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A GENERALIZATION OF THE CONCEPT OF A RING OF QUOTIENTS

BY

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ABSTRACT. Sanderson (Canad. Math. Bull., **8** (1965), 505–513), considering a nonempty collection Σ of left ideals of a ring R, with unity, defined the concepts of "Σ-injective module" and "Σ-essential extension" for unital left modules. Letting Σ be an idempotent topologizing set (called a σ-set below) Sanderson proved the existence of a "Σ-injective hull" for any unital left module and constructed an Utumi Σ-quotient ring of R as the bicommutant of the Σ-injective hull of ${}_RR$. In this paper, we extend the concepts of "Σ-injective module", "Σ-essential extension", and "Σ-injective hull" to modules over arbitrary rings. An overring S of a ring R is a Johnson (Utumi) left Σ-quotient ring of R if ${}_RR$ is Σ-essential (Σ-dense) in ${}_RS$. The maximal Johnson and Utumi Σ-quotient rings of R are constructed similar to the original method of Johnson, and conditions are given to insure their equality. The maximal Utumi Σ-quotient ring U of R is shown to be the bicommutant of the Σ-injective hull of ${}_RR$ when R has unity. We also obtain a σ-set $U\Sigma$ of left ideals of U, generated by Σ, and prove that U is its own maximal Utumi $U\Sigma$-quotient ring. A Σ-singular left ideal $Z_\Sigma(R)$ of R is defined and U is shown to be $U\Sigma$-injective when $Z_\Sigma(R) = 0$. The maximal Utumi Σ-quotient rings of matrix rings and direct products of rings are discussed, and the quotient rings of this paper are compared with these of Gabriel (Bull. Soc. Math. France, **90** (1962), 323–448) and Mewborn (Duke Math. J. **35** (1968), 575–580). Our results reduce to those of Johnson and Utumi when $1 \in R$ and Σ is taken to be the set of all left ideals of R.

A ring S containing a ring R is a left (right) quotient ring of R if ${}_RS$ (resp. S_R) is an essential extension of ${}_RR$ (resp. R_R). Sanderson [9], considering a nonempty collection Σ of left ideals of a ring R with unity, defined the concepts of "Σ-injective module" and "Σ-essential extension" for unital left modules. Letting Σ be an idempotent topologizing set (called a σ-set below) Sanderson proved the existence of a "Σ-injective hull" for any unital left module and constructed an Utumi Σ-quotient ring of R as the bicommutant of the Σ-injective hull of ${}_RR$. In §1 of this paper, we extend the concepts of "Σ-injective module", "Σ-essential extension", and "Σ-injective hull" to modules over arbitrary rings. In §2, we call an overring S of a ring R a Johnson left Σ-quotient ring of R if ${}_RR$ is Σ-essential in ${}_RS$, and construct a maximal Johnson left Σ-quotient ring. The Utumi maximal Σ-quotient

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ring of R is constructed in §3, and characterized as the bicommutant of the Σ-injective hull of R when R has identity in §4. In §5 and §6, the Utumi maximal Σ-quotient ring of a matrix ring and a direct product of rings are discussed, and in §8, the quotient rings of this paper are compared with the quotient rings of Gabriel as discussed in Bourbaki [2]. Our results reduce to those of Johnson [5] and Utumi [11] when $1 \in R$ and Σ is taken as the set of all left ideals of R.

1. **Preliminaries.** Throughout this paper R will denote a ring and all R-modules will be left R-modules unless otherwise stated. A *σ-set* for R is a nonempty set Σ of left ideals of R satisfying the following conditions:

(σ_1) If $I \in \Sigma$, J is a left ideal of R, and $J \supseteq I$, then $J \in \Sigma$.
(σ_2) If $I \in \Sigma$ and $r \in R$, then $Ir^{-1} = \{s \in R \mid sr \in I\} \in \Sigma$.
(σ_3) If I is a left ideal of R, $J \in \Sigma$, and $It^{-1} \in \Sigma$ for all $t \in J$, then $I \in \Sigma$.

It is clear from the following Lemma that a σ-set Σ is what Gabriel calls an idempotent topologizing set of left ideals [2, pp. 157–165].

LEMMA 1.1. *A σ-set Σ is closed under finite intersections.*

Proof. Let $J, K \in \Sigma$. For each $t \in J$, $Kt^{-1} \in \Sigma$ by (σ_2). But $Kt^{-1} \subseteq (K \cap J)t^{-1}$, so $(K \cap J)t^{-1} \in \Sigma$ for all $t \in J$. Hence $K \cap J \in \Sigma$ by (σ_3) and the lemma is proved.

Sanderson [9] attributes Lemma 1.1 to Chew, but indicates no proof.

DEFINITION. A submodule ${}_RM$ of ${}_RN$ is *essential* in ${}_RN$ if for each $0 \neq x \in N$, $(x) \cap M \neq (0)$ where (x) is the submodule of N generated by x.

We generalize this definition as follows:

DEFINITION. Let Σ be a σ-set for R. Then ${}_RN$ is a *Σ-essential extension* of ${}_RM$ (${}_RM$ is *Σ-essential* in ${}_RN$) if ${}_RM$ is essential in ${}_RN$ and $Mx^{-1} = \{r \in R \mid rx \in M\} \in \Sigma$ for all $x \in N$.

We call ${}_RM$ *Σ-injective* if for any module ${}_RA$ and Σ-essential extension ${}_RB$ of ${}_RA$, each R-homomorphism $f: A \to M$ can be extended to an R-homomorphism $\bar{f}: B \to M$.

Now let R be a ring with 1, and *all modules considered be unital.* Then ${}_RM$ is Σ-essential in ${}_RN$ iff for each $0 \neq x \in N$, $Mx^{-1} \in \Sigma$ and $(Mx^{-1})x \neq 0$. Moreover, an essential left ideal $I \in \Sigma$ is Σ-essential in ${}_RR$. Hence for a Σ-injective R-module M, each R-homomorphism from an essential left ideal $I \in \Sigma$ has an extension to all of R. Since each R-homomorphism from a left ideal $I \in \Sigma$ to M can be extended to an essential left ideal, necessarily in Σ, a Σ-injective unital R-module M satisfies the following property:

(S) Each R-homomorphism from a left ideal $I \in \Sigma$ to M can be extended to R.

Let M be a unital R-module satisfying property (S), ${}_RB$ a Σ-essential extension of ${}_RA$, and $f: A \to M$ an R-homomorphism. By Zorn's Lemma, choose ${}_RC$ with

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${}_RA \subseteq {}_RC \subseteq {}_RB$ maximal with respect to the existence of an R-homomorphism $h\colon C \to M$ extending f. If $C \neq B$, let $0 \neq b \in B \backslash C$; then $Cb^{-1} \in \Sigma$ since $Cb^{-1} \supseteq Ab^{-1}$. By property ($S$), the map $g\colon Cb^{-1} \to M$ given by $g(c) = h(cb)$ has an extension $\bar{g}\colon R \to M$. Define $\bar{f}\colon C + Rb \to M$ by $\bar{f}(c + rb) = h(c) + \bar{g}(r)$. Then if $rb \in C$, $r \in Cb^{-1}$ and $h(rb) = g(r) = \bar{g}(r)$; hence f is a well-defined R-homomorphism. This contradicts the maximality of C. Thus $B = C$.

We have proved the following:

PROPOSITION 1.2. *Let R have* 1 *and M be a unital left R-module. Then M is Σ-injective iff M satisfies property* (S).

The definition of Σ-injective for unital R-modules by property (S) is due to Sanderson [9], and so our definitions extend his.

Eckmann and Schopf [3] proved the existence of an injective hull for any R-module. That each R-module has a Σ-injective hull is shown by the following two theorems:

THEOREM 1.3 [9]. *Let R be a ring with* 1 *and M be a unital R-module. There exists an extension $E_\Sigma(M)$ of M, unique up to isomorphism over M, satisfying the following equivalent conditions*:

(1) *$E_\Sigma(M)$ is a maximal Σ-essential extension of M;*
(2) *$E_\Sigma(M)$ is a minimal Σ-injective extension of M;*
(3) *$E_\Sigma(M)$ is a Σ-injective Σ-essential extension of M.*

Let R be a ring without identity and $R^1 = R \times \mathbf{Z}$ be the ring R with identity adjoined. Then R^1 is a ring with identity, each R-module is a unital R^1-module, and for modules ${}_RM$ and ${}_RN$, and $\mathrm{Hom}_R(M, N) = \mathrm{Hom}_{R^1}(M, N)$.

DEFINITION. $\Sigma^1 = \{T \mid T$ is a left ideal of R^1 and there is $J \in \Sigma$ with $J \subseteq T\}$.

PROPOSITION 1.4. *Σ^1 is a σ-set for R^1.*

Proof. For $T \subseteq R^1$ or $T \subseteq R$, let $(Tr^{-1})_R = \{s \in R \mid sr \in T\}$ and $(Tr^{-1})_{R^1} = \{(s, n) \in R^1 \mid (s, n)(r, 0) \in T\}$ when $r \in R$.

(σ_1) is clearly satisfied by Σ^1.

(σ_2): Let $T \in \Sigma^1$, $J \in \Sigma$, $J \subseteq T$ and $(r, n) \in R^1$. By (σ_2), $(Jr^{-1})_R \in \Sigma$ and so $(Jr^{-1})_R \cap J \in \Sigma$. Since $(T(r, n)^{-1})_{R^1} \supseteq J \cap (Jr^{-1})_R$ and $(r, n) \in R^1$ is arbitrary, $(T(r, n)^{-1})_{R^1} \in \Sigma^1$ for $(r, n) \in R^1$.

(σ_3): Let $S \in \Sigma^1$ and suppose $(T(r, n)^{-1})_{R^1} \in \Sigma^1$ for each $(r, n) \in S$. By the definition of Σ^1 there is a $J \in \Sigma$ with $J \subseteq S$. Since $(Tr^{-1})_{R^1} \in \Sigma^1$ for each $r \in J$, $((T \cap R)r^{-1})_R = (Tr^{-1})_{R^1} \cap R \in \Sigma$ for each $r \in J$ and $T \cap R \in \Sigma$. Thus $T \in \Sigma^1$.

REMARK. Clearly, an extension ${}_RM$ of ${}_RN$ is Σ-essential (Σ-injective) iff ${}_{R^1}M$ is a Σ^1-essential (Σ^1-injective) extension of ${}_{R^1}N$. Thus the Σ^1-injective hull of ${}_RM$ is also its Σ-injective hull and, hence, each R-module has a Σ-injective hull. We state this result as a theorem.

THEOREM 1.5. *Let R be a ring and M be a left R-module. Then $E_{\Sigma^1}(M)$ is the unique (up to isomorphism over M) extension of M satisfying*:

(1) *$E_{\Sigma^1}(M)$ is Σ-injective and Σ-essential over M;*
(2) *$E_{\Sigma^1}(M)$ is the maximal Σ-essential extension of M;*
(3) *$E_{\Sigma^1}(M)$ is the minimal Σ-injective extension of M.*

REMARK. If we define Σ-injectivity by property (S), then $E_{\Sigma^1}(M)$ still satisfies (1) and (2).

2. **Construction of R. E. Johnson's maximal quotient ring.** A Σ-essential left ideal of R is called a Σ-large left ideal. Let Δ denote the collection of all Σ-large left ideals of R.

LEMMA 2.1. *$I \in \Delta$ iff $I \in \Sigma$ and I is large left ideal of R.*

Proof. $\Rightarrow$: $Ix^{-1} \in \Sigma$ for all $0 \neq x \in R$ and $I0^{-1} = R \in \Sigma$ so $Ix^{-1} \in \Sigma$ for all $x \in R$. Thus $I \in \Sigma$. I is a large left ideal by definition.

$\Leftarrow$: If I is a large left ideal and $I \in \Sigma$, then for $0 \neq x \in R$, $Ix^{-1} \in \Sigma$ and $I \cap (x) \neq 0$. Thus $I \in \Delta$.

REMARK. It is well known that a finite intersection of large left ideals is large. By Lemma 1.1, Σ is closed under finite intersections. Thus Δ is closed under finite intersections.

Consider the set of all $\mathrm{Hom}_R(I, R)$ where $I \in \Delta$. In what follows we are using Johnson's notation [5]. For $f: I_f \to R$ and $g: I_g \to R$, define

$$f+g: I_f \cap I_g \to R \quad \text{by} \quad (f+g)(x) = f(x)+g(x)$$

and

$$fg: I_g^f \to R \quad \text{by} \quad (fg)(x) = f(g(x))$$

where

$$I_g^f = \{x \in I_g \mid g(x) \in I_f\}.$$

The remark after Lemma 2.1 shows that $I_f \cap I_g \in \Delta$.

LEMMA 2.2. *$I_g^f \in \Delta$.*

Proof. Johnson [5] has shown that I_g^f is a large left ideal of R. Let $0 \neq x \in R$, then $I_g x^{-1} \in \Sigma$ since $I_g \in \Sigma$. For $s \in I_g x^{-1}$, if $g(sx) \neq 0$, then $T = I_f g(sx)^{-1} \in \Sigma$ and $T \subseteq I_g^f x^{-1} s^{-1}$. If $g(sx) = 0$, $R \subseteq I_g^f x^{-1} s^{-1}$, so in any case, $I_g^f x^{-1} s^{-1} \in \Sigma$ for all $s \in I_g x^{-1}$. Thus $I_g^f x^{-1} \in \Sigma$ and so $I_g^f \in \Sigma$.

With the above lemmas, we may apply Johnson's method verbatim to obtain a ring

$$J_\Sigma(R) = \bigcup \{\mathrm{Hom}_R(I, R) \mid I \in \Delta\}/\theta.$$

where $f \,\theta\, g$ iff $f(x) = g(x)$ for $x \in I$ for some $I \in \Delta$.

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DEFINITION. An overring S of R is a *Johnson Σ-quotient ring of R if ${}_RR$ is* Σ-essential in ${}_RS$.

DEFINITION. The *Σ-singular left ideal* of R is

$$Z_\Sigma(R) = \{a \in R \mid Ta = 0 \text{ for some } \Sigma\text{-large left ideal } T\}.$$

THEOREM 2.3. *Let R be a ring, Σ be a σ-set for R and $Z_\Sigma(R)=0$. Then $J_\Sigma(R)$ is a Johnson Σ-quotient ring of R.*

Proof. Consider R as a subring of $J_\Sigma(R)$ by having R operate on itself by right multiplication. Then $J_\Sigma(R)$ is a left R-module. If $0 \neq \alpha \in J_\Sigma(R)$, then $\alpha \in \operatorname{Hom}_R(I_\alpha, R)$ and $r\alpha=\alpha(r)$ for $r \in I_\alpha$. Thus $R\alpha^{-1} \supseteq I_\alpha$ and so $R\alpha^{-1} \in \Sigma$. Since $0 \neq (I_\alpha)\alpha \subseteq (R\alpha^{-1})\alpha$, ${}_RR$ is Σ-essential in ${}_RJ_\Sigma(R)$.

EXAMPLES. (1) If Σ is the collection of all left ideals of R, and $Z_\Sigma(R)=0$, then $J_\Sigma(R)$ is the maximal Johnson quotient ring of R.

(2) If R has 1 and $\Sigma=\{R\}$, then the only Σ-large left ideal of R is R, so $R=J_\Sigma(R)$ and $Z_\Sigma(R)=0$.

(3) If R is a ring and $\Sigma=\{R\}$, then if $Z_\Sigma(R)=r(R)=0$, $J_\Sigma(R)=\operatorname{Hom}_R(R, R)$.

3. **Utumi's Σ-quotient ring.** Utumi [11] constructed another quotient ring using the method of Johnson. When R has identity, Utumi's quotient ring exists regardless of the vanishing of the singular ideal, and agrees with Johnson's quotient ring when $Z(R)=0$. In this section we generalize Utumi's method to obtain Utumi Σ-quotient rings.

DEFINITION. A submodule N of a module M is *rational* in M(${}_RM$ is a *rational extension* of ${}_RN$) if for any pair of elements $x, y \in M$ with $x \neq 0$, there exists $a \in R^1 = R \times \mathbf{Z}$ such that $ay \in N$ and $ax \neq 0$.

We generalize this definition to the following:

DEFINITION. Let Σ be a σ-set. A submodule N of a module M is *Σ-rational* in M (${}_RM$ is a *Σ-rational extension* of ${}_RN$) if N is rational in M and $Ny^{-1} \in \Sigma$ for every $y \in M$.

A left ideal I is *dense* in R (I is a *dense left ideal* of R) if ${}_RI$ is rational in ${}_RR$. We say a left ideal I is *Σ-dense* in R (I is a *Σ-dense left ideal* of R) if ${}_RI$ is Σ-rational in ${}_RR$.

Note that a Σ-rational extension of ${}_RN$ is a rational extension. Moreover, taking $x=y$ in the above definition, it is clear that a Σ-rational extension of ${}_RN$ is a Σ-essential extension. We also note that if $1 \in R$, for unital modules M and N, M is a Σ-rational extension of N if whenever $0 \neq x$, $y \in M$, then $Ny^{-1} \in \Sigma$ and $(Ny^{-1})x \neq 0$.

PROPOSITION 3.1. *A module N is Σ-rational in M if and only if N is Σ-essential in M and if $N \subseteq B \subseteq M$ and $f: B \to M$ is an R-homomorphism with $N \subseteq \ker f$, then $f=0$.*

The proof of Proposition 3.1 follows by a well known characterization of rational extension [4, p. 58].

REMARK. We note that since a Σ-dense left ideal D is Σ-large, $D \in \Sigma$. Moreover, a Σ-dense left ideal is a dense left ideal. Clearly a dense left ideal belonging to Σ is Σ-dense. Thus as the size of Σ increases, the set of Σ-dense left ideals does not decrease in size.

DEFINITION. Let Σ be a σ-set for R. An overring U of R is a *Utumi Σ-quotient ring* of R if ${}_RR$ is Σ-rational in ${}_RU$.

In order to construct a maximal Utumi Σ-quotient ring by Utumi's original method, we require the following lemma. (Again we use the notation of Johnson [5].)

LEMMA 3.2. *Let $f\colon D_f \to R$ and $g\colon D_g \to R$ be R-homomorphisms where D_f and D_g are Σ-dense left ideals of R. Then $D_g^f = \{x \in D_g \mid g(x) \in D_f\}$ is a Σ-dense left ideal of R.*

Proof. Since D_f and D_g are Σ-large, $D_g^f \in \Sigma$ by Lemma 2.2 and Lemma 2.1. It is well known that D_g^f is dense in R, thus D_g^f is Σ-dense in R.

REMARK. Since Σ is closed under finite intersections, and a finite intersection of dense left ideals is a dense left ideal, a finite intersection of Σ-dense left ideals is a Σ-dense left ideal.

Construct the maximal Utumi Σ-quotient ring of R as

$$U_\Sigma(R) = \bigcup \{\mathrm{Hom}_R\,(D, R) \mid D \text{ is } \Sigma\text{-dense in } R\}/\theta$$

where $f\,\theta\,g$ iff $f(x)=g(x)$ for $x \in J$ where J is some Σ-dense left ideal of R. Addition and multiplication are defined as for Johnson's quotient ring.

Consider the map $\phi\colon R \to U_\Sigma(R)$ given by $\phi(r)(s)=sr$. Then $\phi(r) \in \mathrm{Hom}_R\,(D, R)$ for all Σ-dense left ideals of R and so determines an element of $U_\Sigma(R)$ also denoted by $\phi(r)$. $\ker \phi = \{r \in R \mid \phi(r)=0\} = \{r \in R \mid Sr=0$ for some Σ-dense left ideal S of $R\}$. By Proposition 3.1, $r \in \ker \phi$ if and only if $r \in r(R) = \{s \in R \mid Rs=0\}$. Thus $R \subseteq U_\Sigma(R)$ if $r(R)=0$.

Clearly $1 \in U_\Sigma(R)$ since 1 is the element determined by $1_D\colon D \to R$ given by $1_D(d)=d$ for any Σ-dense left ideal D. Then if $U_\Sigma(R)$ is a Utumi Σ-quotient ring of R, $R \subseteq U_\Sigma(R)$ and so if $Rs=0$, $s \in R$, then $U_\Sigma(R)s=0$ by Proposition 3.1, and $1s=s=0$. Thus if $U_\Sigma(R)$ is an Utumi Σ-quotient ring of R, $r(R)=0$. Thus we restrict our rings to have zero right annihilator.

One can prove that $U_\Sigma(R)$ is a maximal Utumi Σ-quotient ring of R by Utumi's original method [11]. However, a more elegant proof may be given using the methods of Lambek [6] as in the next section.

At this point we do note the following

PROPOSITION 3.3. *If $Z_\Sigma(R)=0$, then J is a Σ-dense left ideal of R iff J is a Σ-large left ideal, and so $J_\Sigma(R)=U_\Sigma(R)$.*

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Proof. As noted before, Σ-dense implies Σ-large.

Let J be Σ-essential in R. Then for $0\neq x$, $y\in R$, Jy^{-1} is Σ-large in R, thus $(Jy^{-1})x\neq 0$ since $Z_\Sigma(R)=0$. Hence J is Σ-dense in R.

4. **A characterization of the Utumi Σ-quotient ring.** J. Lambek [6] characterized the Utumi maximal quotient ring of R as the bicommutant of the injective hull of $_RR$ if R has identity. Similarly, when $1\in R$, we show that $U_\Sigma(R)$ is the bicommutant of $E_\Sigma(R)$. This has been done by Sanderson [9]. In this section we obtain a σ-set $U\Sigma$ of left ideals of $U_\Sigma(R)=U$, generated by Σ, and prove that U is $U\Sigma$-injective when $Z_\Sigma(R)=0$, and that U is its own maximal Utumi $U\Sigma$-quotient ring.

Using the methods found in Faith [4], one can prove the following results.

Theorem 4.1. *Let M be an R-module, $E=E_\Sigma(M)$, $\Lambda=\operatorname{Hom}_R(E,E)$, and $M^\Lambda=\{\lambda\in\Lambda \mid \lambda(M)=0\}$. Then*

$$\overline{M} = \bigcap\{\ker\lambda \mid \lambda\in M^\Lambda\}$$

is a maximal Σ-rational extension of M containing each Σ-rational extension of M contained in E. Moreover, $\overline{M}$ is unique up to isomorphism over M.

Theorem 4.2. *Let R be a ring with 1, Σ a σ-set for R, $E=E_\Sigma(_RR)$ and $\bar R$ be the maximal Σ-rational extension of R in E. Then $\bar R$ is a ring whose operations $\bar R\times\bar R\to\bar R$ induce the module operations $R\times\bar R\to\bar R$ in $_R\bar R$, and $\bar R\cong\operatorname{Hom}_\Lambda(E,E)$ (ring iso.) where $\Lambda=\operatorname{Hom}_R(E,E)$.*

Theorem 4.3. *Let R be any ring with $r(R)=0$, R^+ be the subring of $\operatorname{Hom}_R(R,R)$ generated by R and the identity of $\operatorname{Hom}_R(R,R)$ and Σ be a σ-set for R. Then $_R\bar R={}_{R^+}\overline{R^+}$ is a maximal Utumi Σ-quotient ring of R, and has an identity element.*

Note that Theorem 4.2 shows that $\bar R$ is the bicommutant of the Σ-injective hull of R. This was Sanderson's definition [9] of $U_\Sigma(R)$. Since $\bar R=\overline{R^+}$ when $r(R)=0$, $\bar R$ is the bicommutant of $E_{\Sigma^+}(R^+)$. (Cf. Faith [4].) Similarly, using the methods of Lambek [6], it can be shown that when $1\in R$, $U_\Sigma(R)\cong\bar R$ (ring iso.). Lambek's proofs can be used verbatim.

Definition. Let $U=U_\Sigma(R)$ and Σ be a σ-set for R. Define

$$U\Sigma = \{T \mid T \text{ is a left ideal of } U \text{ and there is } J\in\Sigma \text{ with } UJ\subseteq T\}.$$

Lemma 4.4. *$U\Sigma$ is a σ-set for U.*

Proof. (σ_1) is satisfied by definition of $U\Sigma$.

(σ_2): Let $T\in U\Sigma$ and $UJ\subseteq T$, $J\in\Sigma$. Then for $q\in U$, $(Tq^{-1})_U=\{u\in U\mid uq\in T\}\supseteq(UJq^{-1})_U$. Now if $r\in(Rq^{-1})_R=\{r\in R\mid rq\in R\}$, then $rq\in R$ and $(J(rq)^{-1})_R=((Jq^{-1})_Rr^{-1})_R\in\Sigma$. Since $_RR$ is Σ-rational in $_RU$, $(Rq^{-1})_R$ is a Σ-dense left ideal and $(Rq^{-1})_R\in\Sigma$. Thus $(Jq^{-1})_R\in\Sigma$ and $U(Jq^{-1})_R\subseteq(UJq^{-1})_U\subseteq(Tq^{-1})_U$, so $(Tq^{-1})_U\in U\Sigma$.

(σ_3): Let T be a left ideal of U, $L\in U\Sigma$ with $UI\subseteq L$ where $I\in\Sigma$, and suppose

$(Ts^{-1})_U \in U\Sigma$ for all $s \in L$. Then for $i \in I$, $((T \cap R)i^{-1})_R \in \Sigma$ for $i \in I$. Hence $T \cap R \in \Sigma$ and $U(T \cap R) \subseteq T$ so $T \in U\Sigma$.

Lambek [6] and Utumi [11] both proved that $U(R)$ is its own maximal Utumi quotient ring. Analogously we show that $U_\Sigma(R) = U$ is its own Utumi $U\Sigma$-quotient ring.

LEMMA 4.5. *${}_UE$ is the $U\Sigma$-injective hull of ${}_UU$.*

Proof. We use the notation of the proof of Lemma 4.4. Let $0 \neq e \in E$, then $(Ue^{-1})_U \supseteq (Re^{-1})_R$ and $(Re^{-1})_R \in \Sigma$, thus $(Ue^{-1})_U \supseteq U(Re^{-1})_R$ and so $(Ue^{-1})_U \in U\Sigma$. Moreover $(0) \neq R \cap (e) \subseteq U \cap (e)$ so U is $U\Sigma$-essential in ${}_UE$.

Let ${}_UA$ be $U\Sigma$-essential in ${}_UB$ and $f: A \to E$ be a U-homomorphism. Since ${}_UA$ is $U\Sigma$-essential in ${}_UB$, ${}_RA$ is Σ-essential in ${}_RB$. By the Σ-injectivity of ${}_RE$, there is $f': {}_RB \to {}_RE$ extending f. To see that f' is a U-homomorphism, let $b \in B$ and define $g_b: {}_RU \to {}_RE$ by $g_b(u) = f'(ub) - uf'(b)$. Clearly $g_b(R) = 0$, and since ${}_RR$ is Σ-rational in ${}_RU$, $g_b(U) = 0$. Thus f' is a U-homomorphism. Since ${}_UE$ is $U\Sigma$-injective and $U\Sigma$-essential over ${}_UU$, ${}_UE = E_{U\Sigma}(U)$.

COROLLARY. *Let R be a ring with $r(R) = 0$. Then $U_\Sigma(R)$ is its own maximal Utumi $U\Sigma$-quotient ring.*

Proof. Clearly $\mathrm{Hom}_U(E, E) \subseteq \mathrm{Hom}_R(E, E) = \Lambda$. To see the reverse inclusion, let $f \in \mathrm{Hom}_R(E, E)$. For $0 \neq e \in E$, define $g_e \in \mathrm{Hom}_R(U, E)$ by $g_e(u) = f(ue) - uf(e)$. Then $g_e(R) = 0$ and so $g_e(U) = 0$ since R is Σ-rational in U. Thus $f \in \mathrm{Hom}_U(E, E)$ and $U_{U\Sigma}(U) = \mathrm{Hom}_\Lambda(E, E) = U_\Sigma(R) = U$.

We conclude this section by considering the relationship between $J_\Sigma(R)$ and $U_\Sigma(R)$ when $Z_\Sigma(R) = 0$. In Proposition 3.3 we showed that $J_\Sigma(R) = U_\Sigma(R)$ when $Z_\Sigma(R) = 0$. Denote both of these rings by $Q_\Sigma(R)$.

THEOREM 4.6. *Let R be a ring with $Z_\Sigma(R) = 0$. Then*

$$r(R) = 0 \text{ and } (1)\ Q_\Sigma(R) = E_\Sigma(R)$$
$$\text{and } (2)\ Q_\Sigma(R) \text{ is } Q\Sigma\text{-injective.}$$

Proof. (1) Since $Z_\Sigma(R) = 0$, $E_\Sigma(R)$ is the maximal Σ-rational extension of R, and so $E_\Sigma(R) = Q_\Sigma(R)$.

(2) By (1), $Q_\Sigma(R) = E_\Sigma(R)$ and $E_\Sigma(R)$ is the $Q\Sigma$-injective hull of Q by Lemma 4.5. Thus $Q_\Sigma(R)$ is $Q\Sigma$-injective.

5. **Matrix Rings.** Let R be a ring with identity and Σ be a σ-set for R. All modules considered are to be unital. As usual, if R is a ring, S is a nonempty subset of R, and n a positive integer, we let S_n denote the set of $n \times n$ matrices with entries from S; in particular, R_n is the ring of all $n \times n$ matrices over the ring R. Define $\Sigma_n = \{I \mid I$ is a left ideal of R_n, and $I \supseteq J_n$ for some $J \in \Sigma\}$.

LEMMA 5.1. *Σ_n is a σ-set for R_n.*

Proof. (σ_1) is satisfied by definition.

(σ_2): Let $J_n \in \Sigma_n$ for some $J \in \Sigma$ and let $(a_{ij}) \in R_n$. We must show that $J_n(a_{ij})^{-1} \in \Sigma_n$. Now for any $a \in R$ we have $J_n(ae_{kt})^{-1} \supseteq (Ja^{-1})_n$. Thus we write $(a_{ij}) = \sum a_{ij}e_{ij}$ and let $I = \bigcap_{ij} Ja_{ij^{-1}}$, an element of Σ by Lemma 1.1. Clearly $I_n(a_{ij}) \subseteq J_n$ and $I_n \in \Sigma_n$, so $J_n(a_{ij})^{-1} \in \Sigma_n$.

(σ_3): Let P be a left ideal of R_n and $J_n \in \Sigma_n$ for some $J \in \Sigma$. Suppose $P(a_{ij})^{-1} \in \Sigma_n$ for all $(a_{ij}) \in J_n$, we must show that $P \in \Sigma_n$. Now for any $a \in J$, $P(ae_{kt})^{-1} \in \Sigma_n$ so there is an $I \in \Sigma$ with $I_n ae_{kt} \subseteq P$. Let P_t be the collection of tth column matrices of P, then P_t is a left ideal of R_n and $P_t \subseteq P$. Let

$$P_{st} = \{p \in R \mid p \text{ is the } (s, t)\text{th entry in some } (p_{ij}) \in P_t\},$$

then P_{st} is a left ideal of R, and $P_{st}a^{-1} \in \Sigma$ since $I_n ae_{kt} \subseteq P_t$. Since this holds for all $a \in J$, $P_{st} \in \Sigma$. Let $K^t = \bigcap_s P_{st} \in \Sigma$ and let ${}_t(K^t)$ be the collection of tth column matrices with entries from K^t. If $\sum_{j=1}^n k_j e_{jt} \in {}_t(K^t)$ $k_j \in P_{jt}$ and so there is $\sum p_{st}e_{st} \in P_t$ with $p_{jt} = k_j$. Then $e_{jj}(\sum p_{st}e_{st}) = k_j e_{jt} \in P_t$ so ${}_t(K^t) \subseteq P_t$. Now let $N = \bigcap \{K^t \mid K^t = \bigcap_s P_{st} \in \Sigma\}$, then $N \in \Sigma$ and ${}_tN \subseteq {}_t(K^t) \subseteq P_t$ where N is the collection of tth column matrices with entries from N. Hence

$$\sum_1^n {}_tN = N_n \subseteq P_1 + P_2 + \cdots + P_n \subseteq P \text{ so } P \in \Sigma_n.$$

Let $Q(R)$ denote the maximal Utumi quotient ring of R and $Q_\Sigma(R)$ the maximal Utumi Σ-quotient ring of R. Utumi proved that $(Q(R))_n \simeq Q(R_n)$ (ring iso.) [11, p. 5, Theorem 2.4]. Analogously we will show that $(Q_\Sigma(R))_n \simeq Q_{\Sigma_n}(R_n)$ (ring iso.). Note that for the sake of simplicity, here, as elsewhere, we maintain the notation used by Utumi. We require the following

LEMMA 5.2. *Let S be an Utumi Σ-quotient ring of R, then S_n is an Utumi Σ_n-quotient ring of R_n.*

Proof. Let $0 \neq (a_{ij})$, $(b_{ij}) \in S_n$ and suppose $a_{pq} \neq 0$. Then there is $J \in \Sigma$ with $J \subseteq Rb_{ij}^{-1}$; $i, j = 1, 2, \ldots, n$, and $Ja_{pq} \neq 0$ since S is an Utumi Σ-quotient ring of R, and Σ is closed under finite intersections. Since $J_n \in \Sigma_n$, $J_n(b_{ij}) \subseteq R_n$ and $0 \neq Je_{pp}(a_{ij}) \subseteq J_n(a_{ij})$, the result follows.

THEOREM 5.3. $(Q_\Sigma(R))_n \simeq Q_{\Sigma_n}(R_n)$.

Proof. By the Lemma, $(Q_\Sigma(R))_n = Q_n$ is an Utumi Σ_n-quotient ring of R_n.

Let D be a Σ_n-dense ideal of R_n, and let ${}_kM$ be the set of all elements of R each of which is a coefficient of a matrix in $D \cap R_n e_{kk}$. ${}_kM$ is a left ideal of R. Let $0 \neq x$, $y \in R$. Then there is a $J \in \Sigma$ with $J_n \subseteq D(ye_{kk})^{-1}$ and an element $(a_{ij}) \in D(ye_{kk})^{-1}$ with $(a_{ij})(xe_{kk}) = \sum_{i=1}^n a_{ik}xe_{ik} \neq 0$. Suppose $a_{sk}x \neq 0$, then $e_{ks}(\sum_{i=1}^n a_{ik}xe_{ik}) = a_{sk}xe_{kk} \neq 0$ and $a_{sk} \in ({}_kM)y^{-1}$. Hence ${}_kM$ is a dense left ideal of R, and since $J \subseteq ({}_kM)y^{-1}$, $({}_kM)y^{-1} \in \Sigma$ and ${}_kM$ is a Σ-dense left ideal of R. Let $M = \bigcap {}_kM$, then M is a Σ-dense left ideal of R. If $y \in M$, there is matrix $(d_{ie}) \in D \cap R_n e_{kk}$ whose $(1, k)$ entry is y. Thus $ye_{jk} = e_{j1}(d_{ie}) \in D$, and so $M_n \subseteq D$.

Now let $\theta: D_\theta \to R_n$ be an R_n-homomorphism where D_θ is a Σ_n-dense left ideal of R_n, then $J_n \subseteq D_\theta$ for some Σ-dense left ideal J of R. For $x \in J$, denote $(xe_{1k})\theta = e_{11}(xe_{1k})\theta$ by $\Sigma_j(x\theta_{kj})e_{1j}$. Then $\theta_{kj}: J \to R$ is an R-homomorphism, and so there are $q_{kj} \in Q_\Sigma(R)$ with $x\theta_{kj} = xq_{kj}$ for $x \in J$. For each $\Sigma x_{ij}e_{ik} \in J_n$,

$$\begin{aligned}\left(\sum x_{ik}e_{ik}\right)\theta &= \Sigma_{ik}e_{i1}(x_{ik}e_{1k})\theta \\ &= \Sigma_{ikj}e_{i1}(x_{ik}\theta_{kj})e_{1j} \\ &= \Sigma_{ij}(\Sigma_k x_{ik}q_{kj})e_{ij} \\ &= \left(\sum x_{ik}e_{ik}\right)\left(\sum q_{ik}e_{ik}\right)\end{aligned}$$

and so $(Q_\Sigma(R))_n \cong Q_{\Sigma_n}(R_n)$ over R_n.

6. **Direct products and direct sums.** Let $\{R_\alpha \mid \alpha \in A\}$ be a collection of rings, and let for each $\alpha \in A$, Σ_α be a σ-set for R. Define

$$\oplus\,\Sigma_\alpha = \{K \mid K \text{ is a left ideal of } \Pi_\alpha R_\alpha,\ K \supseteq \oplus\, J_\alpha,\ J_\alpha \in \Sigma_\alpha\}.$$

In proving that $\oplus\Sigma_\alpha$ is a σ-set for $\Pi_\alpha R_\alpha$, we require the following.

LEMMA 6.1. *Let R be a ring, and Σ be a nonempty collection of left ideals of R satisfying properties (σ_1) and (σ_2). The following properties for Σ are equivalent*:

(σ_3): If K is a left ideal of R, $J \in \Sigma$ and $Ka^{-1} \in \Sigma$ for all $a \in J$, then $K \in \Sigma$.
(σ_3'): If for some $J \in \Sigma$ there is associated to each $a \in J$ a $K_a \in \Sigma$, then $\sum K_a a \in \Sigma$.

Proof. $(\sigma_3') \Rightarrow (\sigma_3)$: For each $a \in J$ set $K_a = Ka^{-1}$, then $\sum K_a a \in \Sigma$ by (σ_3'), but $\sum K_a a \subseteq K$ so $K \in \Sigma$ by (σ_1).

$(\sigma_3) \Rightarrow (\sigma_3')$: Let $K = \sum K_a a$; for each $a \in J$ we have $Ka^{-1} \supseteq K_a$, since $K_a a \subseteq K$; hence $Ka^{-1} \in \Sigma$ and it follows from (σ_3) that $K \in \Sigma$.

LEMMA 6.2. *$\oplus\Sigma_\alpha$ is a Σ-set for ΠR_α.*

Proof. (σ_1) is clearly satisfied.

(σ_2) is satisfied since $(\oplus J_\alpha)(a_\alpha)^{-1} = \oplus(J_\alpha a_\alpha{}^{-1})$ for any $(a_\alpha) \in \Pi R_\alpha$ and $\oplus J_\alpha \in \oplus\Sigma_\alpha$.

(σ_3): Let K be a left ideal of ΠR_α, $\oplus I_\alpha \in \oplus\Sigma_\alpha$ and suppose $K(a_\alpha)^{-1} \in \oplus\Sigma_\alpha$ for each $(a_\alpha) \in \oplus I_\alpha$. Then $K(a_\alpha)^{-1} \supseteq (\oplus J_\alpha)_{(a_\alpha)}$ where $J_\alpha \in \Sigma_\alpha$ for each $\alpha \in A$. Hence

$$\sum_{(a_\alpha)\in\oplus I_\alpha} (\oplus J_\alpha)_{(a_\alpha)}(a_\alpha) = \oplus\left(\sum_{a_\alpha\in I_\alpha} J_\alpha a_\alpha\right) \in \oplus\Sigma_\alpha \text{ since } \sum_{a_\alpha\in I_\alpha} J_\alpha a_\alpha \in \Sigma_\alpha$$

by the previous lemma.

We next prove the main theorem of this section. The proof is modelled after that of Lambek [7, p. 100].

THEOREM 6.3. *Let $\{R_\alpha \mid \alpha \in A\}$ be a collection of rings with 1, and let, for each $\alpha \in A$, Σ_α be a σ-set for R_α. Then*

$$Q_{\oplus\Sigma_\alpha}\left(\prod R_\alpha\right) \cong \prod Q_{\Sigma_\alpha}(R_\alpha).$$

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Proof. Regard $R=\prod R_\alpha$ as a subring of $\prod Q_{\Sigma_\alpha}(R_\alpha)=Q'$. Let $0\neq q=(q_\alpha)\in Q'$ and set $D'=\bigoplus R_\alpha q_\alpha{}^{-1}$. Then $D'q\subseteq R$ and $D'\subseteq Rq^{-1}$. If $0\neq x$, $y\in R$, then since $R_\alpha q_\alpha{}^{-1}$ is a Σ_α-dense left ideal of R_α for each $\alpha\in A$, there is a $J_\alpha\in\Sigma_\alpha$ with $J_\alpha y_\alpha \subseteq R_\alpha q_\alpha{}^{-1}$ and $J_\alpha x_\alpha\neq 0$. Hence $(\bigoplus J_\alpha)y\subseteq D'$ and $(\bigoplus J_\alpha)x\neq 0$. Thus D' is a $\bigoplus\Sigma_\alpha$-dense left ideal of R, and Q' is an Utumi $\bigoplus\Sigma_\alpha$-quotient ring of R.

Now let D be a $\bigoplus\Sigma_\alpha$-dense left ideal of R and $D_\alpha=\pi_\alpha(D)$ with $\pi_\alpha\colon R\to R_\alpha$ the canonical projection. Then D_α is a Σ_α-dense left ideal of R_α for each $\alpha\in A$. Let $f\in \operatorname{Hom}_R(D, R)$ and $e_\alpha\colon D_\alpha\to D$ be the canonical injection, then $\pi_\alpha\circ f\circ e_\alpha\colon D_\alpha \to R_\alpha$ and so there is a $q_\alpha\in Q_{\Sigma_\alpha}(R_\alpha)$ with $d_\alpha q_\alpha=(\pi_\alpha\circ f\circ e_\alpha)(d_\alpha)$ for all $d_\alpha\in D_\alpha$. Define $q\in Q'$ by requiring $\pi_\alpha(q)=q_\alpha$ for each $\alpha\in A$; then for $d\in D$, we have

$$\pi_\alpha(dq) = d_\alpha q_\alpha = (\pi_\alpha\circ f\circ e_\alpha)(d_\alpha) = \pi_\alpha(f(d)),$$

and hence $dq=f(d)$ and the theorem holds.

REMARK. (1) From the above proofs, it is clear that $\bigoplus\Sigma_\alpha$ is a σ-set for $\bigoplus R_\alpha$ and that $\prod Q_{\Sigma_\alpha}(R_\alpha)=Q_{\bigoplus\Sigma_\alpha}(\bigoplus R_\alpha)$.

(2) When, for each $\alpha\in A$, Σ_α is the collection of all left ideals of R_α, then $\bigoplus\Sigma_\alpha$ is the set of all left ideals of $\prod R_\alpha$. Hence our result extends the original result of Utumi [11].

7. **Classical Σ-quotient rings.**

In 1949, Asano [1] gave necessary and sufficient conditions for a ring to have a classical left quotient ring. In [6], Lambek showed that under Asano's conditions, the classical left quotient ring is a subring of Utumi's maximal quotient ring.

DEFINITION. If R is a subring of a ring Q with identity (and R contains a regular element) then Q is a *left* (*right*) *classical quotient ring* of R if (1) each regular element of R is a unit in Q, and (2) each $q\in Q$ can be written as $q=b^{-1}a$ ($q=ab^{-1}$) where a, $b\in R$ and b is regular in R.

We generalize this definition to the following:

DEFINITION. Let R be a ring and Σ be a σ-set of left ideals for R. An overring Q of R is a *left classical* Σ-*quotient* ring of R if Q is a left classical quotient ring of R and Q is a left Σ-quotient ring of R.

Analogous to Lambek's result, we show that under certain conditions, the left classical Σ-quotient ring of R is a subring of Utumi's maximal left Σ-quotient ring of R.

THEOREM 7.1. *Let R be a ring containing a regular element, and Σ be a σ-set for R. R has a left classical Σ-quotient ring $Q_{\Sigma_c}(R)$ if and only if* (1) *for each regular $b\in R$, Rb is a Σ-large left ideal of R, and* (2) *for each $0\neq a$, $b\in R$, b regular, $(Rb)a^{-1}$ contains a regular element.*

Proof. The requirement that $(Rb)a^{-1}$ contains a regular element whenever $0\neq a\in R$ and b is regular is a restatement of Ore's condition: If $0\neq a$, $b\in R$,

b regular, then there exist $c, d \in R$, c regular, with $ca = db$. As is well known [1], this condition is necessary and sufficient for R to have a classical left quotient ring.

If $b \in R$ is regular, then Rb is Σ-large by hypothesis. Hence if $0 \neq x, y \in R$, $(Rb)y^{-1} \in \Sigma$ and contains a regular element. Hence $[(Rb)y^{-1}]x \neq 0$, and Rb is Σ-dense. Define $\phi \in \mathrm{Hom}_R(Rb, R)$ by $\phi(sb) = s$, then ϕ defines $q \in Q_\Sigma(R)$ and $bq = 1$. The regularity of b yields $qb = 1$ and, hence, b is a unit in $Q_\Sigma(R)$. Let $Q_{\Sigma_c}(R)$ be the subring of $Q_\Sigma(R)$ generated by all elements of the form $b^{-1}a$ for $a, b \in R$, b regular. (Note that $R \subseteq Q_\Sigma(R)$ since $r(R) = 0$.) Then $Q_{\Sigma_c}(R)$ is the left classical Σ-quotient ring of R.

8. **Gabriel's ring of quotients and Mewborn's generalized centralizer.** In a recent paper, Mewborn [8] constructed a "generalized centralizer" of a nonzero left R-module M with respect to a collection τ of submodules of M satisfying

(τ_1): If $T \in \tau$ and $T \subseteq N$, N a submodule of M, then $N \in \tau$.

(τ_2): If $S, T \in \tau$, then $S \cap T \in \tau$.

(τ_3): If $S, T \in \tau$ and $\alpha: T \to M$ is an R-homomorphism such that $\alpha^{-1}(S) = U$, then $U \in \tau$.

The "generalized centralizer" of M is

$$P(M) = \varinjlim \{\mathrm{Hom}_R(T, M) \mid T \in \tau\}.$$

If Σ is a σ-set of left ideals for R, then Σ satisfies conditions (τ_1)–(τ_3) for R. It is clear that (σ_1) is (τ_1). Lemma 1.1 shows that Σ satisfies (τ_2). To see that (τ_3) holds, let $S, T \in \Sigma$ and $\alpha \in \mathrm{Hom}_R(T, R)$. Then for $a \in T$, $[\alpha^{-1}(S)]a^{-1} = S(\alpha(a))^{-1} \in \Sigma$ by (σ_2) since $S \in \Sigma$. Thus $\alpha^{-1}(S) \in \Sigma$ by (σ_3). In a series of exercises [2, pp. 157–165] Gabriel constructs a "ring of quotients" for a ring R with respect to a σ-set Σ for R as

$$G_\Sigma(R) = \varinjlim \{\mathrm{Hom}_R(J, R) \mid J \in \Sigma\}.$$

By what has been shown above, $G_\Sigma(R) = P(R)$ since Σ satisfies (τ_1)–(τ_3) for R.

Now let τ be a collection of submodules of ${}_RM$ satisfying conditions (τ_1)–(τ_3). Let $f \in \mathrm{Hom}(T_f, M)$, $T_f \in \tau$. By Zorn's Lemma, we can find $\bar{f}: J_f \to M$, $J_f \in \tau$, extending f and J_f is maximal with respect to this property. Then J_f is essential in M and $J_f \in \tau$. Hence we see that $P(M)$ is a homomorphic image of $J(M) = \varinjlim \{\mathrm{Hom}_R(S, M) \mid S \in \tau$, S is essential in $M\}$ by the map taking $\bar{f} \in J(M)$ to the image of $\bar{f}$ in $P(M)$. The kernel of this map is $\{\bar{f} \in J(M) \mid \bar{f}^{-1}(0) \in \tau\}$.

In case R is a ring and Σ a σ-set for R, we have shown that $G_\Sigma(R)$ is ring homomorphic image of $J_\Sigma(R)$. If we define $\Sigma(M) = \{m \in M \mid Jm = 0$ for some $J \in \Sigma\}$ for any R-module M, then the kernel of $J_\Sigma(R) \to G_\Sigma(R)$ is $\Sigma(J_\Sigma(R))$. If $\Sigma(R) = 0$, then for each $I \in \Sigma$ and $0 \neq a, b \in R$, $Ib^{-1} \in \Sigma$ and $(Ib^{-1})a \neq 0$. Hence each element of Σ is both a Σ-large and Σ-dense left ideal of R, and so $G_\Sigma(R) = J_\Sigma(R) = U_\Sigma(R)$. When $\Sigma(R) \neq 0$, the difference between $G_\Sigma(R)$ and $J_\Sigma(R)$ can be great; in fact, if Σ is the set of all left ideals of R, and $Z_{\Sigma}(R)=0$, then $G_{\Sigma}(R)=0$ but $J_{\Sigma}(R)$ need not be zero (e.g. let R be a field). Gabriel has shown that $R \subseteq G_{\Sigma}(R)$ if and only if $\Sigma(R)=0$. We note, as have many other authors, that the set of Σ-large left ideals Σ_c and the set of Σ-dense left ideals Σ_d satisfy (σ_1)–(σ_3), and $J_{\Sigma}(R)=G_{\Sigma_c}(R)$ and $U_{\Sigma}(R) = G_{\Sigma_d}(R)$.

Finally we let M be an R-module and τ a collection of submodules of M satisfying conditions (τ_1)–(τ_3).

Definition [8]. M is *τ-complemented* if for each submodule N of M, there is a submodule N' of M such that $N \cap N'=(0)$ and $N+N' \in \tau$.

If M is τ-complemented and N is an essential submodule of M, $N \cap N'=(0)$ only if $N'=(0)$ for any submodule N' of M, thus $N \in \tau$. Since τ contains all large submodules of M, $J(M)$ is Johnson's extended centralizer of M and so is von Neumann regular [5]. Hence $P(M)$ is von Neumann regular. This proves the following theorem due to Mewborn.

Theorem 8.1. *Let M be an R-module and τ a collection of submodules of M satisfying (τ_1)–(τ_3). Then $P(M)$ is von Neumann regular if M is τ-complemented.*

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