Can. J. Math., Vol. XXXII, No. 2, 1980, pp. 310–316

RATIONAL APPROXIMATION TO x^n II

D. J. NEWMAN AND A. R. REDDY

Introduction. In 1858 Chebyshev showed that x^{n+1} can be approximated uniformly on $[-1, 1]$ by polynomials of degree at most n with an error 2^{-n}. Let $0 \leqq \sigma \leqq (n + 1)\tan^2(\pi/2n + 2)$. In 1868 Zolotarev established that $x^{n+1} - \sigma x^n$ can be approximated uniformly on $[-1, 1]$ by polynomials of degree at most $(n - 1)$ with an error $2^{-n}(1 + \sigma/n + 1)^{n+1}$. It is interesting to note that for the case $\sigma = 0$, Zolotarev's result includes Chebyshev's result. Achieser ([**1**], p. 279) proved the following analogue for rational approximation. Let $a_0 \neq 0$, $a_1, a_2, a_3, \ldots, a_n$ be any given real numbers. Then for every $N > n$,

$$\min_{\alpha_i, \beta_i} \max_{-1 \leqq x \leqq 1} \left| \sum_{\nu=0}^{n} a_\nu 2^{-\nu} x^{N-\nu} - \frac{\sum_{i=0}^{N-1} \alpha_i x^i}{\sum_{i=0}^{n} \beta_i x^i} \right| = \frac{|\lambda|}{2^{N-1}},$$

where λ is numerically the smallest root of the polynomial

$$\begin{vmatrix} c_n - \lambda & c_{n-1} & \cdots & c_1 & c_0 \\ c_{n-1} & c_{n-2} - \lambda & \cdots & c_0 & 0 \\ \cdot & \cdot & & & \\ \cdot & \cdot & \cdots & & \\ \cdot & \cdot & & & \\ c_1 & c_0 & \cdots & -\lambda & 0 \\ c_0 & 0 & \cdots & 0 & -\lambda \end{vmatrix}$$

with

$$c_m = \sum_{i=0}^{[m/2]} a_{m-2i} \binom{N - m + 2i}{i}, \quad (m = 0, 1, 2, 3, \ldots, n).$$

Achieser's result fails to give information when one wishes to approximate x^{n+1} on $[-1, 1]$ by rational functions of the form $p_{n-1}(x)/q_m(x)$, where $m > n$. In this connection Newman [**2**] has proved the following:

THEOREM N. *Let s and n be any non-negative integers; we have then*

I. *There is a $p(x)$ of degree $<n$ and a $q(x)$ of degree $2s$ such that throughout* $[-1, 1]$

$$(1) \qquad \left| x^n - \frac{p(x)}{q(x)} \right| \leqq 2^{1-n} \binom{s + n - 3}{s}^{-1}.$$

Received March 21, 1977 and in revised form January 15, 1980.

II. *If $p(x)$ is of degree $<n$ and $q(x)$ is of degree $\leqq 2s$ then, somewhere in* $[-1, 1]$

$$(2) \qquad \left| x^n - \frac{p(x)}{q(x)} \right| \geqq 2^{-2-n} \binom{s+n+1}{s}^{-1}.$$

The above results of Achieser [**1**] and Newman [**2**] fail to provide information regarding the approximation of x^n on $[0, 1]$ by reciprocals of polynomials of degree n. When n is small and s is large the bounds obtained in (1) and (2) do not match each other.

In Theorems 1 and 2 of this paper we obtain error estimates to x^n on $[0, 1]$ by reciprocals of polynomials of degree n. In Theorem 3 we obtain a lower estimate to x^n on $[0, 1]$ by rational functions of the form $p_{l-1}(x)/q_m(x)$ for each $0 \leqq l \leqq n-1$, and $m \geqq 0$. In Theorem 4 we obtain an upper estimate to x^k on $[0, 1]$ by rational functions of the form $x^{k-1}/p_n(x)$.

Notation. Let $g(x) = \sum_{k=-\infty}^{\infty} a_k x^k$. We denote the analytic part of the series as $A(g(x)) = \sum_{k=0}^{\infty} a_k x^k$. As usual $T_n(x)$ denotes the Chebyshev polynomial of degree n. Throughout our work we use $\|p(x)\|$ to denote $\max_{-1 \leqq x \leqq 1} |p(x)|$.

LEMMA 1. [**2**] *Let $p(x)$ be any polynomial of degree $\leqq m$, and $\|p(x)\| \leqq 1$. Then*

$$\left\| A\left(\frac{p(x)}{x^1}\right) \right\| \leqq 2^{n+2} \binom{N+1}{n+1},$$

where

$$N = \left[\frac{m+n}{2}\right].$$

LEMMA 2. ([**4**], p. 68) *Let $p(x)$ be a polynomial of degree at most n satisfying the assumption that* $\max |p(x)| \leqq L$ *on the segment $[a, b]$. Then at any point outside the segment we have*

$$|p(x)| \leqq L \left| T_n\left(\frac{2x-a-b}{b-a}\right) \right|.$$

Theorems.

THEOREM 1. *For all $n \geqq 4$*

$$(6) \qquad \left\| x^n - \frac{1}{\sum_{k=0}^{2n-1} \binom{n+k-1}{k} (1-x)^k} \right\|_{L_\infty[0,1]} \leqq 16 n^2 \left(\frac{27}{64}\right)^n.$$

https://doi.org/10.4153/CJM-1980-023-9 Published online by Cambridge University Press

Proof. For convenience we prove

$$(7)\qquad \left\| (1-y)^n - \frac{1}{\sum_{k=0}^{2n-1} \binom{n+k-1}{k} y^k} \right\|_{L_\infty[0,1]} \leqq 16n^2 \left(\frac{27}{64}\right)^n.$$

It is well known that

$$(1-y)^{-n} = \sum_{k=0}^{\infty} \binom{n-1+k}{k} y^k.$$

Set

$$(8)\qquad \begin{aligned} p(y) &= \sum_{k=0}^{2n-1} \binom{n-1+k}{k} y^k, \\ q(y) &= (1-y)^{-n} - p(y). \end{aligned}$$

Then for $0 \leqq y \leqq 2/3$

$$(9)\qquad 0 \leqq \frac{1}{p(y)} - (1-y)^n = \frac{1}{(1-y)^{-n} - q(y)} - (1-y)^n$$

$$= \frac{q(y)}{(1-y)^{-n} p(y)} = \frac{\sum_{k=2n}^{\infty} \binom{n+k-1}{k} y^k}{(1-y)^{-n} \sum_{k=0}^{2n-1} \binom{n+k-1}{k} y^k}$$

$$\leqq \frac{\binom{3n-1}{2n} y^{2n} \sum_{k=0}^{\infty} \left(\frac{3n}{2n+1}\right)^k y^k}{\binom{2n-1}{n}^2 y^{2n}}$$

$$\leqq (2n+1) \binom{3n-1}{2n} \binom{2n-1}{n}^{-2}.$$

On the other hand, for $2/3 \leqq y \leqq 1$,

$$(10)\qquad 0 \leqq \frac{1}{p(y)} - (1-y)^n \leqq \frac{1}{p(y)} \leqq \frac{1}{p(2/3)} \leqq \frac{2}{3} \binom{3n-2}{2n-1}^{-1} \left(\frac{3}{2}\right)^{2n}.$$

Hence for $0 \leqq y \leqq 1$,

$$\left\| (1-y)^n - \frac{1}{\sum_{k=0}^{2n-1} \binom{n-1+k}{k} y^k} \right\|_{L_\infty[0,1]} \leqq 16n^2 \left(\frac{27}{64}\right)^n.$$

Our result (7) follows from (8), (9) and (10). (6) follows from (7) by choosing $1 - y = x$.

https://doi.org/10.4153/CJM-1980-023-9 Published online by Cambridge University Press

THEOREM 2. *Let $p(x)$ be any polynomial of degree at most m. Then for all $m \geqq 1$ and $n \geqq 1$,*

$$(11)\quad \left\| x^n - \frac{1}{p(x)} \right\|_{L\infty[0,1]} \geqq 2^{-n-1}(3+2\sqrt{2})^{-m}.$$

Proof. For any given $p(x)$ of degree at most m, let

$$(12)\quad \left\| x^n - \frac{1}{p(x)} \right\|_{L\infty[0,1]} = \delta.$$

From (12), we get on $[1/2, 1]$

$$(13)\quad \frac{1}{p(x)} \geqq x^n - \delta \geqq 2^{-n} - \delta.$$

Two cases will arise in (13), for if $2^{-n} - \delta \leqq 0$, then

$$(14)\quad \delta \geqq 2^{-n}.$$

Otherwise

$$(15)\quad \max_{[1/2,1]} |p(x)| \leqq \frac{2^n}{1-2^n\delta}.$$

By applying Lemma 2 to (15) we obtain

$$(16)\quad |p(0)| \leqq \max_{[0,1]} |p(x)| \leqq \frac{2^n(3+2\sqrt{2})^m}{1-2^n\delta}.$$

On the other hand we get from (12)

$$(16')\quad 1/\delta \leqq |p(0)|.$$

We obtain from (16) and (16′)

$$(17)\quad \delta^{-1} \leqq \frac{2^n(3+2\sqrt{2})^m}{1-2^n\delta}.$$

A simple calculation based on (17) will give us

$$(18)\quad \delta \geqq 2^{-n-1}(3+2\sqrt{2})^{-m}.$$

(11) follows from (14) and (18).

THEOREM 3. *Let $p(x)$ and $q(x)$ be any polynomials of degrees at most l $(0 \leqq l \leqq n-1)$ and m $(m \geqq 0)$ respectively. Then*

(i) *For $l = n - 1$*

$$(19)\quad \left\| x^n - \frac{p(x)}{q(x)} \right\|_{L\infty[0,1]} \geqq \frac{m!(2n)!}{(m+2n-1)!2^{2n}(m+n)}.$$

(ii) *For $0 \leqq l \leqq n-1$, and $m = 2s$ (s is any positive integer),*

$$(20)\quad \left\| x \;- \frac{p(x)}{q(x)} \right\|_{L\infty[0,1]} \geqq \frac{(2s+n-l-1)!(2l+2)!2^{-2n-2}}{(2s+n+l)!(2s+n)\binom{2s+2n-2l}{2n-2l-1}}.$$

https://doi.org/10.4153/CJM-1980-023-9 Published online by Cambridge University Press

Proof. Set

$$(21)\quad \left\| x^n - \frac{p(x)}{q(x)} \right\|_{L\infty[0,1]} = \epsilon .$$

Denote

$$(22)\quad f(x) = x^n q(x) - p(x), \quad g(x) = x^n q(x).$$

Normalize $f(x)$ such that

$$(23)\quad \max_{0\leqq x\leqq 1}|f(x)| = 1.$$

It is easy to verify that

$$(24)\quad f^{(l+1)}(x) = g^{(l+1)}(x), \quad g^{(k)}(0) = 0, \quad k = 1, 2, \ldots, l.$$

Now by applying the well known Markov inequality ([**5**], p. 279) to (23), we get

$$(25)\quad \max_{0\leqq x\leqq 1} |f^{(l+1)}(x)| \leqq \frac{2^{2l+2}(l+1)!(m+n)(m+n+l)!}{(m+n-1-l)!(2l+2)!} .$$

From (24) one can easily write

$$(26)\quad g(x) = \int_0^x \int_0^{y_l} \cdots \int_0^{y_3} \int_0^{y_2} \int_0^{y_1} f^{(l+1)}(y)dydy_1 \ldots dy_l .$$

Then we obtain from (22), (25) and (26)

$$(27)\quad |x^n q(x)| = |g(x)| \leqq \frac{x^{l+1}}{(l+1)!} \max_{0\leqq x\leqq 1} |f^{(l+1)}(x)|$$
$$\leqq \frac{x^{l+1}(m+n+l)!2^{2l+2}(m+n)}{(m+n-1-l)!(2l+2)!} ,$$

if $l = n - 1$, then we get from (27)

$$(28)\quad \max_{0\leqq x\leqq 1} |q(x)| \leqq \frac{(m+2n-1)!2^{2n}(m+n)}{m!(2n)!} .$$

If $0 \leqq l \leqq -2$ then choose $T(x) = x^{n-l-1}q(x)$. It is obvious that $T(x)$ is a polynomial of degree $m + n - l - 1$. Now by applying Lemma 1 over the interval $[0, 1]$ instead of $[-1, 1]$, to $T(x)$ we get along with (27),

$$(29)\quad \max_{0\leqq x\leqq 1} |q(x)| \leqq \frac{2^{2n}(m+n+l)!(m+n)}{(m+n-l-1)!(2l+2)!} \binom{m+2n-2l}{2n-2l-1},$$

From (21) and (23) we get

$$(30)\quad \epsilon = \max_{0\leqq x\leqq 1} \left| x^n - \frac{p(x)}{q(x)} \right| = \max_{0\leqq x\leqq 1} \left| \frac{x^n q(x) - p(x)}{q(x)} \right| \geqq \frac{1}{\max_{0\leqq x\leqq 1} |q(x)|} .$$

https://doi.org/10.4153/CJM-1980-023-9 Published online by Cambridge University Press

If $l = n - 1$, then we get from (28) and (30),

$$(31) \quad \epsilon \geqq \frac{m!(2n)!}{(m + 2n - 1)!2^{2n}(m + n)}.$$

If $0 \leqq l \leqq n - 2$, then we get from (29) and (30), for $m = 2s$

$$(32) \quad \epsilon \geqq \frac{(2s + n + l - 1)!(2l + 2)!2^{-2n-2}}{(2s + n + l)!(2s + n)\binom{2s + 2n - 2l}{2n - 2l - 1}}.$$

Hence (19) follows from (31) and (20) follows from (32).

THEOREM 4. *Let k be a real positive integer satisfying the assumption that $0 < m^{-1}4k \log m < 1$. Then there exists a polynomial $q(x)$ of degree m and a positive constant c satisfying*

$$(33) \quad \left\| x^k - \frac{x^{k-1}}{q_m(x)} \right\|_{L\infty[0,1]} \leqq c\left(\frac{\log m}{m}\right)^{2k-2}.$$

Proof. Choose m to be even and $\delta = (4km^{-1} \log m)^2$. Set

$$(34) \quad q_m(x) = \frac{T_{m+1}(1 + \delta) - T_{m+1}(1 + \delta - (2 + \delta)x)}{xT_{m+1}(1 + \delta)},$$

where as usual $T_m(x)$ denotes the Chebyshev polynomial of degree m.

It is easy to verify that $q_m(x)$ is a polynomial of degree at most m. Then for $0 \leqq x \leqq \delta(2 + \delta)^{-1}$,

$$(35) \quad \left| x^k - \frac{x^{k-1}}{q_m(x)} \right|$$

$$= \left| x^k - \frac{x^k T_{m+1}(1 + \delta)}{T_{m+1}(1 + \delta) - T_{m+1}((1 + \delta) - (1 + 2\delta)x)} \right|$$

$$\leqq x^k \left| \frac{T_{m+1}(1 + \delta - (1 + 2\delta)x)}{T_{m+1}(1 + \delta) - T_{m+1}(1 + \delta - (1 + 2\delta)x)} \right| = x^{k-1}L$$

$$\leqq c_1\left(\frac{\delta}{2 + \delta}\right)^{k-1} \leqq c_2\left(\frac{\log m}{m}\right)^{2k-2}$$

since for $0 \leqq x \leqq \delta(2 + \delta)^{-1}$, $L \leqq C_1$. For $\delta(2 + \delta)^{-1} \leqq x \leqq 1$,

$$(36) \quad x^k \left| \frac{T_{m+1}(1 + \delta - (1 + 2\delta)x)}{T_{m+1}(1 + \delta) - T_{m+1}(1 + \delta - (1 + 2\delta)x)} \right|$$

$$\leqq \frac{1}{T_{m+1}(1 + \delta) - 1} \leqq \left(\exp\left(\frac{m}{2}\sqrt{\delta}\right) - 1\right)^{-1} \leqq 2m^{-2k}.$$

(33) follows from (35) and (36).

Remarks on Theorems 3 *and* 4. It is interesting to note that the error estimates obtained in (33) cannot be improved very much. From (32) we can get with $n = k$, $l = k - 1$, for some constant $c_3 > 0$,

$$\left\| x^k - \frac{p(x)}{q(x)} \right\|_{L_\infty[0,1]} \geqq \frac{c_3}{m^{2k}}.$$

Concluding remarks. The approximation to x^n on [0, 1] by polynomials and rational functions of degree at most n having only non-negative coefficients has been considered in [**3**].

REFERENCES

1. N. I. Achieser, *Theory of approximation* (Frederick Ungar Publishing Co., New York, 1956).
2. D. J. Newman, *Rational approximation to x^n*, J. Approximation Theory *22* (1978), 285–288.
3. D. J. Newman and A. R. Reddy, *Rational approximation to x^n*, Pacific J. Mathematics *67* (1976), 247–250.
4. A. F. Timan, *Theory of approximation of functions of a real variable*, International series of monographs in pure and applied mathematics *34* (Macmillan, New York, 1963).
5. E. V. Voronovskaja, *The functional method and its applications* (English translation), American Math. Society, Providence, RI (1970).

Temple University,
Philadelphia, Pennsylvania;
University of Hyderabad,
Hyderabad, India

https://doi.org/10.4153/CJM-1980-023-9 Published online by Cambridge University Press