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THE CONVEX INTERSECTION BODY OF A CONVEX BODY

MATHIEU MEYER

Université Paris-Est - Marne-la-Vallée, Laboratoire d'Analyse et de Mathématiques Appliquées (UMR 8050), Cité Descartes, 5 Bd Descartes, Champs-sur-Marne, 77454 Marne-la-Vallée cedex 2, France
e-mail: mathieu.meyer@univ-mlv.fr

and SHLOMO REISNER†

Department of Mathematics, University of Haifa, Haifa 31905, Israel
e-mail: reisner@math.haifa.ac.il

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Abstract. Let L be a convex body in $\mathbb{R}^n$ and z an interior point of L. We associate with L and z a new, convex and centrally symmetric, body $CI(L, z)$. This generalizes the classical *intersection body* $I(L, z)$ (whose radial function at $u \in S^{n-1}$ is the volume of the hyperplane section of L through z, orthogonal to u). $CI(L, z)$ coincides with $I(L, z)$ if and only if L is centrally symmetric about z. We study the properties of $CI(L, z)$.

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1. Introduction. Let L be a convex body in $\mathbb{R}^n$ containing 0 in its interior. The intersection body $I(L)$ of L, defined by its radial function $\rho_{I(L)}$ on the sphere S^{n-1}, which is

$$\rho_{I(L)}(u) = \mathrm{vol}(L \cap u^{\perp})$$

and the cross-section body $C(L)$ of L, defined by

$$\rho_{C(L)}(u) = \max_t \mathrm{vol}((L \cap (tu + u^{\perp}))$$

are not, in general, convex bodies, although they are identical and, moreover, convex in the case when L is centrally symmetric. This follows from Brunn–Minkowski theorem (see [**21**, p. 309]) and from Busemann's theorem (see [**3**]). We introduce here a new convex body associated with L, generalizing both the intersection body and the cross-section body. More precisely, we define the *convex intersection body* $CI(L)$ of L by its radial function

$$\rho_{CI(L)}(u) = \min_{z \in P_u(L^{*g(L)})} \mathrm{vol}\big(\big[P_u(L^{*g(L)})\big]^{*z}\big), \tag{1}$$

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where $g(L)$ denotes here the centroid of L, and if E is a linear subspace of $\mathbb{R}^n$ and M is a convex body in E, we define for $z \in E$,

$$M^{*z} = \{y \in E; \langle y - z, x - z\rangle \leq 1 \text{ for every } x \in M\}.$$

Thus (1) means: first apply duality with respect to the point $g(L)$, then project onto $u^\perp$, finally apply duality with respect to z and minimize the $(n-1)$-dimensional volume over z.

In Section 2, we shall attach to any convex body K in $\mathbb{R}^n$ a body $J(K)$ constructed with the help of its projections, and prove that it is always convex. In Section 3, we prove that $CI(L)$ is convex, and we study the inclusions

$$CI(L) \subset I(L) \subset C(L),$$

when $g_L = 0$. Finally, in Section 4, a few open problems are listed, with proposed ideas concerning some of them.

Notations. For $x, y \in \mathbb{R}^n$, we denote by $\langle x, y\rangle$ the canonical scalar product in $\mathbb{R}^n$, $|x|$ denotes the Euclidean norm defined by it. If u and v are two non-zero vectors in $\mathbb{R}^n$ that are not orthogonal to one another, we define $\Pi_{w,u^\perp} : \mathbb{R}^n \to u^\perp$ to be the projection parallel to w onto $u^\perp = \{x \in \mathbb{R}^n; \langle x, u\rangle = 0\}$, we denote

$$P_u = \Pi_{u,u^\perp}.$$

If L is a subset of $\mathbb{R}^n$, let $[L]$ be the affine subspace of $\mathbb{R}^n$ that it spans. If B is a convex subset of $\mathbb{R}^n$, we denote its k-dimensional volume by $\text{vol}(B)$ (where $k = \dim[B]$). By $\overline{\text{conv}}\,(A)$, we denote the closed convex hull of A.

For L a convex set in $\mathbb{R}^n$ and $z \in [L]$, we denote the polar body of L with respect to z by

$$L^{*z} = \{y \in [L]; \langle y - z, x - z\rangle \leq 1 \text{ for every } x \in L\}.$$

It is well known that the function $z \to \text{vol}(L^{*z})$ is strictly convex on the relative interior of L and tends to $+\infty$ as z approaches the realtive boundary of L in $[L]$ (see [**17**, **20**]). It follows that it reaches its minimum at a unique point $s(L) \in \text{int}(L)$. This point is called the *Santaló point of* L. We shall denote

$$L^{*s} := L^{*s(L)}.$$

Moreover, $s(L)$ is also characterized as the unique point $z \in \text{int}(L)$, which is the centroid of L^{*z} (see [**20**] and also [**21**, p. 419]). Let us denote by $g(M)$ the centroid of a convex body M in $[M]$, and set

$$M^{*g} = M^{*g(M)}.$$

One has

$$L^{*s} = M \text{ if and only if } M^{*g} = L.$$

Observe that, in general, $s(L) \neq g(L)$ (see the recent [**18**], where a lower bound to how far apart these two points can be is given). Finally if $0 \in \text{int}(M)$, we shall write $M^* = M^{*0}$, so that $M^{*x} = x + (M - x)^*$ for every $x \in \text{int}(M)$.

We adopt the following notation: if K is a star body with respect to 0, we denote

$$||x||_K = \inf\{\lambda > 0; x \in \lambda K\}$$

to be the *gauge* of K. Then for $u \in S^{n-1}$,

$$\rho_K(u) = \frac{1}{||u||_K}$$

is the *radial function* of K.

2. A convexity theorem.

THEOREM 1. *If K is a convex body in $\mathbb{R}^n$, let $N_K : \mathbb{R}^n \to \mathbb{R}_+$ be defined by the formula*

$$N_K(u) = \frac{1}{\mathrm{vol}((P_u K)^{*s})} = \frac{1}{\min_{z\in u^\perp} \mathrm{vol}((P_u K)^{*z})} \text{ for } u \in S^{n-1},$$

and extended to all $\mathbb{R}^n$ by $N_K(ru) = rN_K(u)$ for $r \geq 0$ and $u \in S^{n-1}$. Then N_K is a norm on $\mathbb{R}^n$.

Before proving Theorem 1, we need some preliminary results.

DEFINITION . Let $v \in S^{n-1}$, B be a bounded subset of $\mathbb{R}^n$ and $V : B \to \mathbb{R}$ be a bounded map. The *shadow system* (L_t), $t \in [a, b]$ of convex bodies in $\mathbb{R}^n$, with *direction* the vector v, with *basis* the set B and with *speed* the function V, is the family of convex bodies

$$L_t = \overline{\mathrm{conv}}\,\{b + tV(b)v; b \in B\}, \text{ for } t \in [a, b].$$

For the sake of completeness, we prove the following result, which appears in [**23**] and is used, for example, in [**4**, **5**].

PROPOSITION 2. *Let K be a convex body in $\mathbb{R}^n$. Then, for $u, v \in S^{n-1}$, such that $\langle u, v\rangle = 0$, the family $L_t = \Pi_{u+tv,u^\perp}K$, $t \in \mathbb{R}$, is a shadow system of convex bodies in $u^\perp$, in the direction v.*

Proof. To simplify notation, one may suppose that $u = e_n$ and $v = e_{n-1}$, where $e_1, \ldots, e_n$ is an orthonormal basis of $\mathbb{R}^n$ (we shall write $\mathbb{R}^j$ for $[e_1, \ldots, e_j]$). Then, for all $t \in \mathbb{R}$,

$$\begin{aligned}
\Pi_{u+tv,u^\perp}K &= \{X + ze_{n-1}\,;\, X \in \mathbb{R}^{n-2}, X + ze_{n-1} + r(e_n + te_{n-1}) \in K \text{ for some } r \in \mathbb{R}\} \\
&= \{X + ze_{n-1}\,;\, X \in \mathbb{R}^{n-2}, X + (z + rt)e_{n-1} + re_n \in K \text{ for some } r \in \mathbb{R}\} \\
&= \{X + (x - rt)e_{n-1}\,;\; X \in \mathbb{R}^{n-2},\; r \in \mathbb{R} \text{ such that } X + xe_{n-1} + re_n \in K\} \\
&= \{U - rte_{n-1}\,;\, (U, r) \in P_u K \times \mathbb{R} \text{ such that } U + re_n \in K\}.
\end{aligned}$$

For $U \in P_u K$, define

$$\mathcal{I}(U) = \{r \in \mathbb{R};\; U + re_n \in K\}.$$

Then $\mathcal{I}(U) = [a(U), b(U)]$ is a closed interval of $\mathbb{R}$. Define also $x(U) \in \mathbb{R}$ such that

$$\langle U - x(U)e_{n-1}, e_{n-1}\rangle = 0,$$

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and let

$$D_1 = \{U \in P_u K; x(U) \in \mathbb{Q}\} \text{ and } D_2 = \{U \in P_u K; x(U) \in \mathbb{R} \setminus \mathbb{Q}\}.$$

Define $V : P_u K \to \mathbb{R}$ by

$$v(U) = -b(U) \text{ if } U \in D_1 \text{ and } v(U) = -a(U) \text{ if } U \in D_2.$$

By the continuity of the two concave functions $-a, b : P_u(K) \to \mathbb{R}$, it is easy to see that for every $t \in \mathbb{R}$

$$\Pi_{u+tv,u^\perp} K = \overline{\text{conv}}\,\{U + tV(U)e_n; U \in P_u K\}.$$

□

REMARK. The converse assertion of Proposition 2 is true : every shadow system L_t in $\mathbb{R}^n$ can be represented as $L_t = \Pi_{u+tv,u^\perp}(K)$ with an appropriate convex body $K \subset \mathbb{R}^{n+1}$ and $u, v \in S^n$. This was shown, for example, in [**4**].

The following result was proved in [**16**].

THEOREM 3. *Let $t \in [a, b] \to L_t$ be a shadow system in $\mathbb{R}^n$, then the function ϕ : $\mathbb{R}^n \to \mathbb{R}$, defined by the formula*

$$\phi(t) = \frac{1}{\text{vol}((L_t)^{*s})} = \frac{1}{\min_z \text{vol}((L_t)^{*z})},$$

is convex.

LEMMA 4. *Let $N : \mathbb{R}^n \to \mathbb{R}_+$ satisfy*
- *$N(x) > 0$ for $x \neq 0$,*
- *$N(\alpha x) = |\alpha| N(x)$ for every $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^n$,*
- *for all $u, v \in S^{n-1}$ such that $\langle u, v\rangle = 0$, $t \mapsto N(u + tv)$ is convex on $\mathbb{R}$.*

Then N is a norm on $\mathbb{R}^n$.

Proof. Let us show that $N(x + y) \le N(x) + N(y)$ for every $x, y \in \mathbb{R}^n \setminus \{0\}$. Let $\alpha = |\frac{x}{|x|} + \frac{y}{|y|}|$ and $\beta = |\frac{x}{|x|} - \frac{y}{|y|}|$, $u = \frac{1}{\alpha}(\frac{x}{|x|} + \frac{y}{|y|})$ and $v = \frac{1}{\beta}(\frac{x}{|x|} - \frac{y}{|y|})$. We may suppose that $\alpha \neq 0$ and $\beta \neq 0$. Then $u, v \in S^{n-1}$, $\langle u, v\rangle = 0$ and

$$x = \frac{|x|}{2}(\alpha u + \beta v), \quad y = \frac{|y|}{2}(\alpha u - \beta v).$$

We get

$$\begin{aligned} N(x + y) &= \frac{\alpha(|x| + |y|)}{2} N\left(u + \frac{\beta(|x| - |y|)}{\alpha(|x| + |y|)} v\right) \\ &= \frac{\alpha(|x| + |y|)}{2} N(u + (\lambda t + (1 - \lambda)s)v), \end{aligned}$$

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where $\lambda = \frac{|x|}{|x|+|y|}$, $t = \frac{\beta}{\alpha}$ and $s = -\frac{\beta}{\alpha}$. Under the assumption of the lemma, we get

$$N(x+y) \le \frac{\alpha(|x|+|y|)}{2}(\lambda N(u+tv) + (1-\lambda)N(u+sv))$$
$$= \frac{\alpha}{2}\left(|x|N\left(u+\frac{\beta}{\alpha}v\right) + |y|N\left(u-\frac{\beta}{\alpha}v\right)\right) = N(x)+N(y).$$

□

Proof of Theorem 1. In view of Lemma 4, we need to prove that $t \to g_{u,v}(t) = N(u+tv)$ is convex, whenever $u, v \in S_{n-1}$ satisfy $\langle u, v\rangle = 0$. It is easy to see that for any $t \in \mathbb{R}$, $P_{u+tv}K$ is an affine image of $\Pi_{u+tv,u^\perp}K$ and satisfies

$$\mathrm{vol}(P_{u+tv}K) = \frac{1}{\sqrt{1+t^2}}\mathrm{vol}(\Pi_{u+tv,u^\perp}K).$$

Hence

$$\min_{z\in\{u+tv\}^\perp} \mathrm{vol}((P_{u+tv}K)^{*z}) = \sqrt{1+t^2}\min_{z\in u^\perp}\mathrm{vol}((\Pi_{u+tv,u^\perp}K)^{*z}))$$

It follows that

$$N(u+tv) = \frac{|u+tv|}{\min_{z\in\{u+tv\}^\perp}\mathrm{vol}((P_{u+tv}K)^{*z}))} = \frac{1}{\min_{z\in u^\perp}\mathrm{vol}((\Pi_{u+tv,u^\perp}K)^{*z})}.$$

Now by Proposition 2, $t \to \Pi_{u+tv,u^\perp}K$ is a shadow system on $\mathbb{R}$ and thus by Theorem 3, $g_{u,v}$ is convex on $\mathbb{R}$. □

REMARKS. (1) If K is centrally symmetric (and centred at 0), then all its projections P_uK are centrally symmetric (and centred at 0) so that

$$\min_{z\in u^\perp}\mathrm{vol}(P_uK)^{*z}) = \mathrm{vol}(P_uK)^{*0}) = \mathrm{vol}(K^{*0}\cap u^\perp).$$

Under this hypothesis, we proved that $u \to \frac{1}{\mathrm{vol}(K^{*0}\cap u^\perp)}$ is the restriction to S^{n-1} of a norm on $\mathbb{R}^n$. This is Busemann **[3]** theorem on the sections of convex centrally symmetric bodies, applied to K^*.

(2) For every convex body K in $\mathbb{R}^n$, Theorem 1 defines a centrally symmetric convex body $J(K)$ in $\mathbb{R}^n$ by

$$J(K) = \{x\in\mathbb{R}^n; N_K(x)\le 1\}.$$

Notice that for every $x_0\in\mathbb{R}^n$, $J(K+x_0) = J(K)$ and that for $A:\mathbb{R}^n\to\mathbb{R}^n$ a linear isomorphism, we have

$$J((AK)) = |\det(A)|\,(A^*)^{-1}(J(K)).$$

(3) If $n=2$ and if R is the rotation by angle $\pi/2$ in $\mathbb{R}^2$, then

$$\mathrm{vol}(P_uK) = h_K(Ru) + h_K(-Ru) = h_K(Ru) + h_{-K}(Ru),$$

so that

$$J(K) = \tfrac{1}{4} R(K - K).$$

3. The convex intersection bodies $IC(L, z)$ of a convex body L. Let L be a convex body in $\mathbb{R}^n$. For a point $z \in \mathrm{int}(L)$, the *intersection body* $I(L, z)$ *of* L *with respect to* z is the centrally symmetric star body in $\mathbb{R}^n$ whose radial function $\rho_{I(L,z)}$ is given for $u \in S^{n-1}$ by

$$\rho_{I(L,z)}(u) = \mathrm{vol}(\{x \in L; \langle x - z, u\rangle = 0\}) = \mathrm{vol}(L \cap (z + u^{\perp})).$$

The body $C(L)$ is the star body in $\mathbb{R}^n$ defined by its radial function

$$\rho_{C(L)}(u) = \max_{x \in L} \mathrm{vol}(L \cap (x + u^{\perp})).$$

Of course, one has $I(L, z) \subset C(L)$ for every $z \in \mathrm{int}(L)$. It was proved in [**13**] that these bodies coincide if and only if L is centrally symmetric about z (the 'if' part follows easily from Brunn–Minkowski theorem). We define now the *convex intersection body of* L *with respect to* $z \in \mathrm{int}(L)$, which we denote by $CI(L, z)$, by

$$CI(L, z) = J(L^{*z}).$$

When $z = g(L)$ is the centroid of L, we shall denote $CI(L) = CI(L, g(L))$. The radial function of $CI(L, z)$ is thus given for $u \in S^{n-1}$ by

$$\rho_{CI(L,z)}(u) = \min_{x \in u^{\perp}} \mathrm{vol}((P_u(L^{*z}))^{*x}) = \mathrm{vol}((P_u(L^{*z}))^{*s}).$$

In view of Theorem 1, one has

THEOREM 5. *Let L be a convex body. Then for every $z \in \mathrm{int}(L)$, $CI(L, z)$ is a centrally symmetric convex body such that $CI(L, z) \subset I(L, z)$.*

REMARKS. (1) It is easy to see that one has for every one-to-one affine map $A : \mathbb{R}^n \to \mathbb{R}^n$, $I(AL, Az) = |\det(A)| A^{*-1}\big(I(L, z)\big)$, as well as

$$CI(AL, Az) = |\det(A)|\, A^{*-1}\big(CI(L)\big).$$

(2) In the case $n = 2$, for $u \in S^1$, denoting by R the rotation around 0 of angle $\pi/2$, one has

$$\begin{aligned} ||u||_{C(L)} &= ||Ru||_{K-K} \\ &\le ||u||_{I(L,z)} = \left(\frac{1}{||Ru||_{K-z}} + \frac{1}{||-Ru||_{K-z)}}\right)^{-1} \\ &\le ||u||_{CI(L,z)} = 4(||Ru||_{K-z} + ||-Ru||_{K-z}). \end{aligned}$$

(3) The inclusion $CI(L, z) \subset I(L, z)$ is exact in the sense that their boundaries touch; there always exists $u \in S^{n-1}$ such that

$$\mathrm{vol}(L \cap (z + u^{\perp})) = \mathrm{vol}((P_u(L^{*z}))^{*s}).$$

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As a matter of fact, this equality means that the centroid of $L \cap (z + u^\perp)$ in $z + u^\perp$ is at z. To see that such u exists, define $\phi : S^{n-1} \to \mathbb{R}$ by

$$\phi(v) = \mathrm{vol}(\{x \in L; \langle x - z; v\rangle \geq 0\}).$$

Since ϕ is continuous, it reaches its maximum at some point $u \in S^{n-1}$. Then, by [**15**], z is the centroid of $L \cap (z + u^\perp)$. See also [**10**].

(4) It was proved by Grünbaum ([**10**, Section 6.2]) that for every convex body $L \in \mathbb{R}^n$, there exists some $z_0 \in \mathrm{int}(L)$ such that $(n+1)$ different hyperplanes through z_0, with normals $u_1, \ldots, u_{n+1}$, satisfy that z_0 is the centroid of $L \cap (z + u_i^\perp)$. For this z_0, the boundaries of $CI(L, z_0)$ and of $I(L, z_0)$ have at least $2(n+1)$ contact points.

(5) We have seen above that, in the case when L is centrally symmetric about z, $CI(L, z) = I(L, z)$ and Theorem 5 is nothing else but the classical Busemann's theorem [**1**]. Conversely, the following result holds.

Proposition 6. *One has $CI(L, z) = I(L, z)$ if and only if L is centrally symmetric about z.*

It is a consequence of the following lemma.

Lemma 7. *Let L be a convex body and $z \in L$. Suppose that z is the centroid of every hyperplane section of L through itself. Then L is centrally symmetric about z.*

Proof. Fix some $z_0 \in \mathrm{int}(L)$, $z_0 \neq z$, and define $F : \mathbb{R}^n \to \mathbb{R}$ by

$$F(y) = \mathrm{vol}(\{x \in L - z_0; \langle x, y\rangle \geq 1\}).$$

By [**15**], F is C^1 on $\{F > 0\} = \mathbb{R}^n \setminus \{0\}$ and one has for $y \neq 0$

$$\begin{aligned}\nabla F(y) &= \langle \nabla F(y), y\rangle\, g(\{w \in L - z_0; \langle w, y\rangle = 1\}) \\ &= \langle \nabla F(y), y\rangle \big(g(\{x \in L; \langle x - z_0, y\rangle = 1\}) - z_0\big).\end{aligned} \tag{2}$$

Let H be the affine hyperplane in $\mathbb{R}^n$ defined by

$$H = \{y \in \mathbb{R}^n; \langle z - z_0, y\rangle = 1\}.$$

If $y \in H$, the hyperplane $\{x \in \mathbb{R}^n; \langle x - z_0, y\rangle = 1\}$ passes through z, so that by the hypothesis, one has

$$g(\{w \in L; \langle x - z_0, y\rangle = 1\}) = z,$$

thus, by (2) we get

$$\nabla F(y) = \langle \nabla F(y), y\rangle (z - z_0).$$

Now if $y, y' \in H$, one has

$$\langle y' - y, z - z_0\rangle = 0 \text{ and for every } t \in \mathbb{R},\ (1-t)y' + ty \in H$$

so that

$$F(y') - F(y) = \int_0^1 \langle y' - y, \nabla F((1-t)y + ty'))\rangle dt = 0.$$

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Thus, F is equal to some constant c on H. Define a function $G : S^{n-1} \to \mathbb{R}$ by

$$G(u) = \mathrm{vol}\{x \in L; \langle x - z, u\rangle \geq 0\}.$$

and let

$$U = \{u \in S^{n-1}; \langle u, z - z_0\rangle > 0\}.$$

Then $u \to y(u) := \frac{u}{\langle u, z-z_0\rangle}$ is a one-to-one mapping from U onto H, and it is easy to check that

$$G(u) = F\big(y(u)\big) \text{ for every } u \in U.$$

It follows that $G = c$ on U, and since $G(u) + G(-u) = \mathrm{vol}(L)$ for every $u \in S^{n-1}$, $G = \mathrm{vol}(L) - c$ on $-U$. Now, $S^{n-1} \cap (z - z_0)^{\perp}$ is contained in the closures of both U and of $-U$ in S^{n-1}. Since G is continuous on S^{n-1}, $G = c = 1 - c = \frac{\mathrm{vol}(L)}{2}$ on S^{n-1}. The fact that L is centrally symmetric about z now follows by a classical result (see [**8**] or [**6**]). □

4. Additional comments and some open problems. We know that although $C(L)$ and $I(L, z)$ are not in general convex bodies (by [**14**], $C(L)$ is convex for $n \leq 3$ and by [**2**], if Δ_n is the simplex in $\mathbb{R}^n$, $C(\Delta_n)$ is not convex if $n \geq 4$). However $C(L)$ and $I(L, g(L))$, where $g(L)$ is the centroid of L, are *almost convex*, and even *almost ellipsoids*, in the sense that there exist some constants $c > d > 0$, independent on n and L, such that for every $u \in S^{n-1}$, one has

$$\begin{aligned}\frac{d}{\mathrm{vol}(L)^{\frac{3}{2}}}\left(\int_{L-g(L)} \langle x, u\rangle^2 dx\right)^{\frac{1}{2}} &\leq \frac{1}{\max_t \mathrm{vol}\big(L \cap (tu + u^{\perp})\big)} = \rho_{C(L)}(u)\\ &\leq \frac{1}{\mathrm{vol}(L \cap u^{\perp})} = \rho_{I(L,g(L))}(u)\\ &\leq \frac{c}{\mathrm{vol}(L)^{\frac{3}{2}}}\left(\int_{L-g(L)} \langle x, u\rangle^2 dx\right)^{\frac{1}{2}}.\end{aligned}$$

In the centrally symmetric case, this was proved by Hensley, and Ball [**1**] (for sharp constants, see also [**19**]), and in the general case by Schütt [**22**] and Fradelizi [**7**] (the latter with sharp constants). We have seen that $\rho_{I(L,g(L))} \leq \rho_{CI(L,g(L))}$. A natural question to ask now is

OPEN PROBLEM 1. Does there exist a universal constant $C > 0$, independent on the convex body L in $\mathbb{R}^n$ and on $n \geq 1$, such that $\rho_{CI(L,g(L))} \leq C\rho_{I(L,g(L))}$? Of course, in view of the preceding inequalities, an affirmative answer to this question would say that the radial functions of $C(L)$, $CI(L)$ and $I(L, g(L))$ are all equivalent (with absolute constants).

Observe that an equivalent way of formulating this problem is the following. Let K be a convex body in $\mathbb{R}^n$ such that its Santaló point is at 0. Does there exist an absolute constant $C > 0$, independent on n and K such that

$$\mathrm{vol}\big((P_u K)^{*P_u z}\big) \geq C\,\mathrm{vol}\big((P_u K)^{*0}\big) \text{ for every } z \in \mathrm{int}(K)\,?$$

Equivalently, given a convex $M \subset u^{\perp}$, with Santaló point $s(M)$, and a convex body K in $\mathbb{R}^n$, with Santaló point $s(K)$, such that $P_u K = M$, does

$$\mathrm{vol}(M^{*s(M)}) \geq C\mathrm{vol}(M^{*P_u s(K)})$$

for some universal constant $C > 0$?

If one could prove that in this situation, for some universal constant $c > 0$, the following is true:

$$P_u s(K) - s(M) \in \frac{c}{n}\big(M - s(M)\big),$$

then an affirmative answer could be given, using the following lemma.

LEMMA 8. *Let V be a convex body in $\mathbb{R}^n$ and $x, y \in \mathrm{int}(V)$. Then*

$$(1 - ||x - y||_{V-y})^n \,\mathrm{vol}(V^{*x}) \leq \mathrm{vol}(V^{*y}) \leq \frac{\mathrm{vol}(V^{*x})}{\big(1 - ||y - x||_{V-x}\big)^n}$$

Proof. One has

$$\begin{aligned}\mathrm{vol}(V^{*y}) &= \mathrm{vol}(V^{*y} + y) = \mathrm{vol}\big((V - y)^*\big)\\ &= \mathrm{vol}\big(\big(V - x - (y - x)\big)^*\big) = \int_{(V-x)^*} \frac{1}{(1 - \langle y - x, z\rangle)^{n+1}} dz\end{aligned}$$

because by a formula given in [**18**], if L is a convex body with 0 in its interior, one has for every w in the interior of L,

$$\mathrm{vol}(L^{*w}) = \int_{L^*} \frac{1}{(1 - \langle w, z\rangle)^{n+1}} dz.$$

Since $\langle y - x, z\rangle \leq ||z||_{(V-x)^*} ||y - x||_{V-x}$, we get

$$\mathrm{vol}(V^{*y}) \leq \int_{(V-x)^*} \frac{1}{(1 - ||z||_{(V-x)^*} ||y - x||_{V-x})^{n+1}} dz\,. \tag{3}$$

Now, if L is a convex body with 0 in its interior, and $0 < c < 1$, then

$$\begin{aligned}\int_L \frac{1}{(1 - c||z||_L)^{n+1}} dz &= n v_n \int_{S^{n-1}} \left(\int_0^{\frac{1}{||\theta||_L}} \frac{r^{n-1}}{(1 - cr||\theta||_L)^{n+1}} dr \right) d\theta\\ &= n v_n \int_{S^{n-1}} \frac{1}{n} \left[\left(\frac{r}{1 - cr||\theta||_L} \right)^n \right]_0^{\frac{1}{||\theta||_L}} d\theta\\ &= v_n \int_{S^{n-1}} \frac{1}{(1 - c)^n ||\theta||_L^n} d\theta = \frac{\mathrm{vol}(L)}{(1 - c)^n}.\end{aligned}$$

From which, together with (3), we get

$$\mathrm{vol}(V^{*y}) \leq \frac{\mathrm{vol}(V^{*x})}{(1 - ||y - x||_{V-x})^n}.$$

Applying the same formula while interchanging the roles of x and y, one has

$$\mathrm{vol}(V^{*x}) \leq \frac{\mathrm{vol}(V^{*y})}{(1-||x-y||_{V-y})^n}. \qquad \square$$

It is well known (see, e.g. [**19**]) that there is an affine transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $AL = M$ is *isotropic*, that is, it satisfies $\mathrm{vol}(M) = 1$ and for every $u \in S^{n-1}$,

$$\frac{1}{\mathrm{vol}(M)}\left(\int_{M-g(M)} \langle x, u\rangle^2 dx\right)^{\frac{1}{2}} = L_M\,,$$

where L_M is *the isotropic constant* of M. In that context, Problem 1 is equivalent to

OPEN PROBLEM 2. Let M be an isotropic convex body. Is $CI(M)$ equivalent to the Euclidean ball (with absolute constant independent on $M \subset \mathbb{R}^n$ and on n) ?

Of course, Problems 1 and 2 are non-trivial only if L or M are not centrally symmetric. The particular case of the simplex is still open.

OPEN PROBLEM 3. Let Δ_n be a simplex in $\mathbb{R}^n$. Is there a constant c independent on n such that for every $u \in S^{n-1}$

$$\mathrm{vol}(\Delta_n \cap u^\perp) \leq c\,\mathrm{vol}(((\Delta_n \cap u^\perp)^{*0})^{*s}) = c\,\mathrm{vol}((P_u(\Delta_n^{*g}))^{*s})?$$

Observe that when Δ_n is a regular simplex inscribed in the Euclidean ball, since $(\Delta_n)^{*0} = -n\Delta_n$, one has

$$(\Delta_n \cap u^\perp)^{*0} = P_u((\Delta_n)^{*0}) = P_u(-n\Delta_n)$$

and thus

$$\mathrm{vol}(((\Delta_n \cap u^\perp)^{*0})^{*s}) = \frac{1}{n^{n-1}}\mathrm{vol}((P_u\Delta_n)^{*s}).$$

About Problem 3, one may remark that by affine invariance, we may suppose without loss of generality that Δ_n is the regular simplex with vertices $e_1, \dots, e_{n+1}$, $|e_i| = 1$, centred at $0 = e_1 + \cdots + e_{n+1}$. For $1 \leq i \neq j \leq n+1$, one has then $\langle e_i, e_j\rangle = -\frac{1}{n}$.

FACT. *Let* $A \subset \{1, \dots, n+1\}$ *be such that* $1 \leq k := \mathrm{card}(A) \leq n$ *and define*

$$u_A = \frac{\sum_{i\in A} e_i}{|\sum_{i\in A} e_i|} = \sqrt{\frac{n}{k(n+1-k)}}\sum_{i\in A} e_i \in S^{n-1}.$$

Then 0 *is the centroid of* $\Delta_n \cap u_A^\perp$.

It suffices to show that 0 is the Santaló point of $P_{u_A}\Delta_n$. Let

$$e_l' = P_{u_A}e_l,\ 1 \leq l \leq n+1,\ E = [e_i', i \in A],\ \text{and } F = [e_j'; j \notin A].$$

Then E and F are linear subspaces of $u_A^\perp$ such that $\dim(E) = k-1$, $\dim(F) = n-k$, $\langle x, y\rangle = 0$ for every $x \in E$ and $y \in F$, and

$$S_A := \overline{\mathrm{conv}}\,(\{e_i', i \in A\}) \subset E \text{ and } T_A := \overline{\mathrm{conv}}\,(\{e_j', j \notin A\}) \subset F$$

are regular simplices with centre of mass at 0. It follows that 0 is the Santaló point of S_A in E and of T_A in F, when 0 is the Santaló point of $P_{u_A}\Delta_n = \mathrm{conv}(S_A, T_A)$. It

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follows that 0 is the centroid of $\Delta_n \cap u_A^\perp$, which can be described as

$$\Delta_n \cap u_A^\perp = S_A^* \times T_A^*,$$

where S_A^* and T_A^* are the polars of S_A and of T_A, respectively, in E and F. □

A corollary of this result is the following.

PROPOSITION 9. *For every $A \subset \{1, \dots, n+1\}$, such that $1 \le k := \text{card}(A) \le n$, one has*

$$||u_A||_{CI(\Delta_n,0)} = ||u_A||_{I(\Delta_n,0)}.$$

Moreover, in the particular case when $u^\perp \cap \Delta_n$ is itself a simplex, one has the following computational proposition.

PROPOSITION 10. *Let $u \in S^{n-1}$, and if $u = \sum_{i=1}^{n+1} u_i e_i \in S^{n-1}$ with $\sum_{i=1}^{n+1} u_i = 0$ and $u_1, \dots, u_n \ge 0 > u_{n+1}$, then $u^\perp \cap \Delta_n$ is a simplex and*

$$\rho_{I(\Delta_n,0)(u)} = \text{vol}(\Delta_n \cap u^\perp) = \frac{1}{(n-1)!} \frac{(n+1)^{\frac{n+1}{2}}}{n^{\frac{n}{2}-1}} \frac{1}{\prod_{i=1}^n (u_i + \sum_{j=1}^n u_j)}$$

and

$$\rho_{CI(\Delta_n,0)(u)} = \text{vol}((\Delta_n \cap u^\perp)^{*0})^{*s}) = \frac{1}{(n-1)!} \frac{n^{\frac{n}{2}+1}}{(n+1)^{\frac{n+1}{2}}} \frac{1}{\sum_{i=1}^n u_i}.$$

It follows from Proposition 10 that $CI(\Delta_n, 0)$ has $2n+2$ small faces around $u = \pm e_i$, $1 \le i \le n+1$. Nevertheless, it is easy to check that for such directions $u \in S^{n-1}$ one has

$$1 \le \frac{\text{vol}(\Delta_n \cap u^\perp)}{\text{vol}((\Delta_n \cap u^\perp)^{*0})^{*s})} \le \frac{e}{2}.$$

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