Notes on Number Theory IV

ON THE MULTIPLICITY OF REPRESENTATIONS OF NUMBERS BY SUMS

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Let $A: \{a_1 \le a_2 \le \ldots \le a_t\}$ be a set of non-negative integers not exceeding n, and let $r_n(A;m) = r_k(m)$ denote the numbers of representations of m as the sum of k a's, order taken into account, i.e.

$$(1) \qquad r_k(m) = \sum_{a_{i_1}+a_{i_2}+\ldots+a_{i_k}=m} 1 .$$

Since $0 \le a_{i_1} + a_{i_2} + \ldots + a_{i_k} \le kn$, we have

$$(2) \qquad \sum_{m=o}^{kn} r_k(m) = t^k$$

and $r_k(m) = 0$ for m outside the interval $[0, kn]$. Hence it seems natural to define the mean value $\bar{r}_k$ by

$$(3) \qquad \bar{r}_k = t^k / (kn+1) .$$

If we further define

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(4) $$R_k = \max_m r_k(m)$$

then it follows trivially that

(5) $$R_k \geq \bar{r}_k .$$

In this note we will prove the inequality

THEOREM 1. $R_k \geq \sqrt{\frac{k}{3}}\, \bar{r}_k$.

For $k \leq 3$ theorem 1 is implied by (5), but it does yield new information for $k > 3$. Using a method developed in [1] we can prove that for every $k > 1$ there is an $\varepsilon_k > 0$ such that $R_k > (1+\varepsilon_k)\bar{r}_k$ but we will content ourselves here with a proof of theorem 1.

Our method is based on use of the identity

(6) $$\frac{k}{t} \sum_{i=1}^{t} (a_i - \bar{a})^2 = \frac{1}{t^k} \sum (a_{i_1} + a_{i_2} + \ldots + a_{i_k} - k\bar{a})^2 ,$$

where $\bar{a} = \frac{1}{t} \sum_{i=1}^{t} a_i$, and the sum on the right hand side of (6) runs over the t^k possible sets of subscripts $i_1, i_2, \ldots, i_k$. In the language of probability-statistics, (6) is nothing more than the statement that for independent variables the sum of the variances is equal to variance of the sums. Furthermore there is no difficulty in verifying (6) directly.

We now observe that since $\Sigma(a_i - \bar{a})^2$ is the minimum value of $\Sigma(a_i - x)^2$ and since $0 \leq a_i \leq n$, we have

(7) $$\frac{k}{t} \sum_{i=1}^{t} (a_i - \bar{a})^2 \leq \frac{k}{t} \sum_{i=1}^{t} (a_i - \frac{n}{2})^2 \leq k n^2 / 4 .$$

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We next obtain a lower bound for the right hand side of (6). For this purpose we make use of the fact that the variance of a set of numbers is unchanged if we replace each number in the set by s such numbers and is further invariant with respect to a translation of all the numbers through a fixed distance. Since distinct values of $a_{i_1} + a_{i_2} + \ldots + a_{i_k} - k\bar{a}$ differ by integers, and since no one of these integers can appear more than R_k times, it follows that the right hand side of (6) is not less than the variance of the set of integers $1, 2, \ldots, \left[\frac{t^k}{R}\right]$. The variance of this last set is $\frac{1}{12}\left(\left[\frac{t^k}{R}\right]^2 - 1\right)$, so using this together with (6) and (7) we obtain

$$\frac{1}{12}\left(\left[\frac{t^k}{R_k}\right]^2 - 1\right) \leq k n^2/4 \tag{8}$$

or

$$\frac{R_k}{t^k} \geq \frac{1}{\sqrt{3kn^2+1}} . \tag{9}$$

Theorem 1 now follows from (3) and (4).

Our result can be generalized in a variety of ways. For one thing, if instead of taking the k numbers each from the same set, we choose these numbers from k different sets, each lying in an interval $[0, n]$ essentially the same argument shows that theorem 1 is still true. Further we can get a similar theorem for the case where each of the different sets lies on a different interval. In closing we might remark that the idea behind our proof was first used by Chebychef in a probability setting and has been used in many situations since. However it seems that the present application is a new one and that applications to additive number theory have not yet been exhausted.

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REFERENCE

1. L. Moser, On the representation of 1, 2, ..., n by sums, Acta Arithmetica 6(1960) pp. 11-13.

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