathbb{Q}_p)$ of $G_{E_{\mathfrak{q}}} = \mathrm{Gal}(\bar{E}_{\mathfrak{q}}/E_{\mathfrak{q}})$ is potentially semi-stable for all q.

(ii) Let $\sigma \in W^+ = \{\sigma \in W(\bar{E}_{\mathfrak{q}}/E_{\mathfrak{q}}) \mid n(\sigma) \geqslant 0\}$, $T \in T^{\mathfrak{n}}$, $P \in T_0^{\mathfrak{m}}$, $\tau \in \mathfrak{S}_{g(w-2)}$ and let $\psi : X \to X$ be an endomorphism of degree prime to p. Then for the composite $\Gamma = T \circ R \circ \tau \circ \psi$ as an algebraic correspondence, we have an equality in $\mathbb{Q}$

$$\sum_q (-1)^q \,\mathrm{Tr}(\sigma \circ \Gamma | H^q(X \otimes_E \bar{E}_{\mathfrak{q}}, \mathbb{Q}_\ell)) = \sum_q (-1)^q \,\mathrm{Tr}(\sigma \circ \Gamma | D(H^q(X \otimes_E \bar{E}_{\mathfrak{q}}, \mathbb{Q}_p))). \quad (6.8)$$

(iii) Let e be the algebraic correspondence in Lemma 6.3 and let $\mu | p$ be a finite place of $L \supset E_0$. Then the monodromy filtration of the representations $e \cdot H^q(X \otimes_E \bar{E}_{\mathfrak{q}}, L_\lambda)$ and $D(e \cdot H^q(X \otimes_E \bar{E}_{\mathfrak{q}}, L_\mu))$ of the Weil–Deligne group $'W(\bar{E}_{\mathfrak{q}}/E_{\mathfrak{q}})$ are pure of weight q.

We deduce each assertion in Claim 3.2 from the corresponding assertion in Claim 6.4. Since we identify $F_{\mathfrak{p}} = E_{\mathfrak{q}}$, it is sufficient to consider the representations of the Weil–Deligne group $'W(\bar{E}_{\mathfrak{q}}/E_{\mathfrak{q}})$. The representation $H^q(M_{\bar{E}_{\mathfrak{q}}}, \mathcal{F}_\lambda^{(k)})$ is a direct summand of $e \cdot H^{q+q_0}(X_{\bar{E}_{\mathfrak{q}}}, L_\lambda)((g-1)(w-2))$ by Lemma 6.3. Therefore the assertions (i) and (iii) in Claim 3.2 follows from the assertions (i) and (iii) in Claim 6.4 respectively. We deduce the equality (3.9) from the equality (6.8). By the definition of $\mathcal{F}(\chi_0)$ given in the middle of § 4, we find easily an element $e^\circ \in T_0^{\mathfrak{m}}$ acting as a projector $H^0(N_H \otimes_E \bar{E}, \mathcal{F}(\chi_0^{(g-1)(w-2)})) \to L_\lambda(-(g-1)(w-2))$. Thus by Lemma 6.3, there is an isomorphism

$$e^\circ \circ e \cdot H^q(X_{K,K^\circ} \otimes_E \bar{E}, L_\lambda) \to H^{q-q_0}(M_K \otimes_F \bar{F}, \mathcal{F}_\lambda^{(k)})(-(g-1)(w-2)) \quad (6.9)$$

compatible with the actions of the Galois group $G_E = \mathrm{Gal}(\bar{E}/E)$ and of the Hecke algebra $T^{\mathfrak{n}}$. Hence the equality (6.8) implies the equality (3.9). Thus Theorems 2.2 and 2.4, are reduced to Claim 6.4.

We may deduce the assertion Claim 6.4(i) using alterations [dJo96]. We will give a proof without using alterations by constructing a semi-stable model of X.

For later use, we describe the Hecke operators $T_{\mathfrak{r}} \in T^{\mathfrak{n}}$ and $P_{\mathfrak{s}} \in T_0^{\mathfrak{m}}$ for primes $\mathfrak{r} \nmid \mathfrak{n}$ of O_F and $\mathfrak{s} \nmid \mathfrak{m}$ of O_E respectively. Write $X = X_{K,H,\hat{T},\hat{T}^\circ}$ and $M \times N = M_K \times N_H$ for short. For $\mathfrak{r}$, it is defined as $T_{\mathfrak{r}} = p_{1*} \circ q^* \circ p_2^*$ where p_1, p_2, q are as in the following diagram.

$$\begin{array}{ccccccc}
X & \xrightarrow{p_1} & X_{K_g,H,\hat{T},\hat{R}} & \xrightarrow{q} & X_{K_g,H,g\hat{T},\hat{R}} & \xrightarrow{p_2} & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M \times N & \xleftarrow{p_1} & M_{K_g} \times N_H & = & M_{K_g} \times N_H & \xrightarrow{p_2} & M \times N
\end{array} \quad (6.10)$$

In the diagram, $g = g_{\mathfrak{r}} \in G(\mathbb{A}^\infty)$ is an element whose $\mathfrak{r}$-component is $\begin{pmatrix} \pi_{\mathfrak{r}}^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ and other components are 1 and $K_g = K \cap gKg^{-1}$. The map p_1 is induced by the inclusion $K_g \to K$, the map $p_2 = g_*$ is induced by g and the left and right squares are cartesian. The map q is an isogeny corresponding to the inclusion $\hat{T} \to g\hat{T}$.

Similarly for $\mathfrak{s}$, the operator is defined as $P_{\mathfrak{s}} = q^* \circ p_2^*$ where p_2, q are as in the following diagram.

$$\begin{array}{ccccc} X & \xrightarrow{\ q\ } & X_{K,H,g\hat{T},N_{E/E_0}g\hat{R}} & \xrightarrow{\ p_2\ } & X \\ \downarrow & & \downarrow & & \downarrow \\ M \times N & =\!=\!=\!=\!= & M \times N & \xrightarrow{\ p_2\ } & M \times N \end{array} \tag{6.11}$$

In the diagram, $g \in \mathbb{A}_E^{\infty\times}$ denotes an element whose $\mathfrak{s}$-component is the inverse of a prime element $\pi_{\mathfrak{s}}^{-1}$ and the other components are 1. The map $p_2 = g_*$ is induced by g and the right square is cartesian. The map q is an isogeny corresponding to the inclusions $\hat{T} \to g\hat{T}$ and $\hat{R} \to N_{E/E_0}g \cdot \hat{R}$.

7. Semi-stable model

In the last section, we defined an analogue of the Kuga–Sato variety as an abelian scheme on a Shimura curve. The goal of this section, Lemma 7.1, is to extend it to a semi-stable model of the Shimura curve.

We introduce some terminology. Let $K, H, \hat{T}, \hat{R}$ be as in the last section. We assume that each component of the generic fiber $M_K \otimes_F \bar{F}$ is of genus greater than 1. Then by the stable reduction theorem for curves [DM69], for a sufficiently large finite extension V of the maximal unramified extension $\widehat{F_{\mathfrak{p}}^{\mathrm{nr}}}$, the base change $M_{K,V} = M_K \otimes_F V$ admits a semi-stable model (not necessarily connected) over the integer ring O_V. We do not need to go to the maximal unramified extension to get a semi-stable model. However, since we will work over the maximal unramified extension in the following sections, we state the result as such already in this section. We take the minimal one among the semi-stable models over O_V and denote it by M_{K,O_V}. Recall that we identified the local field $E_{\mathfrak{q}}$ with $F_{\mathfrak{p}}$. From now on, we consider V as an extension of $E_{\mathfrak{q}}$ by this identification. Since N_H is the disjoint union of the spectrum of finite extensions of E, the base change $(M_K \times_F N_H) \otimes_E V$ also admit semi-stable models over the integer ring O_V. We also take the minimal one among them and name it $(M_K \times_F N_H)_{O_V}$. We claim the following.

Lemma 7.1. *Let K, H and V be as above.*

(i) *Let $g \in G(\mathbb{A}^\infty)$, $h \in \mathbb{A}_E^{\infty\times}$ and let $K_1 \subset gKg^{-1}$, $H_1 \subset H$ be open compact subgroups. Assume that the groups are of the form $K = K_{\mathfrak{p}}K^{\mathfrak{p}}$, $g^{-1}K_1g = K_{\mathfrak{p}}(g^{-1}K_1g)^{\mathfrak{p}}$, $H = H_{\mathfrak{q}}H^{\mathfrak{q}}$ and $H_1 = H_{\mathfrak{q}}H_1^{\mathfrak{q}}$. Then the pull-back of the map $(g, h)_* : M_{K_1} \times_F N_{H_1} \to M_K \times_F N_H$ to V extends uniquely to a finite étale morphism $(g, h)_* : (M_{K_1} \times_F N_{H_1})_{O_V} \to (M_K \times_F N_H)_{O_V}$ of the minimal semi-stable model.*

(ii) *Let $\hat{T}, \hat{R}$ be as above. Then the pull-back of the abelian scheme $X_{K,H,\hat{T},\hat{R}} \to M_K \times_F N_H$ to the base extension $(M_K \times_F N_H) \otimes_E V$ extends uniquely to an abelian scheme over a semi-stable model.*

(iii) *Let $\hat{T}_1, \hat{R}_1$ be sublattices in $\hat{T}$ and $\hat{R}$ in assertion (ii), respectively. Assume that their p-components are the same. Then the pull-back of the isogeny $X_{K,H,\hat{T}_1,\hat{R}_1} \to X_{K,H,\hat{T},\hat{R}}$ on $M_K \times_F N_H$ to the base extension $(M_K \times_F N_H) \otimes_E V$ extends uniquely to an étale isogeny over a semi-stable model.*

Proof. (i) We may assume $g = 1$ and $h = 1$. Further we may assume $H = H_1$. In fact, the map $N_{H_1} \to N_H$ is unramified at $\mathfrak{q}$ by the assumption that their $\mathfrak{q}$-components are the same by class field theory. Further, it is sufficient to show that the map $M_{K_1} \to M_K$ extends to a finite étale

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morphism of minimal semi-stable models $M_{K_1,O_V} \to M_{K,O_V}$. In fact, then the fiber product $M_{K_1,O_V} \times_{M_{K,O_V}} (M_K \times N_H)_{O_V}$ is a semi-stable model of $(M_{K_1} \times N_H)_V$ and does not have a (-1)-curve. Hence it is the minimal semi-stable model $(M_{K_1} \times N_H)_{O_V}$ and $(M_{K_1} \times N_H)_{O_V} \to (M_K \times N_H)_{O_V}$ is finite étale.

In the case where the $\mathfrak{p}$-components of $K_{\mathfrak{p}} = K_{1,\mathfrak{p}}$ are $GL_2(O_{F_{\mathfrak{p}}})$, it is shown in [Car86a, Propositions 6.1, 6.2] that the canonical map $M_{K_1,F_{\mathfrak{p}}} \to M_{K,F_{\mathfrak{p}}}$ extends to a finite étale morphism $M_{K_1,O_{F_{\mathfrak{p}}}} \to M_{K,O_{F_{\mathfrak{p}}}}$ of proper, smooth models. We consider the general case. Let $\bar{K} \supset K$, $\bar{K}_1 \supset K_1$ be the groups obtained by replacing their $\mathfrak{p}$-components $K_{\mathfrak{p}} = K_{1,\mathfrak{p}}$ by $GL_2(O_{F_{\mathfrak{p}}})$. First, we show that the canonical map $M_{K,V} \to M_{\bar{K},V}$ extends to the minimal semi-stable model $M_{K,O_V} \to M_{\bar{K},O_V}$. In fact, it extends on a suitable blow-up. However, the exceptional divisors are contracted to points in the image and hence the map is defined on the semi-stable model. We consider the fiber product $M_{\bar{K}_1,O_V} \times_{M_{\bar{K},O_V}} M_{K,O_V}$. It is a semi-stable model of $M_{K_1,V}$ and does not have a (-1)-curve. Hence it is minimal and the assertion is proved.

(ii) We assume there exists an open compact subgroup $K'' \subset G''(\mathbb{A}^\infty)$ containing $K'' \supset KH$ and satisfying the following conditions (a) and (b).

(a) The open subgroup K'' satisfies the conclusion of Lemma 6.1. Namely for a complete set Σ of representatives $\tilde{G}\backslash G''(\mathbb{A}^\infty)/K''$, the map $\amalg g : \coprod_g M'_{K'_g} \to M''_{K''}$ is an isomorphism.

To state the other condition (b), we identify the group $G'(\mathbb{Q}_p)$. By the assumption that E_0 splits at p, we have an isomorphism

$$\begin{array}{ccccc} G'(\mathbb{Q}_p) & = & \mathbb{Q}_p^\times \times (B \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times & \xrightarrow{\sim} & \mathbb{Q}_p^\times \times GL_2(F_{\mathfrak{p}}) \times (B \otimes_F F_p^{\mathfrak{p}})^\times \\ \cap\downarrow & & \cap\downarrow & & \\ G''(\mathbb{Q}_p) & = & (F \otimes \mathbb{Q}_p)^\times \times (B \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times & & \end{array} \tag{7.1}$$

(see [Car86a, (2.6.3)]). Here $F_p^{\mathfrak{p}}$ denote the product $\prod_{\mathfrak{p}'|p,\mathfrak{p}'\neq p} F_{\mathfrak{p}'}$. The second condition is the following.

(b) The intersection $K' = K'' \cap G'(\mathbb{A}^\infty)$ is of the form $K' = \mathbb{Z}_p^\times \times GL_2(O_{F_{\mathfrak{p}}}) \times K_p'^{\mathfrak{p}} \times K'^p$ for some choice of isomorphism as above.

Here $K_p'^{\mathfrak{p}}$ denotes $\prod_{\mathfrak{p}'|p,\neq\mathfrak{p}} K'_{\mathfrak{p}'}$. It is shown in [Car86a, Proposition 5.4] using a modular interpretation that the condition (b) implies that $M'_{K'_g}$ has good reduction over $O_{E_{\mathfrak{q}}}$ and the abelian scheme $A_{K'_g,g\hat{T}}$ on the generic fiber extends to a (unique) proper, smooth model $M'_{K'_g,O_{E_{\mathfrak{q}}}}$. We will recall this modular interpretation in § 9. Hence by the condition (a), $M''_{K''}$ has also good reduction over $O_{E_{\mathfrak{q}}}$ and the abelian scheme $A_{K'',\hat{T}}$ extends to the proper, smooth model $M''_{K'',O_{E_{\mathfrak{q}}}}$. By the same argument as in the proof of (i), the map $(M_K \times N_H)_V \to M''_{K'',V}$ extends uniquely to a map $(M_K \times N_H)_O \to M''_{K'',O_{E_{\mathfrak{q}}}} \otimes O_V$. Hence we obtain the extension of an abelian scheme $X_{K,H,\hat{T},\hat{R}}$, by taking the pull-back.

(iii) Since $(M_K \times_F N_H)_{O_V}$ is normal, an endomorphism on the generic fiber extends to the integral model by a theorem of Grothendieck (see [Gro66]). □

In the proof of assertion (iii), we could also use the modular interpretation recalled in § 9.

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8. Proof of Theorems 2.2 and 2.4

We prove Theorems 2.2 and 2.4 by showing the assertions in Claim 6.4. The argument is the same as in [Sai97] complemented in [Sai00]. Let the notation be as in Claim 6.4. We fix sufficiently small open compact subgroups $K \subset G(\mathbb{A}^\infty)$, $H \subset \mathbb{A}_{E_0}^{\infty\times}$, an $\hat{O}_D$-lattice $\hat{T}$ and $\hat{O}_{E_0}$-lattice $\hat{R}$. To simplify the notation, we will write $M \times N$ for $M_K \times N_H$ and X for $X_{K,H,\hat{T},\hat{R}}$. Recall that we identify $E_\mathfrak{q} = F_\mathfrak{p}$.

We prove that the p-adic representation $H^q(X \otimes_E \bar{E}_\mathfrak{q}, \mathbb{Q}_p)$ of the Galois group $\mathrm{Gal}(\bar{E}_\mathfrak{q}/E_\mathfrak{q})$ is potentially semi-stable. Since we have a semi-stable model X_{O_V} of the base change X_V to an extension V of $E_\mathfrak{q}$ by Lemma 7.1, we may apply the C_{st}-conjecture proved by Tsuji [Tsu99].

We compute $D(H^q(X \otimes_E \bar{E}_\mathfrak{q}, \mathbb{Q}_p))$ in terms of the minimal semi-stable model X_O of X defined in Lemma 7.1. Let Y denote the closed fiber of the minimal semi-stable model X_O with the natural log structure. Then further by [Tsu99], we have a canonical isomorphism

$$D(H^q(X \otimes_E \bar{E}_\mathfrak{q}, \mathbb{Q}_p)) \simeq H^q_{\log\ \mathrm{crys}}(Y/W) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p. \tag{8.1}$$

It follows from the functoriality of the comparison isomorphism for finite étale morphism and from the compatibility with the Poincaré duality that the isomorphism is compatible with the action of endomorphisms and permutations that appeared in Claim 6.4. We define Hecke operators on the log crystalline cohomology and compare them with those on the left-hand side of (8.1) induced by the Hecke operators on the étale cohomology. Let $\mathfrak{n} \subset O_F$ and $\mathfrak{m} \subset O_E$ be sufficiently divisible ideals as in § 6.3. Let $\mathfrak{r} \nmid \mathfrak{n}$ be a prime ideal of O_F. Then the projections p_1, p_2 and the isogeny q described at (6.10) is extended to a finite étale morphism of the minimal semi-stable model by Lemma 7.1. On log crystalline cohomology, we define the Hecke operator $T_\mathfrak{r}$ as the composite $p_{1*} \circ q^* \circ p_2^*$. Similarly we define the Hecke operator $P_\mathfrak{s}$ for a prime ideal $\mathfrak{s} \nmid \mathfrak{m}$ of O_E as the composite $q^* \circ p_2^*$, using the description at (6.11) . Then it follows from the functoriality that the isomorphism is compatible with the Hecke operators thus defined.

We define the Galois action on the log crystalline cohomology and compare it with that on the left-hand side defined in § 2. We may and do assume that the finite extension V of $\widehat{E_\mathfrak{q}^{\mathrm{nr}}}$ is the completion of a Galois extension of $E_\mathfrak{q}$. We have a natural action of the Galois group $G_{E_\mathfrak{q}} = \mathrm{Gal}(\bar{E}_\mathfrak{q}/E_\mathfrak{q})$ on V and hence on the base change M_V. Since the minimal semi-stable model is unique, the action of $G_{E_\mathfrak{q}}$ on the generic fiber M_V extends to the minimal semi-stable model M_{O_V}. Further it uniquely extends to the abelian scheme X_{O_V}. It induces a semi-linear action of the Weil group $W_\mathfrak{q}$ on the log crystalline cohomology. By modifying the action of $\sigma \in W_{E_\mathfrak{q}}$ by $\varphi^{n(\sigma)} \circ \sigma$ as in § 2 and together with the monodromy operator N, we define a linear action of the Weil–Deligne group $W'_{E_\mathfrak{q}}$ on the log crystalline cohomology. We verify the compatibility of the isomorphism with the action of Weil–Deligne group defined above. By transport of the structure, it is compatible with the semi-linear action of the Weil group before modification. Since the comparison isomorphism is compatible with the action of F and N, the compatibility is established.

Therefore, Claim 6.4 is reduced to the following.

Claim 8.1. Let the notation be as in Claim 6.4. Then, the following holds.

(i) Let $\sigma \in W^+ = \{\sigma \in W(\bar{E}_\mathfrak{q}/E_\mathfrak{q}) \mid n(\sigma) \geqslant 0\}, T \in T^\mathfrak{n}, R \in T_0^\mathfrak{m}, \tau \in \mathfrak{S}^g_{w-2}$ and let $\psi : X \to X$ be an endomorphism of degree prime to p. Then for the composite $\Gamma = T \circ R \circ \tau \circ \psi$ as an algebraic correspondence, we have an equality in $\mathbb{Q}$

$$\sum_q (-1)^q \,\mathrm{Tr}(\sigma \circ \Gamma | H^q(X \otimes_E \bar{E}_\mathfrak{q}, \mathbb{Q}_\ell)) = \sum_q (-1)^q \,\mathrm{Tr}(\sigma \circ \Gamma | H^q_{\log\ \mathrm{crys}}(Y/W)). \tag{8.2}$$

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(ii) Let e be the algebraic correspondence in Lemma 6.3 and let $\lambda \nmid p, \mu | p$ be finite places of $L \supset E_0$. Then the monodromy filtration of the representations $e \cdot H^q(X \otimes_E \bar{E}_{\mathfrak{q}}, L_\lambda)$ and $e \cdot (H^q_{\log \text{crys}}(Y/W) \otimes \widehat{L^{\text{nr}}_\mu})$ of the Weil–Deligne group ${}'W(\bar{E}_{\mathfrak{q}}/E_{\mathfrak{q}})$ are pure of weight q.

In assertion (ii), the tensor product is taken with respect to the map $W = O_{\widehat{E^{\text{nr}}_{\mathfrak{q},0}}} \subset \widehat{E^{\text{nr}}_{\mathfrak{q},0}} \xleftarrow{\sim} \widehat{F^{\text{nr}}_{\mathfrak{p},0}} \to \widehat{L^{\text{nr}}_\mu}$ where the last map was fixed in § 2. We prove Claim 8.1 by studying the weight spectral sequences for $\ell \neq p$ and for p.

We prove assertion (i). It suffices to apply Lemma 2 of [Sai97]. However, since we will use the weight spectral sequences in the proof of assertion (ii), we give more details. First, we compute the ℓ-adic side. We consider the weight spectral sequence [RZ82, Ill94]

$$E_1^{i,j} = \bigoplus_{r \geqslant \max(0,-i)} H^{j-2r}(Y^{(i+2r)}, \mathbb{Q}_\ell(-r)) \Rightarrow H^{i+j}(X_{\bar{V}}, \mathbb{Q}_\ell) \tag{8.3}$$

for the semi-stable model X_{O_V}. Here $Y^{(i)}$ denotes the disjoint union of $i+1$ by $i+1$ intersections of the irreducible components of the closed fiber $Y = X \otimes_O \bar{\mathbb{F}}_{\mathfrak{q}}$. The schemes $Y^{(i)}$ are projective and smooth over $\bar{\mathbb{F}}_{\mathfrak{q}}$. We have $Y^{(i)} = \emptyset$ for $i > 1$ since the semi-stable model X_O is proper smooth over the semi-stable model $(M \times N)_O$ of a curve. The spectral sequence degenerates at E_2-terms as a consequence of the Weil conjecture.

Since the action of the Galois group $G_{E_{\mathfrak{q}}}$ extends to the semi-stable model X_{O_V}, the spectral sequence is compatible with its action by transport of structure. It is also compatible with the action of Hecke operators, endomorphisms and permutations by the same argument as in the case of the p-adic comparison isomorphism (8.1). Hence, from the spectral sequence, we immediately deduce that the left-hand side of the equality (8.2) is equal to

$$\sum_i (-1)^i \sum_{r=0}^{i} q_{\mathfrak{q}}^{n(\sigma)r} \sum_q (-1)^q \operatorname{Tr}(\sigma \circ \Gamma | H^q(Y^{(i)}, \mathbb{Q}_\ell)) \tag{8.4}$$

where $q_{\mathfrak{q}}$ denotes the norm of $\mathfrak{q}$.

Let $\sigma \in W^+_{\mathfrak{q}}$ be an element in the Weil group with $n(\sigma) \geqslant 0$. The action of σ on $Y^{(i)}$ is compatible with the action on the base field $\bar{\mathbb{F}}_{\mathfrak{q}}$ and hence is not geometric. Thus, in order to apply the Lefschetz trace formula, we modify it and define an endomorphism σ_{geom} of $Y^{(i)}$ to be $\sigma_{\text{geom}} = \sigma \circ (\text{abs. Frob.})^{[\mathbb{F}_{\mathfrak{q}}:\mathbb{F}_p]\cdot n(\sigma)}$ for each i. It is a geometric endomorphism of a scheme $Y^{(i)}$ over the base field $\bar{\mathbb{F}}_{\mathfrak{q}}$. Since the absolute Frobenius acts trivially on étale cohomology $H^q(Y^{(i)}, \mathbb{Q}_\ell)$, we have $\sigma_* = \sigma_{\text{geom}*}$ as an operator acting on it.

Let Γ_σ denote the composite of σ_{geom} with Γ as an algebraic correspondence and let (Γ_σ, Δ) be the intersection number. We apply the Lefschetz trace formula to a proper, smooth scheme $Y^{(i)}$ and an algebraic correspondence Γ_σ. Then we obtain

$$\sum_q (-1)^q \operatorname{Tr}(\sigma \circ \Gamma | H^q(Y^{(i)}, \mathbb{Q}_\ell)) = (\Gamma_\sigma, \Delta). \tag{8.5}$$

Next we compute the p-adic side. For log crystalline cohomology, we also have the weight spectral sequence (see [Mok93])

$$E_1^{i,j} = \bigoplus_{r \geqslant \max(0,-i)} H^{j-2r}_{\text{crys}}(Y^{(i+2r)}/W)(-r) \Rightarrow H^{i+j}_{\log \text{crys}}(Y/W). \tag{8.6}$$

Here the Tate twist $(-r)$ means that we replace the Frobenius φ by $p^r\varphi$. Since the maps involved in the definitions of the Hecke operators are finite étale, by the same argument as in the ℓ-adic case, we see that the spectral sequence is compatible with the action of Hecke operators, endomorphisms and permutations. It is also compatible with the semi-linear action of the Galois group and the Frobenius operator. Hence by modifying it in the same way on both sides, it is also compatible with the linear action of the Weil group. For $\sigma \in W^+$, the modified action $\sigma_* \circ F^{n(\sigma)[\mathbb{F}_\mathfrak{q}:\mathbb{F}_p]}$ is the same as the action of the geometric endomorphism $\sigma_{\mathrm{geom}} = \sigma \circ (\text{abs. Frob.})^{[\mathbb{F}_\mathfrak{q}:\mathbb{F}_p]\cdot n(\sigma)}$. Hence the right-hand side of (8.2) is equal to

$$\sum_i (-1)^i \sum_{r=0}^{i} q_\mathfrak{q}^{n(\sigma)r} \sum_q (-1)^q \operatorname{Tr}(\sigma_{\mathrm{geom}*} \circ \Gamma | H^q_{\mathrm{crys}}(Y^{(i)}/W)) \tag{8.7}$$

where $q_\mathfrak{q}$ denotes the norm of $\mathfrak{q}$. Again by the Lefschetz trace formula (see [GM87, Gro85]), we have

$$\sum_q (-1)^q \operatorname{Tr}(\sigma_{\mathrm{geom}*} \circ \Gamma | H^q_{\mathrm{crys}}(Y^{(i)}/W)) = (\Gamma_\sigma, \Delta). \tag{8.8}$$

Thus both sides give the same answer and the equality (8.2) is proved.

Finally we prove assertion (ii), the monodromy-weight conjecture. The algebraic correspondence e in Lemma 6.3 acts as an projector on the spectral sequences. We consider their e-parts. We compute the E_1-terms of the e-parts. Let C denote the closed fiber of the semi-stable model $(M \times N)_O$. Then the disjoint union $C^{(0)}$ of the components is the same as the normalization of C and the disjoint union $C^{(1)}$ of the their intersections is the singular locus of C. To describe the E_1-terms, we introduce some sheaves on $C^{(i)}$.

For a place $\lambda | \ell$ of L, we define a smooth L_λ-sheaf $\mathcal{F}^{(k)}_\lambda$ to be $\bigotimes_i (\mathrm{Sym}^{k_i-2} \otimes \det^{(w-k_i)/2})(e_i R^1 a_* L_\lambda) \otimes (e_0 R^1 b_* L_\lambda)^{\otimes (w-2)(g-1)}$. It is the restriction of the extension of $\mathcal{F}^{(k)}_\lambda$ on M to M_O. Similarly for a place $\mu | p$ of L, we define an F-isocrystal $\mathcal{E}^{(k)}_\mu$. We consider F-isocrystals $R^1 a_* \mathcal{O}_{\mathrm{crys}} \otimes_W \widehat{L^{\mathrm{nr}}_\mu}$ and $R^1 b_* \mathcal{O}_{\mathrm{crys}} \otimes_W \widehat{L^{\mathrm{nr}}_\mu}$ where the tensor product is taken as remarked after Claim 8.1. We regard them as an $O_D \otimes_\mathbb{Z} L$-module and an $O_{E_0} \otimes_\mathbb{Z} L$-module respectively. Then we define $\mathcal{E}^{(k)}_\mu$ to be

$$\bigotimes_i (\mathrm{Sym}^{k_i-2} \otimes \det^{(w-k_i)/2})(e_i R^1 a_* \mathcal{O}_{\mathrm{crys}} \otimes_W \widehat{L^{\mathrm{nr}}_\mu}) \otimes (e_0 R^1 b_* \mathcal{O}_{\mathrm{crys}} \otimes_W \widehat{L^{\mathrm{nr}}_\mu})^{\otimes (w-2)(g-1)}. \tag{8.9}$$

Similarly as in Lemma 6.3, we have $eR^q c_* L_\lambda = \mathcal{F}^{(k)}_\lambda$ if $q = q_0 = (w-2)(2g-1)$ and zero otherwise. Also in the p-adic case, we have $eR^q c_* \mathcal{O}_{\mathrm{crys}} \otimes_W \widehat{L^{\mathrm{nr}}_\mu} = \mathcal{E}^{(k)}_\mu$ if $q = q_0$ and zero otherwise. The e-part of the Leray spectral sequence $E_2^{p,q} = H^p(C^{(i)}, R^q c_* L_\lambda) \Rightarrow H^p(Y^{(i)}, L_\lambda)$ degenerates at E_2-terms and defines an isomorphism $H^p(C^{(i)}, \mathcal{F}^{(k)}_\lambda) \xrightarrow{\simeq} e \cdot H^{q_0+p}(Y^{(i)}, L_\lambda)$ as in Lemma 6.3. We have a similar assertion in the p-adic case. Since $H^p(C^{(i)}, \mathcal{F}^{(k)}_\lambda)$ is zero except for $i = 0, p = 0, 1, 2$ and for $i = 1, p = 0$, there are only five non-vanishing E_1-terms

$$\begin{matrix} E_1^{-1,q_0+2} & E_1^{0,q_0+2} & \\ & E_1^{0,q_0+1} & \\ & E_1^{0,q_0} & E_1^{1,q_0} \end{matrix} \tag{8.10}$$

where $q_0 = (2g-1)(w-2)$. Each term is described as follows. In the ℓ-adic setting, we have

$$E_1^{0,q_0+q} = H^q(C^{(0)}, \mathcal{F}^{(k)}_\lambda), \quad E_1^{1,q_0} = E_1^{-1,q_0+2}(1) = H^q(C^{(1)}, \mathcal{F}^{(k)}_\lambda). \tag{8.11}$$

In the crystalline setting, we replace $\mathcal{F}^{(k)}_\lambda$ by $\mathcal{E}^{(k)}_\mu$. The map d_1^{-1,q_0+2} is the Gysin map and d_1^{1,q_0} is the restriction map.

By the Weil conjecture, the eigenvalues of a lifting of the geometric Frobenius acting on each E_1-term $E_1^{i,j}$ are algebraic integers purely of weight j and the spectral sequence degenerates at E_2-terms. Hence the monodromy-weight conjecture is equivalent to the statement that the monodromy filtration is equal to the filtration defined by the weight spectral sequence. By the definition of the monodromy filtration, it is further equivalent to the statement that the monodromy operator induces an isomorphism $E_2^{-1,q_0+2}(1) \to E_2^{1,q_0}$. Since the monodromy operator N on the E_2-terms is induced by the canonical isomorphism $N : E_1^{-1,q_0+2}(1) \to E_1^{1,q_0}$ [Mok93, RZ82], it is further equivalent to the statement that the isomorphism N on the E_1-term induces an isomorphism on E_2-terms. Thus we are reduced to showing the following claim.

Claim 8.2. Let $q_0 = (2g-1)(w-2)$. The canonical map

$$N : E_2^{-1,q_0+2} = \mathrm{Ker}(E_1^{-1,q_0+2} \to E_1^{0,q_0+2})(1) \to E_2^{1,q_0} = \mathrm{Coker}(E_1^{0,q_0} \to E_1^{1,q_0}) \tag{8.12}$$

is an isomorphism.

First we prove it in the case where the multiweight k is of the form $k = (2, 2, \ldots, 2, w)$. In this case, the sheaves $\mathcal{F}_\lambda^{(k)}$ and $\mathcal{E}_\mu^{(k)}$ are constant. Let I be the set of irreducible components and J be the set of singular points. Then it is enough to show that $\mathrm{Ker}(\mathbb{Q}^J \to \mathbb{Q}^I) \to \mathrm{Coker}(\mathbb{Q}^I \to \mathbb{Q}^J)$ is an isomorphism. It is proved easily by extending scalars to $\mathbb{R}$.

We assume the multiweight k is not of the form $k = (2, 2, \ldots, 2, w)$. To show Claim 8.2, we prove Proposition 8.3 below in the next section. To state it, we introduce some terminology. Take a sufficiently small open compact subgroup K'' such that $M''_{K''}$ has a proper, smooth model $M''_{K'',O}$ and that $KH \subset K''$. We consider the natural map $(M \times N)_O \to M''_{K'',O}$. We say a component C_i in $C = (M \times N)_O \otimes_O \bar{\mathbb{F}}_p$ is ordinary, if it dominates a component C'' of the closed fiber of $M''_{K'',O}$. Otherwise, we say it is supersingular.

Proposition 8.3. *Let C_i be an ordinary irreducible component of $(M \times N)_O \otimes_O \bar{\mathbb{F}}_p$. Then we have*

$$H^0(C_i, \mathcal{F}_\lambda^{(k)}) = H^2(C_i, \mathcal{F}_\lambda^{(k)}) = 0, \tag{8.13}$$

$$H^0(C_i, \mathcal{E}_\mu^{(k)}) = H^2(C_i, \mathcal{E}_\mu^{(k)}) = 0 \tag{8.14}$$

unless $k = (2, 2, \ldots, 2, w)$.

The proof will be given in the next section.

We show Claim 8.2, admitting Proposition 8.3. Let $\Sigma \subset (M \times N)_O$ be the union of the image of supersingular components and of singular points. Then for each $s \in \Sigma$ the sheaves $\mathcal{F}_\lambda^{(k)}$ and $\mathcal{E}_\mu^{(k)}$ are constant in the inverse image. Let I_s be the set of supersingular components and J_s be the set of singular points in the inverse image. Then the claim holds if $\mathrm{Ker}(\mathbb{Q}^{J_s} \to \mathbb{Q}^{I_s}) \to \mathrm{Coker}(\mathbb{Q}^{I_s} \to \mathbb{Q}^{J_s})$ is an isomorphism, which is proved in the same way as in the case $k = (2, \ldots, 2, w)$.

9. Vanishing of H^0

We prove Proposition 8.3. First we restate it in terms of the closed fiber of the proper, smooth model of $M'_{K'}$ and Tate modules. Let $K' \subset G'(\mathbb{A}^\infty)$ be a sufficiently small open subgroup satisfying the condition (b) in the proof of Lemma 7.1(ii) in § 7: $K' = \mathbb{Z}_p^\times \times GL_2(O_{F_\mathfrak{p}}) \times K_p'^{\mathfrak{p}} \times K'^p$. Then as is recalled there, Carayol has shown that $M'_{K'}$ has good reduction and the abelian variety $A'_{K',\hat{T}}$ extends to the proper, smooth model $M'_{K',O_{E_\mathfrak{q}}}$. Let C be an irreducible component of the geometric closed fiber $M'_{K',O_{E_\mathfrak{q}}}\otimes\bar{\mathbb{F}}_\mathfrak{q}$. We define a smooth ℓ-adic sheaf $\mathcal{F}_\lambda^{*(k)}$ and an F-isocrystal $\mathcal{E}_\mu^{*(k)}$ on C in a similar way as for $\mathcal{F}_\lambda^{\prime(k)}$: For a place $\lambda|\ell$ of L, we define a smooth L_λ-sheaf $\mathcal{F}_\lambda^{*(k)}$ to be

$$\bigotimes_i(\mathrm{Sym}^{k_i-2}(e_iT_\ell(A)\otimes_{\mathbb{Z}_\ell}L_\lambda)\otimes(\det(e_iT_\ell(A)\otimes_{\mathbb{Z}_\ell}L_\lambda))^{\otimes\frac{w-k_i}{2}}). \tag{9.1}$$

Here the idempotents $e_i\in\mathrm{End}_{M'}(A)\otimes L$ act on $T_\ell(A)\otimes_{\mathbb{Z}_\ell}L_\lambda$ by the covariant functoriality of Tate modules. We define an F-crystal. Let $\mathcal{T}_p(A)$ denote the F-crystal associated to the p-divisible group $A[p^\infty]$ on M'. Let $\mu|p$ be a place of L. We regard the crystal $\mathcal{T}_p(A)\otimes_W\widehat{L_\mu^{\mathrm{nr}}}$ as an $O_D\otimes_{\mathbb{Z}}L$-module by the covariant functoriality as above. For each i, we define an F-isocrystal $\mathcal{E}_i$ to be $e_i(\mathcal{T}_p(A)\otimes_W\widehat{L_\mu^{\mathrm{nr}}})$ and put $\mathcal{E}_\mu^{*(k)}=\bigotimes_i(\mathrm{Sym}^{k_i-2}\mathcal{E}_i\otimes(\det\mathcal{E}_i)^{\otimes(w-k_i)/2})$.

PROPOSITION 9.1. *Let $C'\to C$ be a finite covering of proper smooth curves and assume the multiweight k is not of the form $(2,2,\ldots,2,w)$. For $\lambda|\ell\neq p$, the pull-back to C' of the smooth sheaf $\mathcal{F}_\lambda^{*(k)}$ has no non-trivial (geometrically) constant subsheaf or quotient smooth sheaf. For $\mu|p$, the pull-back to C' of the underlying isocrystal $\mathcal{E}_\mu^{*(k)}$ has no non-trivial constant sub-isocrystal or quotient isocrystal.*

We show that Proposition 9.1 implies Proposition 8.3. Let $C'=C_i$ be an ordinary component as in Proposition 8.3 and $C''\subset M''_{K'',O_{E_\mathfrak{q}}}\otimes\bar{\mathbb{F}}_\mathfrak{q}$ be the image. By the construction of $\mathcal{F}_\lambda^{(k)}$ and $\mathcal{E}_\mu^{(k)}$, we may assume that $C=C''$ is in $M'_{K',O_{E_\mathfrak{q}}}\otimes\bar{\mathbb{F}}_\mathfrak{q}\subset M''_{K'',O_{E_\mathfrak{q}}}\otimes\bar{\mathbb{F}}_\mathfrak{q}$ where K' is as above. Then, Proposition 9.1 implies a similar statement where we replace C', $\mathcal{F}_\lambda^{*(k)}$ and $\mathcal{E}_\mu^{*(k)}$ by C_i, $\mathcal{F}_\lambda^{(k)}$ and $\mathcal{E}_\mu^{(k)}$. It immediately implies the assertion for H^0 in Proposition 8.3. For H^2, it suffices to use Poincaré duality,

Proof of Proposition 9.1 for $\lambda\nmid p$. First, we prove the ℓ-adic case. The argument is similar to the proof of vanishing of H^0 and H^2 in the reduction of the equality (2.8) to (3.9) given in § 3. It is enough to show that the image of the action of $\pi_1(C)$ is sufficiently large. We show that the action on the Tate module defines a surjection $\pi_1(C)\to SK'_\ell=\mathrm{Ker}(\nu:K'_\ell\to\mathbb{Z}_\ell^\times\times(O_E\otimes\mathbb{Z}_\ell)^\times)$. Let V denote the maximal unramified extension of $E_\mathfrak{q}$ and M'^{+}_{K',O_V} be the connected component of the proper smooth model whose closed fiber is C. Since $T_\ell(A)$ is locally constant on M'^{+}_{K',O_V}, the map $\pi_1(M'^{+}_{K',\bar V})\to(\hat O_D^\times)^p$ factors through a surjection $\pi_1(M'^{+}_{K',\bar V})\to\pi_1(M'^{+}_{K',O_V})\simeq\pi_1(C)$. Since $\pi_1(M'^{+}_{K',\bar V})\simeq\pi_1(M'^{+}_{K',\mathbb{C}})\simeq SK'_\ell$, we obtain the surjection. The rest of the argument is identical to the reduction to Claim 3.2 in § 3 and we will not repeat it here. □

To proceed to the crystalline case, we recall the modular interpretation due to Carayol of the integral model of M' over the integer ring $O=O_{E_\mathfrak{q}}$ (see [Car86a]). Let $K'\subset G'(\mathbb{A}^\infty)$ be a sufficiently small open subgroup satisfying the condition (b) in the proof of Lemma 7.1(ii) in § 7: $K'=\mathbb{Z}_p^\times\times GL_2(O_{F_\mathfrak{p}})\times K_p'^{\mathfrak{p}}\times K'^p$. Recall that $K_p'^{\mathfrak{p}}=\prod_{\mathfrak{p}'|p,\neq\mathfrak{p}}K'_{\mathfrak{p}'}$. We take an order $O_D\subset D$ such that $K'\subset\hat O_D^\times$. We take an $\hat O_D$-lattice $\hat T\subset D\otimes\mathbb{A}^\infty$. We assume they satisfy the following conditions:

$$\begin{cases} O_D \text{ is stable under the involution } *,\\ O_D\otimes_{\mathbb{Z}}\mathbb{Z}_p \text{ is maximal in } D\otimes_{\mathbb{Q}}\mathbb{Q}_p,\\ \mathrm{Tr}\,\psi(\hat T,\hat T)\subset\hat{\mathbb{Z}},\\ \text{and }\ \mathrm{Tr}\,\psi(\hat T\otimes_{\hat O_E}O_{E_p},\hat T\otimes_{\hat O_E}O_{E_p})\to\mathbb{Z}_p \text{ is perfect.}\end{cases} \tag{9.2}$$

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We put $\hat{\mathbb{Z}}^p = \prod_{q \neq p} \mathbb{Z}_q$, $\hat{O}_E^p = O_E \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p$, $\hat{T}^p = \hat{T} \otimes_{\hat{\mathbb{Z}}} \hat{\mathbb{Z}}^p$ etc. Then, $\hat{T}^p$ is a free $\hat{O}_E^p$-module of rank four and has a symmetric bilinear form $\operatorname{Tr}\psi : \hat{T}^p \times \hat{T}^p \to \hat{\mathbb{Z}}^p$. We also put $O_{F,p}^{\mathfrak{p}} = \prod_{\mathfrak{p}'|p, \mathfrak{p}' \neq \mathfrak{p}} O_{F_{\mathfrak{p}'}}$. We define a free $O_{F,p}^{\mathfrak{p}}$-module $T_p^{\mathfrak{p}}$ of rank 4 as follows. By the isomorphism $O_E \otimes \mathbb{Z}_p \xrightarrow{\simeq} \prod_{\mathfrak{p}'|p} (O_{F_{\mathfrak{p}'}} \times O_{F_{\mathfrak{p}'}})$, we have a direct sum decomposition $\hat{T} \otimes \mathbb{Z}_p = \prod_{\mathfrak{p}'|p} (\hat{T}_{\mathfrak{p}'_1} \times T_{\mathfrak{p}'_2})$. Here the first factors correspond to the embedding $O_{E_0} \to \mathbb{Z}_p$ fixed in § 4. We put

$$T_p^{\mathfrak{p}} = \prod_{\mathfrak{p}'|p, \neq \mathfrak{p}} \hat{T}_{\mathfrak{p}'_1}. \tag{9.3}$$

For an O_D-abelian scheme A on an $O_{E_{\mathfrak{q}}}$-scheme S, we define direct summands $\operatorname{Lie}^2 A$ and $\operatorname{Lie}^{1,2} A$ of $\operatorname{Lie} A$ similarly as in § 5. We define $T_p^{\mathfrak{p}}(A)$ similarly as $\hat{T}_p^{\mathfrak{p}}$ in (9.3). On the category of schemes over $O_{E_{\mathfrak{q}}}$, there is a proper, smooth model $M'_{K', O_{E_{\mathfrak{q}}}}$ of $M'_{K'}$ representing the functor $S \mapsto \{\text{isomorphism classes of } (A, \theta, \bar{k})\}$ where the following hold.

(i) An O_D-abelian scheme A of dimension $4g$ such that $\operatorname{Lie}^2 A = \operatorname{Lie}^{1,2} A$ and it is a locally free $\mathcal{O}_S$-module of rank two.

(ii) An O_D-polarization $\theta \in \operatorname{Hom}(A, A^*)^{\mathrm{sym}}$ of A.

(iii) A pair $\bar{k} = \bar{k}_p^{\mathfrak{p}} \times \bar{k}^p$ of a $K_p'^{\mathfrak{p}}$-equivalent class of a $\prod_{\mathfrak{p}'|p, \neq \mathfrak{p}} O_{D_{\mathfrak{p}'}}$-isomorphism $k_p^{\mathfrak{p}} : T_p^{\mathfrak{p}}(A) \to T_p^{\mathfrak{p}}$ and a K'^p-equivalent class of a $\prod_{\mathfrak{p}' \nmid p} O_{D_{\mathfrak{p}'}}$-isomorphism $k^p : T^p(A) \to T^p$ such that there exists a $\hat{\mathbb{Z}}^p$-isomorphism k' making the diagram

$$\begin{array}{ccccc} T^p(A) \times T^p(A) & \xrightarrow{(1,\theta_*)} & T^p(A) \times T^p(A^*) & \longrightarrow & \hat{\mathbb{Z}}^p(1) \\ \big\downarrow{\scriptstyle k \times k} & & & & \big\downarrow{\scriptstyle k'} \\ T^p \times T^p & & \xrightarrow[\operatorname{Tr}\psi]{\hspace{6em}} & & \hat{\mathbb{Z}}^p \end{array}$$

commutative.

In condition (iii), the $O_E \otimes \hat{\mathbb{Z}}^p$-module $T^p(A)$ is free of rank four and, by the condition (i), the $\prod_{\mathfrak{q}'|p, \mathfrak{q}' \nmid \mathfrak{q}_0, \mathfrak{q}' \neq \mathfrak{q}} O_{E'_{\mathfrak{q}}}$-module $T_p^{\mathfrak{p}}(A)$ is also free of rank four. As is shown in [Car86a], the generic fiber $M'_{K, O_{E_{\mathfrak{q}}}} \otimes_{O_{E_{\mathfrak{q}}}} E_{\mathfrak{q}}$ represents the restriction of the functor M'_K to the schemes over $E_{\mathfrak{q}}$. Hence the smooth, proper scheme $M'_{K, O_{E_{\mathfrak{q}}}}$ is a model of the base change $M'_K \otimes_E E_{\mathfrak{q}}$ and the universal abelian scheme A is a unique extension on $M'_{K, O_{E_{\mathfrak{q}}}}$ of the pull-back.

We state Lemmas 9.2 and 9.3 on the p-divisible group $A[p^\infty]$ on C. We will deduce Proposition 9.1 in the crystal case from the Lemmas 9.2 and 9.3. As in [Car86a, 2.6.3], we put

$$T_p^{\mathfrak{p}}(A) = T_p(A) \otimes_{O_E \otimes \mathbb{Z}_p} \prod_{\mathfrak{q}'|p, \nmid \mathfrak{q}_0, \nmid \mathfrak{p}} O_{E_{\mathfrak{q}'}}. \tag{9.4}$$

We identify $\prod_{\mathfrak{q}'|p, \nmid \mathfrak{q}_0, \nmid \mathfrak{p}} O_{E_{\mathfrak{q}'}} = \prod_{\mathfrak{p}'|p, \neq \mathfrak{p}} O_{F_{\mathfrak{p}}} = O_{F,p}^{\mathfrak{p}}$ and regard $T_p^{\mathfrak{p}}(A)$ as an $O_{B,p}^{\mathfrak{p}} = \prod_{\mathfrak{p}'|p, \neq \mathfrak{p}} O_{B_{\mathfrak{p}}}$-module. By the modular interpretation recalled above, it is a smooth étale sheaf on the proper scheme $M'_{K', O_{E_{\mathfrak{q}}}}$ of $O_{B,p}^{\mathfrak{p}}$-modules of rank one. Let $\mathfrak{q}_2 | \mathfrak{p}, \neq \mathfrak{q}$ be the other prime ideal of O_E dividing $\mathfrak{p}$. We identify $O_{D_{\mathfrak{q}_2}} = O_{B_{\mathfrak{p}}}$ and take an isomorphism $O_{B_{\mathfrak{p}}} \simeq M_2(O_{F_{\mathfrak{p}}})$. Let $e \in O_{D_{\mathfrak{q}_2}}$ be the idempotent corresponding to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \simeq M_2(O_{F_{\mathfrak{p}}})$. Similarly as in [Car86a, 5.4], let $\mathbb{E}_\infty$ be the p-divisible group

$$\mathbb{E}_\infty = e(A[p^\infty] \otimes_{O_E \otimes \mathbb{Z}_p} O_{E_{\mathfrak{q}_2}}). \tag{9.5}$$

In the terminology of [Car86a, Appendix 1], it is an $O_{F_{\mathfrak{p}}}$-divisible group of height 2. As a p-divisible group, it is of height $2[F_{\mathfrak{p}}:\mathbb{Q}_p]$ and of dimension one. Let $\mathcal{E}_0$ and $\mathcal{T}^{\mathfrak{p}}$ be the F-crystals associated to the p-divisible group $\mathbb{E}_\infty$ and to the Tate module $T_p^{\mathfrak{p}}(A)$, respectively.

The F-isocrystal $\mathcal{E}_\mu^{*(k)} = \bigotimes_i(\mathrm{Sym}^{k_i-2}\mathcal{E}_i \otimes (\det \mathcal{E}_i)^{(w-k_i)/2})$ is related to them in the following way. We regard $\mathcal{E}_0$ and $\mathcal{T}^{\mathfrak{p}}$ as an $O_{F_{\mathfrak{p}}}$-module and an $O_{F,p}^{\mathfrak{p}}$-module, respectively, by the covariant functoriality. Let $I_1 \subset I = \{\tau_i : F \to L\}$ be the subset $I_1 = \{\tau_i : F_{\mathfrak{p}} \to L_\mu\}$. Then for $i \in I_1$, the F-isocrystal $\mathcal{E}_i$ is isomorphic to $\mathcal{E}_0 \otimes_{O_{F_{\mathfrak{p}}}} \widehat{L_\mu^{\mathrm{nr}}}$ with respect to $\tau_i : O_{F_{\mathfrak{p}}} \to \widehat{L_\mu^{\mathrm{nr}}}$. Here we identify $O_{F_{\mathfrak{p}}}$ with $O_{E_{\mathfrak{q}}}$. For $i \in I - I_1$, we take an isomorphism $B \otimes_F L \simeq M_2(L)$ for the tensor product with respect to τ_i and let e be the idempotent corresponding to $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$. Then the F-isocrystal $\mathcal{E}_i$ is isomorphic to $e(\mathcal{T}^{\mathfrak{p}} \otimes_{O_{F,p}^{\mathfrak{p}}} \widehat{L_\mu^{\mathrm{nr}}})$ with respect to $\tau_i : O_{F,p}^{\mathfrak{p}} \to \widehat{L_\mu^{\mathrm{nr}}}$. Here we also identify $O_{E_{\mathfrak{q}'}} = O_{F_{\mathfrak{p}'}}$ for primes $\mathfrak{p}'|p, \neq \mathfrak{p}$ and $\mathfrak{q}'|\mathfrak{p}', \nmid \mathfrak{q}_0$.

It is shown in [Car86a, (6.7), (9.4.3)] that there exists a finite nonempty set $\Sigma \subset C$ of closed points satisfying the following condition.

– At each point in Σ, the p-divisible group $\mathbb{E}_\infty$ is connected. On the complement $U = C - \Sigma$, the p-divisible group $\mathbb{E}_\infty$ is an extension of an étale p-divisible group $\mathbb{E}_\infty^{\text{ét}}$ by a connected p-divisible group $\mathbb{E}_\infty^\circ$.

We call a point in Σ a supersingular point and a point in U an ordinary point. The p-divisible groups $\mathbb{E}_\infty^{\text{ét}}$ and $\mathbb{E}_\infty^\circ$ have natural structures of $O_{F_{\mathfrak{p}}}$-modules. The Tate module $T(\mathbb{E}_\infty^{\text{ét}})$ is a smooth sheaf of $O_{F_{\mathfrak{p}}}$-modules of rank 1.

LEMMA 9.2. *The morphism $\pi_1(U) \to O_{F_{\mathfrak{p}}}^\times \times O_{B,p}^{\mathfrak{p}\times}$ defined by the smooth sheaf $T_p(\mathbb{E}_\infty^{\text{ét}}) \times T_p^{\mathfrak{p}}(A)$ of $O_{F_{\mathfrak{p}}} \times O_{B,p}^{\mathfrak{p}}$-modules of rank 1 defines a surjection*

$$\pi_1(U) \to O_{F_{\mathfrak{p}}}^\times \times SK'^{\mathfrak{q}}. \tag{9.6}$$

Let $\mathcal{E}_0'$ and $\mathcal{E}_0''$ be the F-crystals associated to the p-divisible groups $\mathbb{E}_\infty^{\text{ét}}$ and $\mathbb{E}_\infty^\circ$ on the ordinary locus U, respectively. The restriction of $\mathcal{E}_0$ on U is an extension

$$0 \longrightarrow \mathcal{E}_0' \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{E}_0'' \longrightarrow 0, \tag{9.7}$$

since $\mathbb{E}_\infty$ is an extension.

LEMMA 9.3. *The extension of the underlying isocrystal*

$$0 \longrightarrow \mathcal{E}_0' \otimes \mathbb{Q}_p \longrightarrow \mathcal{E}_0 \otimes \mathbb{Q}_p \longrightarrow \mathcal{E}_0'' \otimes \mathbb{Q}_p \longrightarrow 0 \tag{9.8}$$

is non-trivial.

Proof of Proposition 9.1 for $\mu \mid p$. The argument is similar to that in [Cre92]. First, we prove it, admitting Lemmas 9.2 and 9.3. It is sufficient to show that, on the inverse image $U' \subset C'$ of the ordinary locus U, the restriction of $\mathcal{E}_\mu^{*(k)}$ has no constant sub-isocrystal or quotient isocrystal. Before starting the proof, note that an F-(iso)crystal is constant if and only if the underlying (iso)crystal is constant. In fact, if the underlying (iso)crystal $\mathcal{E}$ is constant, the Frobenius pull-back $F^*\mathcal{E}$ and the Frobenius map $F : F^*\mathcal{E} \to \mathcal{E}$ defining the structure of F-(iso)crystal is constant. The 'only if' part is trivial.

We put $r = [F_{\mathfrak{p}} : \mathbb{Q}_p]$ and $I_1 = \mathrm{Hom}(F_{\mathfrak{p}}, L_\mu) = \{\tau_1, \ldots, \tau_r\} \subset I = \mathrm{Hom}(F, L) = \{\tau_1, \ldots, \tau_g\}$. We define a decreasing filtration on the restriction of $\mathcal{E}_\mu^{*(k)}$ on U with multi-index $\mathbb{Z}^{I_1}$ as follows. On $\mathcal{E}_0$, we define a filtration $F^\bullet$ on $\mathcal{E}_0$ by $F^0\mathcal{E}_0 = \mathcal{E}_0$, $F^1\mathcal{E}_0 = \mathcal{E}_0'$, $F^2\mathcal{E}_0 = 0$. For each $i \in I_1$, it induces a filtration on $\mathcal{E}_i$ and hence on $\mathrm{Sym}^{k_i-2}\mathcal{E}_i$ by the isomorphism $\mathcal{E}_i \simeq \mathcal{E}_0 \otimes_{O_{F_\mathfrak{p}}} L_\mu$. Taking symmetric powers and tensor product, we obtain a filtration on $\mathcal{E}_\mu^{*(k)} = \bigotimes_{i\in I}(\mathrm{Sym}^{k_i-2}\mathcal{E}_i \otimes (\det \mathcal{E}_i)^{\otimes(w-k_i)/2})$. We consider the graded piece $Gr_F^q \mathcal{E}_\mu^{*(k)} = F^q\mathcal{E}_\mu^{*(k)} / \sum_{q'>q} F^{q'}\mathcal{E}_\mu^{*(k)}$ for each $q = (q_1, \ldots, q_r) \in \mathbb{Z}^{I_1}$.

We deduce from Lemma 9.2 that the isocrystal $Gr_F^q\mathcal{E}_\mu^{*(k)}$ has no constant sub-isocrystal or quotient isocrystal except for at most one multi-index $q = (q_1, \ldots, q_r)$ satisfying $(k_1, \ldots, k_g) = (2q_1+2, \ldots, 2q_r+2, 2, \ldots, 2)$. In the exceptional case, we will see that the graded piece is in fact constant. The graded pieces are computed as

$$Gr_F^q\mathcal{E}_\mu^{*(k)} = \bigotimes_{i\in I_1}((\det \mathcal{E}_i)^{\otimes(w-k_i)/2+q_i} \otimes (Gr_F^0\mathcal{E}_i)^{\otimes k_i-2-2q_i}) \otimes \bigotimes_{i\in I-I_1}(\mathrm{Sym}^{k_i-2}\mathcal{E}_i \otimes (\det \mathcal{E}_i)^{\otimes(w-k_i)/2}) \tag{9.9}$$

for $0 \leqslant q_i \leqslant k_i - 2$ for $i \in I_1$ and as zero otherwise. By the Weil pairing of the Drinfeld basis (see [Car86a, 9.2]), the determinant isocrystal $\det \mathcal{E}_i$ is geometrically constant for $i \in I_1$. Similarly, but more easily, $\det \mathcal{E}_i$ is also constant for $i \in I - I_1$. Therefore it is sufficient to show that the isocrystal $\bigotimes_{i\in I_1}(Gr_F^0\mathcal{E}_i)^{\otimes k_i-2-2q_i} \otimes \bigotimes_{i\in I-I_1}\mathrm{Sym}^{k_i-2}\mathcal{E}_i$ has no non-trivial constant sub-isocrystal or quotient isocrystal unless $(k_1, \ldots, k_g) = (2q_1+2, \ldots, 2q_r+2, 2, \ldots, 2)$.

The F-isocrystals $Gr_F^0\mathcal{E}_i$ for $i \in I_1$ and $\mathcal{E}_i$ for $i \in I - I_1$ are defined by smooth p-adic étale sheaves on U. Let $\mathcal{L}_i$ and $\mathcal{F}_i$ be the corresponding smooth p-adic sheaves. Since an F-isocrystal is constant if and only if the underlying crystal is constant, we are reduced to showing that the smooth p-adic sheaf $\bigotimes_{i\in I_1}\mathcal{L}_i^{\otimes k_i-2-2q_i} \otimes \bigotimes_{i\in I-I_1}\mathrm{Sym}^{k_i-2}\mathcal{F}_i$ is irreducible. It follows from the surjectivity of the map $\pi_1(U) \to SK'_p$ (see Lemma 9.2) by the same argument as in the ℓ-adic case.

We complete the proof by using Lemma 9.3. We assume that there exists a non-trivial constant sub-isocrystal of $\mathcal{E}_\mu^{*(k)}$ for $(k_1, \ldots, k_g) \neq (2, \ldots, 2)$ and deduce a contradiction. The proof for the quotient is similar and is omitted. By the study of the graded pieces (9.9), the proof is complete except for the case where k_i are even for $i \in I_1$ and $k_i = 2$ for $i \in I - I_1$. We put $(k_1, \ldots, k_g) = (2q_1+2, \ldots, 2q_r+2, 2, \ldots, 2)$ and assume $q = (q_1, \ldots, q_r) \neq 0$. By the computation of the graded pieces, if we had a non-trivial constant sub-isocrystal, it should be contained in $F^q\mathcal{E}_\mu^{*(k)}$ and mapped isomorphically to $Gr^q\mathcal{E}_\mu^{*(k)}$. Namely, the extension $F^q\mathcal{E}_\mu^{*(k)}$ of $Gr^q\mathcal{E}_\mu^{*(k)}$ is split. Take an index $i \in I_1$ such that $q_i > 0$ and let q' (respectively q'') be the multi-index obtained from q by replacing q_i by $q_i + 1$ (respectively by $q_i + 2$). Then the extension

$$0 \longrightarrow Gr^{q'}\mathcal{E}_\mu^{*(k)} \longrightarrow F^q\mathcal{E}_\mu^{*(k)}/F^{q''}\mathcal{E}_\mu^{*(k)} \longrightarrow Gr^q\mathcal{E}_\mu^{*(k)} \longrightarrow 0$$

is also split. Its extension class is q_i times the class of the extension (9.8) and hence is non-zero. Thus we get a contradiction. We have proved that Lemmas 9.2 and 9.3 imply Proposition 9.1. □

We prove Lemmas 9.2 and 9.3 to complete the proof of Proposition 9.1, hence of Theorems 2.2 and 2.4. We prove Lemma 9.2 using a supersingular point which exists by [Car86a, (9.4.3)]. Lemma 9.3 will be proved using an ordinary point.

Proof of Lemma 9.2. Since $T_p^{\mathfrak{p}}(A)$ is smooth on the proper, smooth model $M'_{K',O_{E_\mathfrak{q}}}$, the same argument as in the ℓ-adic case shows that we have a surjection $\pi_1(C) \to SK_p'^{\mathfrak{p}}$. Take a supersingular point $x \in \Sigma \neq \emptyset$ and let I_x denote the inertia group. It is enough to show that the restriction $I_x \to O_\mathfrak{q}^\times$ is surjective. Let U_n be the finite étale covering $U_n = \mathrm{Isom}(O_{F_\mathfrak{p}}/\mathfrak{p}^n, \mathbb{E}_n^{\text{ét}})$ of U trivializing the $\mathfrak{p}^n$-torsion part $\mathbb{E}_n^{\text{ét}}$. Here an isomorphism means an isomorphism of

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$O_{F_\mathfrak{p}}/\mathfrak{p}^n$-group schemes. The covering U_n is an analogue of an Igusa curve. It is sufficient to show that U_n is totally ramified at a supersingular point. Namely, we show the following lemma. □

LEMMA 9.4. *Let K_x denote the completion of the function field of C at a supersingular point x. Then the base change $U_n \times_C \operatorname{Spec} K_x$ is the spectrum of a totally ramified extension of K_x.*

Proof. Let E denote the formal group associated to the p-divisible group $\mathbb{E}_\infty$ over the completion $\hat{C} = \operatorname{Spec} \hat{\mathcal{O}}_{C,x}$. Let π be a prime element of $O_{F_\mathfrak{p}}$. For an integer n, let $E^{(n)}$ denote the base change of E by the $(q_\mathfrak{p})^n$th power Frobenius and $F^n : E \to E^{(n)}$ be the $(q_\mathfrak{p})^n$th power relative Frobenius over $\hat{C}$. Then the multiplication $[\pi^n] : E \to E$ is factorized as $[\pi^n] = V^n \circ F^n$ for a map $V^n : E^{(n)} \to E$. Outside the closed point x, the map V^n is étale and hence $\operatorname{Ker} V^n$ is a finite flat group scheme over $\hat{C}$ extending the étale quotient $\mathbb{E}_n^{\text{ét}}$ on the generic point. Let $C_n = (\operatorname{Ker} V^n)^\times$ be the scheme of $O_{F_\mathfrak{p}}/\mathfrak{p}^n$-basis of $\operatorname{Ker} V^n$ in the sense of Drinfeld. Namely, it is a closed subscheme of $\operatorname{Ker} V^n$ representing the functor

$$R \mapsto \left\{ s \in \operatorname{Ker} V^n(R) \,\middle|\, \sum_{a \in O_{F_\mathfrak{p}}/\mathfrak{p}^n} [as] = \operatorname{Ker} V^n \text{ as a divisor in } E_R^{(n)} \right\} \tag{9.10}$$

for a ring over $\hat{\mathcal{O}}_{C,x}$. Outside the closed point, the scheme C_n is the same as the base change of U_n. Therefore, it is sufficient to show that C_n is regular and the inverse image of the closed point x by $C_n \to \hat{C}$ contains only one point. The second assertion is clear since C_n is a closed subscheme of a local scheme $\operatorname{Ker} V^n$. We show that the intersection $C_n \cap [0]$ of C_n with the zero-section $[0]$ of the formal group $E^{(n)}$ is equal to $\operatorname{Spec} \kappa(x)$. This will imply that C_n is regular since the zero section is a divisor in $E^{(n)}$.

Let $R = \Gamma(C_n \cap [0], \mathcal{O})$. It is an Artin $\hat{\mathcal{O}}_{C,x}$-algebra. Since $[0]$ is a Cartier divisor of the formal group $E^{(n)}$, it is sufficient to show that the surjection $\hat{\mathcal{O}}_{C,x} \to R$ factors through the surjection $\hat{\mathcal{O}}_{C,x} \to \kappa(x)$. By the assumption, the zero-section is an $O_{F_\mathfrak{p}}/\mathfrak{p}^n$-basis of $\operatorname{Ker} V^n$. Hence, we have $\operatorname{Ker}[\pi^n] = \operatorname{Ker} F^{2n}$ on R and an isomorphism $E_R \simeq E_R^{(2n)} \simeq E_R^{(2mn)}$ for $m \geqslant 1$. Since R is Artinian, for sufficiently large m, the map $a \to a^{(q_\mathfrak{q})^{2mn}}$ factors through $R \to \kappa(x) \to R$ and we obtain $E_R \simeq E_R^{(2mn)} \simeq E_x \otimes_{\kappa(x)} R$. This means that $\hat{\mathcal{O}}_{C,x} \to R$ factors through $\kappa(x)$ since $\mathbb{E}_\infty$ over $\hat{\mathcal{O}}_{C,x}$ is the universal deformation of $\mathbb{E}_\infty|_x$, [Car86a, Proposition 5.4]. Thus we have proved Lemma 9.4 and hence Lemma 9.2. □

To prove Lemma 9.3, we show the following.

LEMMA 9.5. *Let $\hat{C} = \operatorname{Spec} \hat{\mathcal{O}}_{C,x}$ be the completion at an ordinary closed point $x \in U$. Let $[\mathbb{E}] \in \operatorname{Ext}^1(\mathbb{E}^{\text{ét}}, \mathbb{E}^\circ)$ be the class of $\mathbb{E}$ as an extension of $O_{F_\mathfrak{p}}$-divisible groups on $\hat{C}$. Then the class $[\mathbb{E}]$ is not torsion.*

We derive it from the following statement proved in [Car86a, Proposition 5.4, App. Théorème 3].

LEMMA 9.6. *On the completion $\hat{C}$ at an ordinary closed point, the connected part $\mathbb{E}^\circ$ is isomorphic to the pull-back of the Lubin–Tate formal group. The étale part $\mathbb{E}^{\text{ét}}$ is isomorphic to the constant $O_{F_\mathfrak{p}}$-divisible group $F_\mathfrak{p}/O_{F_\mathfrak{p}}$. The completion $\hat{C}$ pro-represents the functor $R \mapsto \operatorname{Ext}_R(F_\mathfrak{p}/O_{F_\mathfrak{p}}, \mathbb{E}_0) = \mathbb{E}_0(R)$ on the category of Artin $\mathbb{F}_\mathfrak{p}$-algebras R together with a surjection $R \to \kappa(x)$. It is isomorphic to $\mathbb{E}^0$ as a formal scheme. The extension $\mathbb{E}$ on $\hat{C} = \mathbb{E}^0$ is identified with the universal extension.*

Proof of Lemma 9.5. We identify the formal schemes $\mathbb{E}^0 = \hat{C}$. By Lemma 9.6, the universal extension $\mathbb{E}$ corresponds to the identity $C \to \mathbb{E}^0$. Hence it is the universal section of the formal group $\mathbb{E}^0$ and is not torsion. □

Proof of Lemma 9.3. It is enough to prove that the restriction to the completion at an ordinary closed point is not the trivial extension. Since the p-divisible groups $\mathbb{E}^\circ$ and $\mathbb{E}^{\text{ét}}$ are constant on $\hat{C}$, the F-isocrystals $\mathcal{E}' \otimes \mathbb{Q}_p$, $\mathcal{E}'' \otimes \mathbb{Q}_p$ and hence their underlying crystals are constant there. If the extension of the underlying isocrystal were trivial, the underlying isocrystal and hence the F-isocrystal $\mathcal{E} \otimes \mathbb{Q}_p$ would be constant. It means that the extension class $[\mathbb{E}] \in \text{Ext}^1(\mathbb{E}^{\text{ét}}, \mathbb{E}^\circ)$ is torsion and contradicts Lemma 9.5. □

Thus the proof of Proposition 9.1 and hence of Theorems 2.2 and 2.4 are now complete.

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Added in proof. The assumption (C) in Theorem 2.2 is also removed in

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