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Secular Resonance Maps

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Abstract. A complete list of combinations of the rates of asteroid perihelia and nodes and the corresponding fundamental frequencies of planets, giving rise to secular resonances and involving up to 4 frequencies, is known from the previous work, while for the resonances with 6 frequencies a systematically derived comprehensive list is given here for the first time. There are 28 divisors in the theory of degree up to 4, not all of which can give rise to resonances, while at degree 6 there are (at least) 33 such possibly resonant frequency combinations.

Mapping the secular resonances by plotting the resonant lines in the phase space of proper elements or of secular frequencies, possibly also against the background of known asteroids, enables to straightforwardly identify resonances causing large long periodic variations of asteroid orbital elements, resonances that interact with known families, those that bound the dynamically distinct regions, deplete or disturb asteroids in these regions, etc.

Keywords. secular resonances, proper elements, asteroid families

1. Introduction

Secular resonances correspond to singularities in the phase space of orbital elements where integer combinations of frequencies of the longitude of perihelion and the longitude of node of perturbed (asteroids) and perturbing (planets) bodies, appearing in divisors of the perturbing terms in the equations of motion, tend to zero. Permitted combinations are determined by D'Alembert rules, but not all of them give rise to secular resonances located in the region of interest.

Roughly, secular resonances can be divided into two broad categories: the linear ones, which are the strongest and most important in terms of the effects they produce in the motion of asteroids, and the nonlinear ones, which affect the motion to a lesser extent, depending on the degree of these resonances themselves. Linear resonances appear in the equations of motion as divisors of the terms linear in eccentricity and/or (sine of) inclination (that is, of degree 2 in the perturbing Hamiltonian) and they include the difference of just two frequencies - one of the asteroid and another of a perturbing planet, while all others are nonlinear resonances, corresponding to divisors of terms with amplitudes containing higher degrees of $(e, \sin I)$ and involving combinations of 4, 6, 8, and so on, frequencies. Limiting our dynamical model to include only the most massive perturbing planets - Jupiter and Saturn - the linear secular resonances are just three: $g - g_5, g - g_6, s - s_6$, while examples of degree 4 and 6 nonlinear ones are, e.g.: $g + s - g_6 - s_6, g + 2s - g_5 - 2s_6$; here g, s denote the secular frequencies of the longitude of perihelion and the longitude of node of the asteroid's orbit, while $g_5 = 4.26$ arcsec/y, $g_6 = 28.25$ arcsec/y, $s_6 = -26.34$ arcsec/y are the corresponding frequencies for Jupiter and Saturn.

Knowledge of the exact positions of secular resonances in the phase space of proper orbital elements or frequencies encompassing the asteroid belt, is of paramount importance for a number of reasons; these include the possibility to reliably assess the interaction with asteroids inside or near the resonant region, the reshaping of asteroid families interacting with resonances, the transport of objects inside or outside the asteroid belt, and so on. Mapping the secular resonances is the most effective tool to pin down and visualize their positions, and to appreciate their importance in terms of the asteroid dynamics.

The first secular resonance maps in the region of asteroid belt were presented by Williams and Faulkner (1981), who plotted secular resonance surfaces projected to $(a, \sin I)$ plane. They showed the lines corresponding to a number of different values of eccentricity, representing the centers of libration only for the three above mentioned linear resonances with Jupiter and Saturn (which they labeled ν_5, ν_6, ν_{16}, respectively, and for one nonlinear resonance located in the very high inclination region. Subsequently, Milani and Knežević (1990), and later Milani and Knežević (1992) and Milani and Knežević (1994), presented improved version of these maps, with resonant positions marked by the contour lines approximately indicating also the width of the resonant strip. They upgraded the mapping by showing the positions of a number of most significant nonlinear secular resonances against the background of known asteroids in the proper elements space, illustrating the interaction of asteroid families with these resonances, studying dynamical behavior of asteroids affected by resonances, etc. Knežević et al. (1991) further extended the representation of positions of linear secular resonances to the interval of semimajor axes from 2 to 50 au, followed by Michel and Froeschle (1997) who mapped the positions of linear secular resonances in the region of semimajor axes less than 2 au. The most recent entries into the literature on secular resonance positions pertain to an attempt to precisely locate the $g - g_5$ secular resonance Knežević (2020), and to a survey of positions of all the resonances in the asteroid belt up to degree 4, as well as of an incomplete list of degree 6 ones and of several resonances of degree 8 Knežević (2022).

To be able to map the secular resonances, one needs to know which are the combinations of frequencies, allowed by the D'Alembert rules, giving rise to resonances. For this purpose in the following we consider the lists of resonant combinations and assess their completeness.

2. Secular resonance census

The most obvious way to assess the completeness of a list of resonances is on a degree by degree basis. We proceed from the linear, degree 2 resonances, involving combinations of two frequencies only, and continue afterwards with resonances of degrees 4, 6, and so on, involving the corresponding number of frequencies, while checking for all of them whether they can achieve zero value in the region of interest, that is, in the asteroid belt.

As already explained above, if we limit ourselves to consider a dynamical model with only Jupiter and Saturn as perturbing planets, we see that the degree two resonances are just three, while in the full planetary system model, with 8 perturbing planets, there are 15 such combinations.

The situation is not so trivial already at the next level, in the degree 4 case. In Table 1 we give a list of 28 divisors appearing in the analytical theory of Milani and Knežević (1990), not all of which, however, can give rise to a resonance - obviously, this is the case with divisor $g_5 - g_6$, which is simply a constant, but also divisors Nos. $22 - 27$ cannot be zero, as frequency s has always a negative value throughout the asteroid belt. Bearing in mind that this list does not include the "fake" divisors, involving s_5, which are removed from the theory by the change of coordinates to express inclinations with respect to the invariable plane, one may state that this list of resonances is complete.

Table 1. Secular resonances of degree 2 and 4

No	Divisor	No	Divisor
1	$g - g_5$	15	$2g - 2s_6$
2	$g - g_6$	16	$g + g_5 - s - s_6$
3	$s - s_6$	17	$g + g_6 - s - s_6$
4	$g - g_5 + s - s_6$	18	$g + g_5 - 2s_6$
5	$g - g_6 + s - s_6$	19	$g + g_6 - 2s_6$
6	$2g - 2s$	20	$s - s_6 + g_5 - g_6$
7	$g - 2g_5 + g_6$	21	$s - s_6 - g_5 + g_6$
8	$g + g_5 - 2g_6$	22	$2s - 2g_5$
9	$2g - g_5 - g_6$	23	$2s - 2g_6$
10	$g - g_5 - s + s_6$	24	$2s - g_5 - g_6$
11	$g - g_6 - s + s_6$	25	$s + s_6 - 2g_5$
12	$2g - s - s_6$	26	$s + s_6 - 2g_6$
13	$g + g_5 - 2s$	27	$s + s_6 - g_5 - g_6$
14	$g + g_6 - 2s$	28	$g_5 - g_6$

While for resonances up to degree 4 we could use the results available from the previous work, for the extension to degree 6 resonances there is no such possibility. Thus, a dedicated effort is needed to get the full list of these resonances suitable for their mapping.

Before we venture to address the problem at hand, let us clarify whether this is a worthwhile effort. Let us, namely, consider whether the degree 6 secular resonances are at all important in the asteroid dynamics: are their effects large enough to be easily recognized and do they offer plausible explanations for some peculiar features of the motion of asteroids and of their distribution in the phase space. For simplicity, we shall recall here only a couple of examples of secular resonances affecting the shapes of asteroid families in the space of proper elements, because these are easy to recognize and understand.

One of the first ever examples of the observable effect of a secular resonance, and to that matter of a degree 6 resonance, on the shape of an asteroid family is due to Bottke et al. (2001): they showed that the so-called "Prometheus surge", consisting of members of the Koronis asteroid family exhibiting systematically larger proper eccentricities than the rest of the family, is a consequence of the interaction of family members migrating outwards from the Sun due to Yarkovsky effect with the secular resonance $g + 2g_5 - 3g_6$.

Another example pertains to Hansa family Milani et al. (2019) affected by the degree 6 secular resonance $2g - g_5 - g_6 + s - s_6$. The effect of this resonance explains a large scatter of proper eccentricity e in the part of the family with semimajor axis $a < a_0$, where a_0 is the proper a of the parent body. The members on this side of the family, also originally ejected in the same direction as suggested by the positive V-shape of the family, are moving preferentially towards the Sun due to the Yarkovsky effect, so that many members currently outside of the secular resonance zone must have passed through it in the past. The eccentricities of the family members being low and inclinations high, the scattering in this case affects much more the proper e than the proper $\sin I$ because of the D'Alembert rule, thus the changes in e are much larger than those in $\sin I$.

There are other examples of this kind, but even just these two are enough to conclude that secular resonances of degree 6 are important and thus worth an effort of identifying and mapping them..

2.1. *Identification of secular resonances of degree 6*

The identification of degree 6 secular resonances is not a straightforward thing to do: the analytical theory based on the expansion of perturbing function beyond degree 4 terms in eccentricity and sine of inclination, but with inclinations expressed with respect to a common coordinate plane, is not available Ellis and Murray (2000), and a computer

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Table 2. Secular resonances of degree 4 as combinations of linear and Kozai ones. Columns gs and ν give the same divisor written in two ways

No	gs	ν	No	gs	ν
1	$g-g_5$	ν_5	15	$2g-2s_6$	$2\nu_{16}+2K$
2	$g-g_6$	ν_6	16	$g+g_5-s-s_6$	$2K-\nu_5+\nu_{16}$
3	$s-s_6$	ν_{16}	17	$g+g_6-s-s_6$	$2K-\nu_6+\nu_{16}$
4	$g-g_5+s-s_6$	$\nu_5+\nu_{16}$	18	$g+g_5-2s_6$	$2K-\nu_5+2\nu_{16}$
5	$g-g_6+s-s_6$	$\nu_6+\nu_{16}$	19	$g+g_6-2s_6$	$2K-\nu_5+2\nu_{16}$
6	$2g-2s$	Kozai	20	$s-s_6+g_5-g_6$	$\nu_6-\nu_5+\nu_{16}$
7	$g-2g_5+g_6$	$2\nu_5-\nu_6$	21	$s-s_6-g_5+g_6$	$\nu_5-\nu_6+\nu_{16}$
8	$g+g_5-2g_6$	$2\nu_6-\nu_5$	22	$2s-2g_5$	$2\nu_5-2K$
9	$2g-g_5-g_6$	$\nu_5+\nu_6$	23	$2s-2g_6$	$2\nu_6-2K$
10	$g-g_5-s+s_6$	$\nu_5-\nu_{16}$	24	$2s-g_5-g_6$	$\nu_5+\nu_6-2K$
11	$g-g_6-s+s_6$	$\nu_6-\nu_{16}$	25	$s+s_6-2g_5$	$2\nu_5-\nu_{16}-2K$
12	$2g-s-s_6$	$\nu_{16}+2K$	26	$s+s_6-2g_6$	$2\nu_6-\nu_{16}-2K$
13	$g+g_5-2s$	$2K-\nu_5$	27	$s+s_6-g_5-g_6$	$\nu_5+\nu_6-\nu_{16}-2K$
14	$g+g_6-2s$	$2K-\nu_6$	28	g_5-g_6	$\nu_6-\nu_5$

algebra system like TRIP Gastineau and Laskar (2011) is both quite demanding to use and available only in a limited version, in any case not yet applied for our purpose. There is, however, a simple and efficient "empirical" way to find most, if not all, secular resonances of degree 6 (and higher, if neeeded) and produce their list; whether such a list is also complete remains to be verified by one of the above methods.

Carruba and Michtchenko (2009) proposed a new notation for nonlinear secular resonances that employs the ν_k labels introduced by Williams and Faulkner (1981); the new notation was based on the idea that nonlinear resonances can be expressed as simple combinations of linear ones (c.f. $g+g5-2g_6=2\nu_6-\nu_5$; $2g-s-g_5-g_6+s_6=\nu_5+\nu_6-\nu_{16}$). Although this conjecture soon proved not to be entirely correct since only 10 resonances of degree 4 could be constructed by means of the three linear ones and 14 of them could not (e.g. resonances like $g-2s+g6$ or $s+s6-2g5$, appearing in Table 1, cannot be obtained as a combination of linear ones), the idea lingered around, to eventually become useful providing an appropriate correction is added. It turned out, namely, that all divisors in Table 1 can be recovered if also Kozai's resonance $2K=2g-2s$ is included in the arithmetic. This is demonstrated in Table 2, where all the resonances of degree 4 listed in Table 1 are expressed as combinations of either the linear ones only, or as combinations that involve Kozai's resonance as well.

It is now straightforward to assume that resonances of higer degree, in particular these of degree 6, can be identified in the same way. The procedure may be considered an extrapolation, but note that D'Alembert rules are at work here, making the higher degree terms logically connected with the lower degree ones. In Table 1 we thus list all 33 resonances of degree 6 identified as combinations of linear resonances only (1–20), and as combinations with Kozai's resonance included too (21–33).

3. Mapping of resonances

Once we have a list of secular resonances at our disposal, it is easy to map them and visualize their positions and interactions with asteroid families and background objects in a suitable phase space, be it a proper elements space or, perhaps more appropriate in this context, a secular frequencies space. In the former case, one uses frequencies computed on a regular grid by means of the polynomial fitting procedure developed by Knežević and Milani (2019), and represents them by means of the contour lines in the plane of two proper elements (e.g. $a, \sin I$), while keeping the third one (e) fixed Knežević (2022). In the latter case, resonance loci are simply plotted as straight lines in the corresponding (g, s) plane. Adding in the same figures the positions of asteroids selected to

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Table 3. Secular resonances of degree 6 as combinations of linear and Kozai ones. Columns gs and ν give the same divisor written in two ways

No	gs	ν	No	gs	ν
1	$3g-2g_5-g_6$	$2\nu_5+\nu_6$	21	$3g-2s-g_5$	ν_5+2K
2	$3g-g_5-2g_6$	$\nu_5+2\nu_6$	22	$3g-2s-g_6$	ν_6+2K
3	$2g+s-g5-g6-s6$	$\nu_5+\nu_6+\nu_{16}$	23	$3s-2g-s_6$	$\nu_{16}-2K$
4	$2g-s-g_5-g_6+s_6$	$\nu_5+\nu_6-\nu_{16}$	24	$3s-g-g_5-s_6$	$\nu_5+\nu_{16}-2K$
5	$2g+s-2g_5-s_6$	$2\nu_5+\nu_{16}$	25	$3g-s-g5-s6$	$\nu_5+\nu_{16}+2K$
6	$2g-s-2g_5+s_6$	$2\nu_5-\nu_{16}$	26	$3s-g-g_6-s_6$	$\nu_6+\nu_{16}-2K$
7	$2g+s-2g_6-s_6$	$2\nu_6+\nu_{16}$	27	$3g-s-g_6-s_6$	$\nu_6+\nu_{16}+2K$
8	$2g-s-2g_6+s_6$	$2\nu_6-\nu_{16}$	28	$2g-2s-g_5+g_6$	$\nu_5-\nu_6+2K$
9	$g+2s-g_5-2s_6$	$\nu_5+2\nu_{16}$	29	$2g-2s+g_5-g_6$	$\nu_6-\nu_5+2K$
10	$g-2s-g_5+2s_6$	$\nu_5-2\nu_{16}$	30	$2s-g-2g_5+g_6$	$2\nu_5-\nu_6-2K$
11	$g+2s-g_6-2s_6$	$\nu_6+2\nu_{16}$	31	$2s-g+g_5-2g_6$	$2\nu_6-\nu_5-2K$
12	$g-2s-g_6+2s_6$	$\nu_6-2\nu_{16}$	32	$g+2s-g_5-2g_6$	$\nu_5+2\nu_6-2K$
13	$2g-3g_5+g_6$	$3\nu_5-\nu_6$	33	$g+2s-2g_5-g_6$	$2\nu_5+\nu_6-2K$
14	$2g+g_5-3g_6$	$3\nu_6-\nu_5$			
15	$g+s-2g_5+g_6-s_6$	$2\nu_5-\nu_6+\nu_{16}$			
16	$g+s+g_5-2g_6-s_6$	$2\nu_6-\nu_5+\nu_{16}$			
17	$g-s-2g_5+g_6+s_6$	$2\nu_5-\nu_6-\nu_{16}$			
18	$g-s+g_5-2g_6+s_6$	$2\nu_6-\nu_5-\nu_{16}$			
19	$2s-g_5+g_6-2s_6$	$\nu_5-\nu_6+2\nu_{16}$			
20	$2s+g_5-g_6-2s_6$	$\nu_6-\nu_5+2\nu_{16}$			

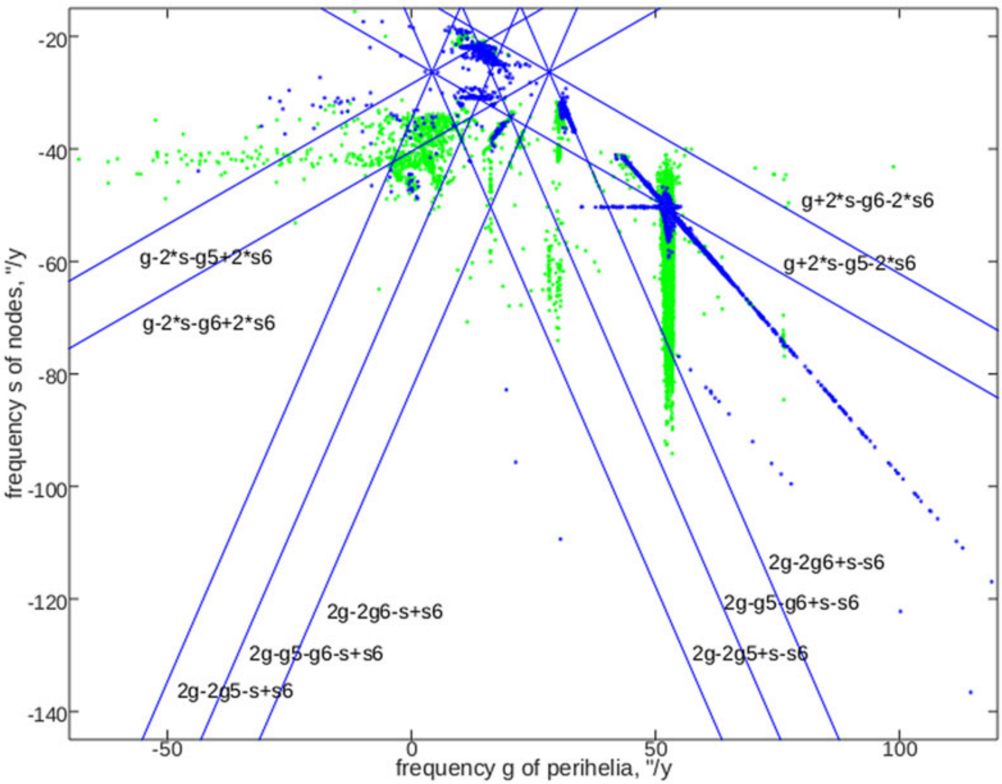

Figure 1. Sample of secular resonances of degree 6 plotted in the plane of frequencies (g, s). Overlapped are asteroids with deteriorated proper elements: $\sigma e > 0.005$ (green dots) and $\sigma \sin I > 0.003$) (blue dots).

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demonstrate interaction of the resonance with a particular asteroid population, (families, asteroids with deteriorated proper elements, or similar), one also enables analysis of the resonant dynamics and effects these interactions cause.

In Figure 1 a sample of secular resonances of degree 6 is shown in the plane of frequencies (g, s). Also shown are asteroids with deteriorated proper elements - these with $\sigma e > 0.005$ (green dots) and with $\sigma \sin I > 0.003$ (blue dots). Although the strips with lower accuracy data align mostly with resonances of different degrees not shown in the figure, in at least one case ($2g - 2g_6 + s - s_6$, at $g \simeq 30$ arcsec/y, $s \simeq -30$ arcsec/y) the alignment of blue dots with the resonance is obvious.

4. Conclusions

In conclusion one can state that the complete list of resonances of degree 6 (and possibly higher) is indeed necessary, since many of them have important effects on the dynamics of asteroids, and that their exact positions must be known to pursue the research. Whether the simple scheme presented in this work solves the issue of completeness of their census remains to be verified in the future with some more sophisticated approaches.

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