INDUCED REPRESENTATIONS AND INVARIANTS

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1. Introduction. The problem of the expression of an invariant matrix of an invariant matrix as a direct sum of invariant matrices is intimately associated with the representation theory of the full linear group on the one hand and with the representation theory of the symmetric group on the other. In a previous paper[1] the author gave an explicit formula for this reduction in terms of characters of the symmetric group. Later J. A. Todd derived[2] the same formula using Schur functions, i.e. characters of representations of the full linear group. Clearly, this formula has no practical application beyond the range of existing character tables, and even within this range it is a cumbersome to use. However, the formula is important since it leads to an *explicit* definition of the representation $[\alpha] \odot [\beta]$ of S_{mn}. In fact, if we write

$$H = S_m \times S_m \times \ldots \times S_m, \qquad n \text{ factors},$$

then[3] $[\alpha] \odot [\beta]$ *is that representation of* S_{mn} *induced by the irreducible representation* $[\alpha; \beta]$ *of the normaliser* $\mathfrak{N}(H)$ *of* H *in* S_{mn}.

Here we show how the original step-by-step process of D. E. Littlewood[4] can be utilized to lead to a practical solution of the problem. An analysis of Todd's procedure in relation to the theory developed here is given in the last section of the paper.

2. The irreducible representations[5] of $\mathfrak{N}(H)$. The group $\mathfrak{N}(H)$ can be written as a sum of cosets in the following manner:

2.1 $$\mathfrak{N}(H) = H + Hs^*_2 + \ldots + Hs^*_{n!},$$

where the operations s^*_i permute the n sets of m symbols, preserving the order of the symbols in each set. Clearly, the factor group

$$\mathfrak{N}(H)/H \sim S^*_n \sim S_n,$$

and the s^*_i generate the subgroups S^*_n of $\mathfrak{N}(H)$.

With regard to the isomorphism between S_n and S^*_n, let us assume that an element s_i of S_n has ω_λ cycles of length λ, so that

2.2 $$n = \omega_1 + 2\omega_2 + \ldots + n\omega_n;$$

Received December 20, 1949.

[1][5, p. 172]. [2][8].

[3][5, §6]. [4][1].

[5]For details of the theory of this section, along with illustrative examples, the reader should consult [5].

https://doi.org/10.4153/CJM-1950-032-4 Published online by Cambridge University Press

then the corresponding element s^*_i of S^*_n has $m\ \omega_\lambda$ cycles of length λ.

If we denote by $\mathfrak{C}(s^*_i)$ the subgroup of those elements of H which commute with s^*_i, then a little consideration will show that

2.3 $$\mathfrak{C}(s^*_i) \sim S_m \times S_m \times \ldots \times S_m, \qquad \omega \text{ factors,}$$

where $0 < \omega = \omega_1 + \omega_2 + \ldots + \omega_n \leqslant n$. Moreover, we can write

2.4 $$H = \mathfrak{C} + \mathfrak{C}g_2 + \ldots + \mathfrak{C}g_r,$$

where $r = (m!)^{n-\omega}$, and the g_i can be chosen to be elements of certain of the original S_m's. In fact we can assume that

2.5 $$G = \{g_i\} = S_m \times S_m \times \ldots \times S_m, \qquad n - \omega \text{ factors,}$$

where any $\lambda - 1$ of the λ factors S_m *linked* in a given factor of $\mathfrak{C}$ appear in G. Collecting together all the omitted factors we construct

2.6 $$K = S_m \times S_m \times \ldots \times S_m, \qquad \omega \text{ factors,}$$

and $H = K \times G$. Now the $(m!)^{n-\omega}$ cosets of K in Hs^* are conjugate to Ks^* under transformation by elements of G, so that *the conjugate sets of* $\mathfrak{N}(H)$ *can be gathered into blocks, each block associated with a definite value of* ω. A glance at the Table of Characters of $\mathfrak{N}(H)$ for $m = 2$, $n = 3$ will make this clear.

In order to study the irreducible representations of $\mathfrak{N}(H)$ we first construct the irreducible Kronecker product representation

2.7 $$[\alpha]^n = [\alpha] \times [\alpha] \times \ldots \times [\alpha], \qquad n \text{ factors,}$$

of H. As has been shown, this can be extended by means of any irreducible representation $[\beta]$ of S_n to yield the irreducible representation $[\alpha; \beta]$ of $\mathfrak{N}(H)$ of degree $x_\alpha{}^n x_\beta$, where $[\alpha]$ and $[\beta]$ are of degrees x_α and x_β respectively. If an element ks^* belongs to an ω-block, i.e. if $\mathfrak{C}(s^*)$ is a product of ω factors S_m, then the character of ks^* in $[\alpha; \beta]$ is

2.8 $$\theta(k) \text{ in } [\alpha]^\omega \ . \ \chi(s) \text{ in } [\beta],$$

where k is an element of K.

The remaining irreducible representations of $\mathfrak{N}(H)$ arise by considering Kronecker products[6]

2.9 $$[\alpha; r_1] \times [\alpha'; r_2] \times \ldots \times [\alpha_1] \times [\alpha_2] \times \ldots \times [\alpha_s],$$

where $\alpha \neq \alpha' \neq \alpha_i$, $[r_i]$ is a diagram containing r_i nodes, and $r_1 + r_2 + \ldots + s = n$. Such a Kronecker product gives rise to others as the variables are permuted by S_n. The number of sets of variables is equal to the number of ways of distributing the n integers $1-n$ in the given partition. The only fact

[6]See the Table of Characters of $\mathfrak{N}(H)$ at the end of the paper. We have placed a bar over the multiplication signs in the generating Kronecker products to designate the representations Γ described here.

concerning these irreducible representations Γ, as we shall call them, which is of significance here is given by the following theorem.

2.10 *The characteristic of every conjugate set of the block associated with* $\omega = 1$ *in an irreducible representation* Γ *of* $\mathfrak{N}(H)$ *is zero.*

This follows immediately, since no set of variables remains invariant under any one of the regular permutations which compose this block.

3. Induced representations. We have already seen that the representation $[\alpha] \odot [\beta]$ of S_{mn} is induced by the irreducible representation $[\alpha; \beta]$ of $\mathfrak{N}(H)$, and that there is a one-to-one correspondence between the irreducible components of $[\alpha] \odot [\beta]$ and the irreducible components of Littlewood's "new product" representation $\{\alpha\} \otimes \{\beta\}$ of the full linear group. Our purpose in this section is to study Littlewood's step-by-step building process[4] with a view to adding a criterion by means of which the irreducible components of $[\alpha] \odot [\beta]$ can be determined in a systematic manner. Our basic tool will be Frobenius' Reciprocity Theorem, which will tie together the two formulae which we shall write side by side, referring in what follows to the one on the left by writing A and to the one on the right by writing B.

Let us assume that we know the frequencies ρ_{il} in the reductions:

3.1 $$[\alpha] \odot [\beta_l] = \sum_i \rho_{il}[\lambda_i]; \qquad [\lambda_i] \approx \sum_l \rho_{il}\,[\alpha; \beta_l] + \ldots,$$

where we denote by $\approx$ the restiction of S_{mn} to the subgroup $\mathfrak{N}(H)$. Littlewood's procedure is to build on each $[\lambda_i]$ with $[\alpha]$ in all possible ways to yield the irreducible representations $[\lambda'_j]$ of $S_{m(n+1)}$. We may describe this process by means of the formulae:[7]

3.2 $$[\lambda_i] \,.\, [\alpha] = \sum_j \sigma_{ji}\,[\lambda'_j]; \qquad [\lambda'_j] \doteq \sum_i \sigma_{ji}[\lambda_i] \times [\alpha] + \ldots,$$

where $\doteq$ denotes the restriction of $S_{m(n+1)}$ to the subgroup $S_{mn} \times S_m$. The quantities σ_{ji} are known from the building process. Finally, we have the analogue of 3.1 for $S_{m(n+1)}$:

3.3 $$[\alpha] \odot [\beta'_k] = \sum_j \rho'_{jk}\,[\lambda'_j]; \quad [\lambda'_j] \approx \sum_k \rho'_{jk}\,[\alpha; \beta'_k] + \ldots,$$

the prime on β' indicating an irreducible representation of S_{n+1}.

Thus, confining our attention for the moment to 3.3B we can write

3.4 $$[\lambda'_j] \approx \sum_k \rho'_{jk}\,[\alpha; \beta'_k] + \ldots \doteq \sum_{k,l} \rho'_{jk}\,\epsilon_{kl}\,[\alpha; \beta_l] \times [\alpha] + \ldots,$$

where $[\beta'_k] \doteq \sum_l \epsilon_{kl}\,[\beta_l]$, and the $[\beta_l]$ are obtained from $[\beta'_k]$ by removing a

[7]We are using the notation $[\lambda_i].[\alpha]$ to denote the representation of S_{mn} induced by the Kronecker product representation $[\lambda_i] \times [\alpha]$ of the direct product $S_{mn} \times S_m$. Cf. [4] in which the notion of the corresponding disjoint diagram denoting this representation was introduced.

single node in all possible ways. The truth of the reduction

$$[a; \beta'_k] \simeq \sum_l \epsilon_{kl}[a; \beta_l] \times [a],$$

where H' is the direct product of $n + 1$ factors S_m and we are restricting from $\mathfrak{N}(H')$ to $\mathfrak{N}(H) \times S_m$, can be easily seen by checking the degrees on both sides of the equation.

On the other hand we can make the restriction in a different manner, using first 3.2B and then 3.1B to yield

3.5 $$[\lambda'_j] \simeq \sum_i \sigma_{ji}[\lambda_i] \times [a] + \ldots \simeq \sum_{i,l} \sigma_{ji}\rho_{il}[a; \beta_l] \times [a] + \ldots .$$

Equating the coefficients of the irreducible representations $[a; \beta_l] \times [a]$ on the right of 3.4 and 3.5, we have

3.6 $$\boxed{\sum_k \rho'_{jk}\epsilon_{kl} = \sum_i \sigma_{ji}\rho_{il}}$$

since no such representations arise from resticting the Γ's, by 2.9.

There is an interesting analogy here with the operations of the ordinary calculus. For example, in the equation

$$[\beta'_k] \simeq \sum \epsilon_{kl}[\beta_l]$$

where $\epsilon_{kl} = 0, 1$, the removal of a node in all possible ways can be thought of as corresponding to *differentiation*.[8] The reverse process is *not* the same as inducing, but consists of finding one or more diagrams which, when differentiated, yield the given *integrand*; e.g. the "integral" of $[2, 1]$ is $[2^2]$, while the integral of

$$[3, 2] + [3, 1^2] + [2^2, 1]$$

is $[3, 2, 1]$. Not every direct sum of irreducible representations can be *integrated*, but the definition of $[a] \odot [\beta']$ as an induced representation provides an existence theorem which makes possible the solution of the equation 3.6 for the ρ'. It is necessary, however, to enquire into the existence of a "boundary condition" which must be satisfied. We shall determine this condition in the following section.

4. The star diagram of $[\lambda']$. In what follows we must assume some knowledge of the theory of *hooks* which has turned out to be of paramount importance in the modular representation theory. Without going into details, we define a hook H_r in a Young diagram $[\lambda]$ to consist of all those nodes to the right of and below (including) a given node of $[\lambda]$. If we write $H_r = [n - r, 1^r]$, then we shall say that H_r is of *length* n and *parity* $(-1)^r$. In the present case we are thinking of removing m hooks of length $n + 1$ from the diagram

[8]Consider the process as applied to the corresponding tensor!

https://doi.org/10.4153/CJM-1950-032-4 Published online by Cambridge University Press

$[\lambda']$. If we denote these hooks by $H_{r_i}(i = 1, 2, \ldots, m)$, then the quantity

$$\sigma = \Pi\sigma_i = (-1)^{\Sigma r_i} = \pm 1$$

is uniquely determined and is independent of the way in which the hooks are removed. We call σ the *parity* of $[\lambda']$.

Now the $(n+1)$-hook structiure of $[\lambda']$ is given by the *star diagram* $[\lambda']^*_{n+1}$ of $[\lambda']$, which contains m nodes and is in general *skew*. A knowledge of this star diagram enables us to write down the characteristic in $[\lambda']$ of every conjugate set appearing in the $\omega = 1$ block of the irreducible representations of $\mathfrak{N}(H)$. In fact[9]

4.1
$$\chi(ks^*) \text{ in } [\lambda'] = \sigma\chi^*(k) \text{ in } [\lambda']^*_{n+1} = \sigma(\chi_{\alpha_1} + \chi_{\alpha_2} + \ldots),$$

where the $[\alpha_i]$ are the irreducible representations of $K = S_m$ which appear as components of $[\lambda']^*_{n+1}$. On the other hand we have from 3.3B

4.2
$$\chi(ks^*) \text{ in } [\lambda'] = \sum_{i,j} \chi(ks^*) \text{ in } [\alpha_i; \beta'_j] = \chi_{\alpha_1}(\delta_{\beta'_1} + \delta_{\beta'_2} + \ldots) + \chi_{\alpha_2}(\delta_{\beta'_1} + \delta_{\beta'_2} + \ldots) + \ldots$$

by 2.8, where $\delta_\beta' = \pm 1, 0$ according as $[\beta']$ is or is not a hook of even or odd parity. No irreducible representations Γ of $\mathfrak{N}(H)$ appear in 4.2 by virtue of 2.10. Equating and multiplying the right-hand sides of 4.1 and 4.2 by χ_α and summing over the elements k of $K = S_m$, we obtain, after dividing through by $m!$,

4.3
$$\sigma = \delta_{\beta'1} + \delta_{\beta'2} + \ldots$$

for every $[\alpha_i]$ in 4.1. We conclude that one $\delta_\beta' = \sigma$, while the sum of the remaining terms vanishes. We state our conclusions in the following

THEOREM. *If the irreducible representation* $[\lambda']$ *of* $S_{m(n+1)}$ *has zero* $(n+1)$*-core*:

4.4 *The* $[\alpha]$'s *appearing in* 3.3B *are just the irreducible components* $[\alpha_i]$ *of the star diagram* $[\lambda']^*_{n+1}$.

4.5 *For a given* $[\alpha]$ *in* 3.3B, *one of the associated* $[\beta'_k]$ *is a hook representation of* S_{n+1} *of parity* σ, *while the sum of the characters of the remaining* $[\alpha; \beta'_k]$ *is zero for every conjugate set in the* $\omega = 1$ *block of* $\mathfrak{N}(H')$.

If the irreducible representation $[\lambda']$ *has non-zero* $(n+1)$*-core then*

4.6
$$\chi(ks^*) \text{ in } [\lambda'] = 0$$

for every conjugate set in the $\omega = 1$ *block of* $\mathfrak{N}(H')$.

[9]This formula is the special case for zero core of that obtained in [4, p. 291]. R. M. Thrall has recently drawn my attention to an error in the formulation of this result; this error will be corrected in a forthcoming Note. The independence of σ of the method of removing hooks is proved in [4, p. 289]; cf. also [6]. A glance at the Table of Characters of $\mathfrak{N}(H)$ will help the reader at this point.

https://doi.org/10.4153/CJM-1950-032-4 Published online by Cambridge University Press

The conclusion 4.4 is of intrinsic interest though unnecessary if we use the method described above, since we are confining our attention to a given $[\alpha]$ and the quantities $\rho_{i,l}$; $\sigma_{j,i}$ determine for us whether a given $[\lambda']$ appears. On the other hand, 4.5 and 4.6 are important, as the following argument shows.

Consider once more the integration process. We have given a representation φ of S^*_n, in general reducible, and we are seeking a representation φ' of S^*_{n+1}, *of the same degree* which when restricted to S^*_n, yields the representation φ. In other words, we are seeking to extend φ by means of a matrix (s^*), where $s^* \sim (1, 2, 3, \ldots, n+1)$, and such that

4.7 $$S^*_{n+1} = S^*_n + S^*_n \cdot s^* + S^*_n \cdot s^{*2} + \ldots + S^*_n \cdot s^{*n}.$$

By resticting the representation $[\lambda']$ of $S_{m(n+1)}$ to $\mathfrak{N}(H')$ we may obtain (s^*) and deduce from 4.7 that φ' is completely determined. The preceding theorem shows that the characteristic of (s^*) can be obtained from the $(n+1)$-hook structure of $[\lambda']$. We conclude that *there is an unique solution of the equation* 3.6 *which satisfies the boundary condition* 4.5 *or* 4.6.

5. Illustrative example. We apply the preceding theory to the problem of determining the irreducible components[10] of $[2] \odot [4]$, $[2] \odot [3, 1]$, $[2] \odot [2^2]$, $[2] \odot [2, 1^2]$, $[2] \odot [1^4]$, taking for granted the reductions:

5.1
$$\begin{aligned} &[2] \odot [3] &&= [6] + [4, 2] + [2^3], &&\text{of degree 15,}\\ &[2] \odot [2, 1] &&= [3, 2, 1] + [5, 1] + [4, 2], &&\text{of degree 30,}\\ &[2] \odot [1^3] &&= [4, 1^2] + [3^2], &&\text{of degree 15.}\end{aligned}$$

Though we have written the equations in the form 3.1A we shall actually use them in the form 3.1B. We calculate first the degrees from the formula

5.2 $$\text{degree } [\alpha] \odot [\beta] = x_\alpha{}^n \cdot x_\beta,$$

obtaining 105, 315, 210, 315, 105 respectively; these provide a useful check on the construction involved. Building

On [6] with [2] we obtain: $[8]_+$, $[7, 1]_-$, $[6, 2]_0$;
On $[2^3]$: $[4, 2^2]_0$, $[3, 2^2, 1]_-$, $[2^4]_+$;
On [4, 2] : $[6, 2]_0$, $[5, 3]_0$, $[5, 2, 1]_0$, $[4^2]_+$, $[4, 3, 1]_-$, $[4, 2^2]_0$;
On [3, 2, 1]: $[5, 2, 1]_0$, $[4, 3, 1]_-$, $[4, 2^2]_0$, $[4, 2, 1^2]_+$, $[3^2, 2]_+$, $[3^2, 1^2]_0$, $[3, 2^2, 1]_-$;
On [5,1] : $[7, 1]_-$, $[6, 2]_0$, $[6, 1^2]_+$, $[4, 3]_0$, $[5, 2, 1]_0$;
On $[4, 1^2]$: $[6, 1^2]_+$, $[5, 2, 1]_0$, $[5, 1^3]_-$, $[4, 3, 1]_-$, $[4, 2, 1^2]_+$;
On $[3^2]$: $[5, 3]_0$, $[4, 3, 1]_-$, $[3^2, 2]_+$.

From the equations 5.1 we have the ρ_{il} and from this building process we have the σ_{ji} as in 3.2. The subscripts $+$, $-$ in the table above indicate the parity σ of $[\lambda']$ if it has no $(n+1)$-core. If on the other hand $[\lambda']$ has a core, the characteristics of the $\omega = 1$ block of $\mathfrak{N}(H')$ all vanish when $S_{m(n+1)}$ is restricted to $\mathfrak{N}(H')$; this fact is indicated by the subscript 0.

[10] [2, p. 289].

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Littlewood made it clear that all the $[\lambda']$ must be distributed amongst the $[\alpha] \odot [\beta']$ and devised several more or less empirical methods for making the distribution. The "integration" method proposed here is rapid and simple to carry through, once some familiarity has been gained with the ideas involved.

Take for example the case of $[4, 3, 1]_-$ which is obtained

5.3
$$\begin{aligned} &\text{once from } [4, 2] &&\asymp [2; 3] + [2; 2, 1] + \ldots, \\ &\text{once from } [3, 2, 1] &&\asymp [2; 2, 1] + \ldots, \\ &\text{once from } [3^2] &&\asymp [2; 1^3] + \ldots, \\ &\text{once from } [4, 1^2] &&\asymp [2; 1^3] + \ldots . \end{aligned}$$

Before going further it may be worth while writing out completely one of the reductions listed above. For example, we have from the character table at the end of the paper the reduction

$$[3^2] \asymp [2; 1^3] + [1^2; 3] + [2; 2] \overline{\times} [1^2],$$

which shows also that $[3^2]$ can be obtained by building with $[1^2]$ since the star diagram $[3^2]^*_3$ of $[3^2]$ is

$$\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix} = [2] + [1^2],$$

illustrating 4.4.

To continue, we note that

$$\begin{aligned} [2; 3, 1] &\asymp [2; 3] \times [2] + [2; 2, 1] \times [2], \\ [2; 2, 1^2] &\asymp [2; 2, 1] \times [2] + [2; 1^3] \times [2], \end{aligned}$$

so that we conclude that

5.4
$$\begin{aligned} [4, 3, 1] &\asymp [2; 3, 1] + [2; 2, 1^2] + [2; 1^4] + \ldots \\ &= \left[2; \begin{smallmatrix} & \bullet & \bullet \\ \bullet & & \\ \bullet & & \end{smallmatrix}\right] + [2; 1^4] + \ldots, \end{aligned}$$

using the convenient notation of disjoint diagrams.[7] The term $[2; 1^4]$, or $[2; 3, 1]$, is of the proper parity $\sigma = -1$ and the character of

$$\left[2; \begin{smallmatrix} & \bullet & \bullet \\ \bullet & & \\ \bullet & & \end{smallmatrix}\right], \text{ or } \left[2; \begin{smallmatrix} & \bullet \\ \bullet & \\ \bullet & \\ \bullet & \end{smallmatrix}\right],$$

vanishes for every conjugate set of the $\omega = 1$ block of $\mathfrak{N}(H')$. This illustrates 4.5 and suggests the following

THEOREM. *The character of a representation* $[\alpha; \beta]$ *of* $\mathfrak{N}(H)$ *vanishes for every conjugate set of the* $\omega = 1$ *block if and only if*:

5.5 $[\beta]$ *is a right diagram, but not a hook*;

5.6 $[\beta]$ *is a skew diagram, but not a skew hook*;

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The theorem follows immediately since in these and only these cases can no n-hook representation appear as an irreducible component[11] of $[\beta]$.

As a further illustration it is worthwhile examining the representation $[4^2]_+$ which is obtained

5.7 $$\text{once only from } [4.2] \approx [2;3] + [2;2,1] + \ldots .$$

One might be tempted to integrate and obtain $[2;3,1]$, but $[3,1]$ is a 4-hook with $\sigma = -1$, contrary to our boundary condition that $\sigma = +1$. Thus instead we must have

5.8 $$[4^2]_+ \approx [2;4] + [2;2^2] + \ldots ,$$

which illustrates 5.5 with $[\beta] = [2^2]$.

From Frobenius' Reciprocity Theorem we conclude that $[4,3,1]$ is an irreducible component of $[2] \odot [3,1]$, $[2] \odot [2,1^2]$, $[2] \odot [1^4]$, from 5.4. Similarly, $[4^2]$ is a component of $[2] \odot [4]$ and $[2] \odot [2^2]$, from 5.8.

To illustrate the case of subscript 0 let us determine the distribution of $[5,2,1]_0$ which is obtained

5.9 $$\begin{aligned} &\text{once from } [4,2] &&\approx [2;3] + [2;2,1] + \ldots ,\\ &\text{once from } [3,2,1] &&\approx [2;2,1] + \ldots ,\\ &\text{once from } [5,1] &&\approx [2;2,1] + \ldots ,\\ &\text{once from } [4,1^2] &&\approx [2;1^3] + \ldots .\end{aligned}$$

Integrating, we obtain

$$\begin{aligned} [5,2,1]_0 &\approx [2;3,1] + [2;2^2] + [2;2,1^2] + \ldots \\ &= \left[2; \begin{smallmatrix} & \bullet & \bullet \\ & \bullet & \\ \bullet & & \end{smallmatrix}\right] + \ldots ,\end{aligned}$$

so that $[5,2,1]$ is an irreducible component of $[2] \odot [3,1]$, $[2] \odot [2^2]$, and $[2] \odot [2,1^2]$. Collecting together the results thus obtained we have

$$\begin{aligned}
[2] \odot [4] &= [8] + [6,2] + [4^2] + [4,2^2] + [2^2], && \text{of degree } 105,\\
[2] \odot [3,1] &= [7,1] + [6,2] + [5,3] + [5,2,1] + [4,3,1] &&\\
&\qquad + [4,2^2] + [3,2^2,1], && \text{of degree } 315,\\
[2] \odot [2^2] &= [6,2] + [5,2,1] + [4^2] + [4,2^2] + [3^2,1^2], && \text{of degree } 210,\\
[2] \odot [2,1^2] &= [6,1^2] + [5,3] + [5,2,1] + [4,3,1] + [4,2,1^2] &&\\
&\qquad + [3^2,2], && \text{of degree } 315,\\
[2] \odot [1^4] &= [5,1^3] + [4,3,1], && \text{of degree } 105.
\end{aligned}$$

6. A comparison with Todd's method. In a recent paper Todd[12] gives a method of determining the irreducible components of $\{\alpha\} \otimes \{\beta\}$ or of $[\alpha] \odot [\beta]$ which is akin to the procedure developed here. We propose to comment briefly upon Todd's method.

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[11] [3, p. 296]. [12] [7].

TABLE OF CHARACTERS OF $\mathfrak{N}(H)$ IN S_6

$H = S_2 \times S_2 \times S_2$

ω	3				2				1	
Sets	1	(12)	(12) (34)	(12) (34) (56)	(13) (24)	(1324)	(13) (24) (56)	(1324) (56)	(135) (246)	(135246)
Conjugates	1	3	3	1	6	6	6	6	8	8
$[2; 3]$	1	1	1	1	1	1	1	1	1	1
$[2: 2, 1]$	2	2	2	2	0	0	0	0	−1	−1
$[2; 1^3]$	1	1	1	1	−1	−1	−1	−1	1	1
$[1^2; 3]$	1	−1	1	−1	1	−1	−1	1	1	−1
$[1^2; 2, 1]$	2	−2	2	−2	0	0	0	0	−1	1
$[1^2; 1^3]$	1	−1	1	−1	−1	1	1	−1	1	−1
$[2; 2]\overline{\times}[1^2]$	3	1	−1	−3	1	1	−1	−1	0	0
$[2; 1^2]\overline{\times}[1^2]$	3	1	−1	−3	−1	−1	1	1	0	0
$[1^2; 2]\overline{\times}[2]$	3	−1	−1	3	1	−1	1	−1	0	0
$[1^2; 1^2]\overline{\times}[2]$	3	−1	−1	3	−1	1	−1	1	0	0

https://doi.org/10.4153/CJM-1950-032-4 Published online by Cambridge University Press

In the first place Todd does not utilize the notion of the star diagram, but derives its irreducible components by another method. These components, for the representations under consideration, are listed in the last column of Table I [7, p. 331] along with the quantity θ_σ which can be identified with the *parity* σ used here. Thus the basis of Todd's method is the same as ours, but he goes on to construct functions

6.1 $$\{\alpha\} \otimes \overline{S}_n,$$

where S_n is the sum of the nth powers of the variables in question. These functions have very interesting analogues in the theory of the symmetric group, namely certain *identities*[13] which are satisfied by the characters of the irreducible representations $[\sigma]$ of the symmetric group S_{mn}, if the elements of S_{mn} whose orders are divisible by n are left out of consideration. These identities are of importance in the modular theory but *they have no group theoretical significance* if attention is confined to the ordinary representation theory. In fact, the same remark applies to the functions 6.1 as regards the representation theory of the full linear group. This explains why their use introduces the cancellation of irreducible representations, whose appearance is always awkward in a purely group-theoretic argument. The use of theorems 4.5 and 4.6 avoids this difficulty.

From the point of view of invariant theory it is desirable to have the dual interpretation in terms of the full linear group and the symmetric group available at all times. This duality throws much light on the role played by the variables in the symbolic representation of invariants as opposed to the symmetry properties of the symbolism The significance of the subgroups H and $\mathfrak{N}(H)$ of S_{mn} has yet to be fully explored in this connection.

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