THE NON-BIPLANAR CHARACTER OF THE COMPLETE 9-GRAPH

W. T. Tutte

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1. Introduction. Let us define a planar partition of a graph G as a pair $\{H_1, H_2\}$ of subgraphs of G with the following properties

(i) Each of H_1 and H_2 includes all the vertices of G.

(ii) Each edge of G belongs to just one of H_1 and H_2.

(iii) H_1 and H_2 are planar graphs.

It is not required that H_1 and H_2 are connected. Moreover either of these graphs may have isolated vertices, incident with none of its edges.

We describe a graph having a planar partition as biplanar.

The problem of characterizing biplanar graphs has been found of interest in connection with the design of computing machinery. The simplest graph for which the problem has proved difficult is the complete 9-graph. This graph has 9 vertices and 36 edges, each pair of vertices being joined by a single edge.

The complete 9-graph has been proved non-biplanar in [2]. The object of this note is to present another proof which, so Professor Harary assures me, has its own points of interest.

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2. <u>Triangulations of the sphere.</u> Assume that the complete 9-graph G has a planar partition $\{H_1, H_2\}$. Then H_1 can be realized as a topological graph K_1 in the 2-sphere. If two vertices of the realization lie in the boundary of the same residual region, and are not joined by an edge of K_1, they can be joined by an arc L in the residual region. Then we can transfer an edge from H_2 to H_1 so that the new edge of H_1 is represented by L. Both H_1 and H_2 remain planar in this operation. Repeating the process sufficiently often we can arrange that K_1 defines a map on the sphere whose faces are all triangular. We call such a map a <u>triangulation</u> (of the sphere).

The <u>complementary</u> graph G' of a graph G is defined as a graph with the same vertices as G, and in which two distinct vertices are joined by an edge if and only if they are not so joined in G. The graph G' has no "loops", that is edges with coincident ends. We deduce from the foregoing considerations the following theorem

(2.1) <u>The assertion that the complete 9-graph is not biplanar is equivalent to the assertion that if the graph of a triangulation has 9 vertices then its complementary graph is not planar.</u>

Let M be a triangulation of the sphere, defined by a graph K. A <u>separating triangle</u> of M is a triangle, made up of three edges of K, which does not bound a face of M. That is, the triangle has vertices of K in both its residual domains. A triangulation with no separating triangles will be called <u>simple</u>.

A triangulation having a separating triangle T can be derived from a triangulation of fewer vertices, in which T bounds a face F, by subdividing F into smaller triangular regions. By repeated application of this observation we obtain

(2.2) <u>If a triangulation is not simple it can be derived from some simple triangulation of 4 or more vertices by subdividing one or more faces into smaller triangular regions.</u>

A subdivision of a face will be said to be of <u>order</u> n if it introduces just n new vertices.

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We shall use (2.2) to reduce our problem to a study of simple triangulations of from 4 to 9 vertices. Fortunately there are so few of these triangulations that they can be studied individually. Figures I, II, III, IV, and V show those of 4, 6, 7, 8 and 9 vertices respectively. There is no simple triangulation of 5 vertices.

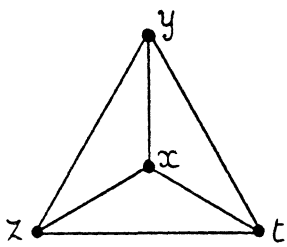

Figure I

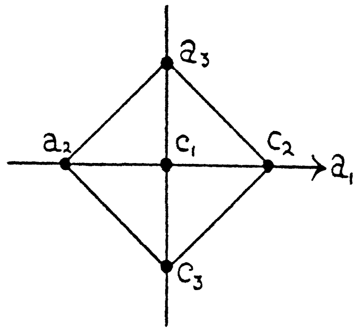

Figure II

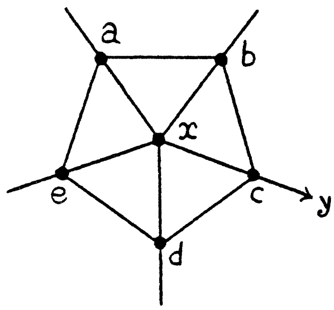

Figure III

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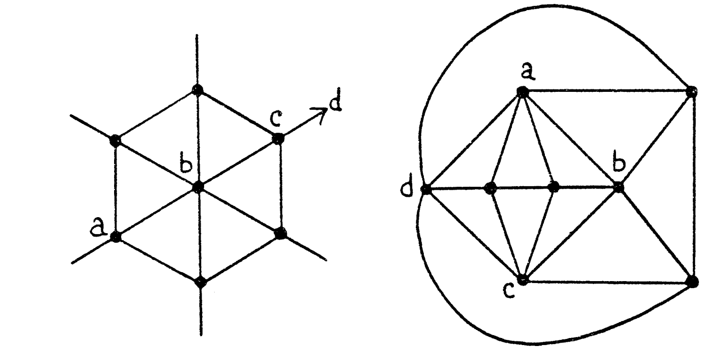

Figure IV

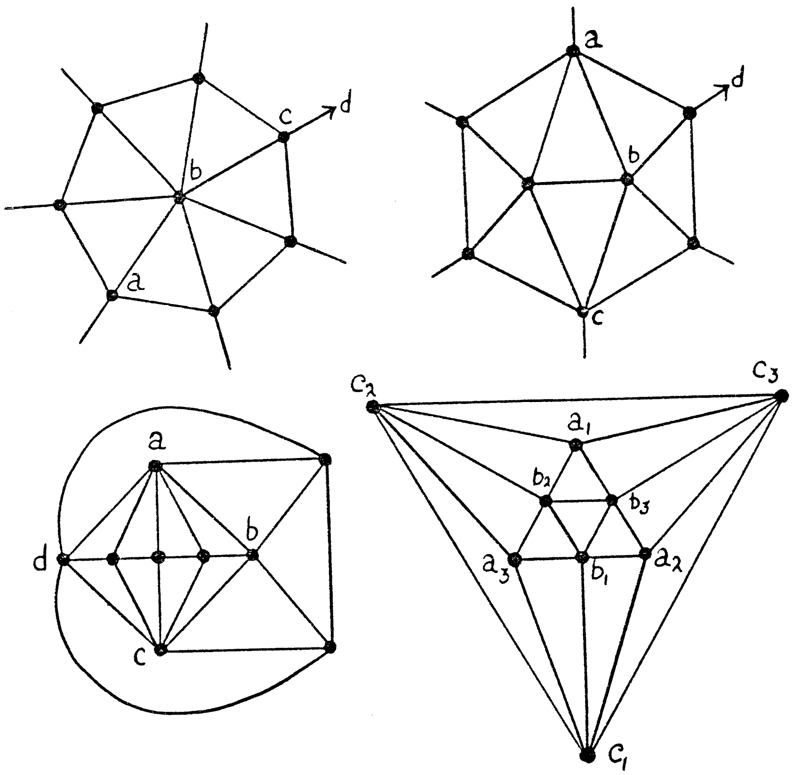

Figure V

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In Figure II the four outer edges are supposed to radiate to a sixth vertex a_1 at infinity (in a sterographic projection of the sphere). The point at infinity is also a vertex in Figure III, in the first diagram of Figure IV and in the first and second diagrams of Figure V.

Before going any further we must show that our list of simple triangulations of from 4 to 9 vertices is complete.

Let M be a simple triangulation defined by a graph K. Let it have n vertices, where $4 \leq n \leq 9$.

We observe that no two faces of M have the same three vertices. Otherwise they would have the same boundary, their union with it would be the whole sphere, and n would be 3. Hence no two faces have two sides in common, and therefore the valency of each vertex is at least 3.

Let v be a vertex of valency k. Its k incident faces, with their incident edges and vertices, constitute a closed simply connected region R_v bounded by a polygon P_v made up of k vertices and k edges of K. If we count v as being "inside" P_v the other $n - k - 1$ vertices of M not on P_v will be outside P_v. We note that P_v has no diagonal in K, that is no edge of K joins two non-consecutive vertices of P_v. For such a diagonal, together with the edges of K joining its two ends to v, would constitute a separating triangle.

Let a be a vertex of M of maximum valency m. Suppose first that there is no vertex outside P_a. Then $m = 3$ and $n = 4$ for otherwise the outside of P_a would be subdivided into two or more faces and P_a would have a diagonal in K. This is the case illustrated in Figure I.

In the remaining case $m \geq 4$ and $n \geq 6$, since otherwise P_a would be a separating triangle.

Suppose there is just one vertex, b say, outside P_a. Each edge of P_a is incident with a face not having a as a vertex. Since P_a has no diagonal in K this face must be incident with b. Hence each vertex of P_a is joined to b by an edge of M. This is the case illustrated by Figures II and III and the first diagrams of Figures IV and V.

In the remaining case there are at least two vertices outside P_a. If b is any one of these it is joined by an edge of K to another. For suppose not. Let c be any vertex, other than b, outside P_a. The edges incident with b join it only to vertices of P_a, and any two consecutive ones are sides of a face having b as a vertex. Since P_a has no diagonal in K it follows that b is joined to each vertex of P_a. The edges incident with b thus subdivide the exterior of P_a into faces. One of these has c in its interior and a in its exterior, contrary to the hypothesis that M is simple.

We have shown that K has edges with both ends outside P_a. At least one such edge must be incident with a face whose third vertex, x say, is on P_a. But then x is at least penta-valent. So in the case still outstanding $m \geq 5$ and $n \geq 8$.

Suppose two vertices b and c outside P_a are joined by an edge bc of K, and that the two faces incident with bc have their third vertices, x and y say, on P_a. The points x and y partition P_a into two arcs L and L'. We may suppose that the arcs xby and xcy subdivide the exterior of P_a into three regions, S_1 bounded by L and xby, S_2 bounded by xby and xcy, and S_3 bounded by xcy and L'. We observe that L must have at least one internal vertex, that is a vertex distinct from x and y, for otherwise L and xcy would define a separating triangle. Similarly L' has at least one internal vertex.

If $m = 5$ we may suppose L to have one internal vertex u and L' to have two, v and w say, w being adjacent to

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y on P_a (Figure VI).

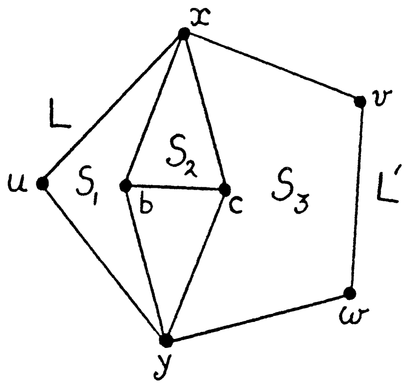

Figure VI

Now x is already joined to a, b, c, u and v. Since its valency is at most 5 we have to complete the triangles xbu and xcv. Similarly, replacing x by y, we must complete the triangle cwy. We cannot insert a third vertex outside P_a without introducing a separating triangle. We have now constructed the second diagram of Figure IV.

If m = 6 there are two possibilities. Either L has one internal vertex and L′ has three, or L and L′ have two each. In each case we can complete the triangulation only by joining b to all the internal vertices of L and c to all those of L′. The two resulting triangulations correspond to the second and third diagrams of Figure V.

In the remaining case m = 5, there are just three vertices b, c and d outside P_a, and these are the vertices of a face of M. The second faces incident with bc, db and cd have vertices x, y and z respectively on P_a. The vertices x, y and z must be distinct. If for example x = y then the edges db, dx and bx define a separating triangle. The vertices x, y and z separate P_a into three arcs L_{xy}, L_{yz} and L_{xz}, where the suffices indicate the ends of the corresponding arc

(Figure VII).

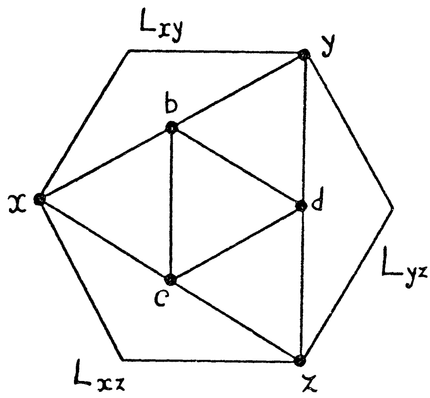

Figure VII

We can only complete the triangulation by joining b to the internal vertices of L_{xy}, c to those of L_{xz} and d to those of L_{yz}. Hence, since $m = 5$, we may suppose L_{xy} and L_{yz} to have one internal vertex each, while L_{xz} has none. We are then forced to construct the fourth diagram of Figure IV.

Another way to establish the completeness of our list of simple triangulations would be to make use of the enumerative results of [4].

3. Tests for non-planarity. The theorem of Kuratowski on non-planar graphs is well known [1, 3]. It states that a graph is non-planar if and only if one of its subgraphs is a subdivision of the Thomsen graph or the complete 5-graph. The Thomsen graph has 6 vertices a_1, a_2, a_3, b_1, b_2, b_3, and 9 edges, each a_i being joined to each b_j by a single edge. The complete 5-graph has 5 vertices a_1, a_2, a_3, a_4, a_5, and 10 edges, each pair of vertices being joined by a single edge. Subdivision is effected by breaking some edges into arcs of two or more new edges by inserting internal vertices.

From this theorem we may derive the following corollary.

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(3.1) <u>A graph G is non-planar if one of the following propositions is true</u>

(i) <u>G has six disjoint connected subgraphs A_1, A_2, A_3, B_1, B_2, B_3 such that for each A_i and each B_j we can find an edge with one end in A_i and the other in B_j.</u>

(ii) <u>G has five disjoint connected subgraphs A_1, A_2, A_3, A_4 and A_5 such that whenever $1 \leq i < j \leq 5$ we can find an edge with one end in A_i and the other in A_j.</u>

The proof is straightforward and may legitimately be left to the reader.

We shall use (3.1) to test the complementary graphs of triangulations for non-planarity. It will be convenient to have the following auxiliary theorems.

(3.2) <u>Let N, defined by a graph J, be a triangulation of the sphere. Let it have a separating triangle T with three vertices a_1, a_2 and a_3 on one side and three vertices b_1, b_2 and b_3 on the other. Then the complementary graph J' of J is non-planar.</u>

<u>Proof.</u> J' contains a Thomsen graph with vertices a_1, a_2, a_3, b_1, b_2, b_3.

(3.3) <u>Let N, defined by a graph J, be a triangulation of the sphere. Let $Q = abcd$ be a quadrilateral in J with two vertices u and v in one residual domain and three vertices x, y and z in the other. Suppose further that no edge of J joins a and c, or b and d. Then either J' is non-planar or one of the vertices x, y and z is joined to all four of a, b, c and d by edges of J.</u>

<u>Proof.</u> Suppose neither x, y nor z is joined to both a and c in J. Let B_1 be the connected subgraph of J' defined by the vertices a and c and the edge joining them. Let A_1,

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A_2, A_3, B_2 and B_3 be subgraphs of J' consisting of the single vertices x, y, z, u and v respectively. Then condition (i) of (3.1) holds for J', and therefore J' is non-planar.

Similarly J' is non-planar if neither x, y nor z is joined to both b and d in J.

In the remaining case we may suppose x joined to a and c in J, and some $w \in \{x, y, z\}$ joined to b and d in J. But the arcs axc and bwd, which lie in the same residual domain of Q, must cross. Hence $w = x$.

4. <u>The triangulations of 9 vertices</u>. Let N, defined by a graph J, be any triangulation of the sphere with 9 vertices. By (2.2) it is either simple or derived from a simple triangulation M, defined by a graph K with at least 4 vertices, by subdividing one or more of its faces. In case of ambiguity M is to be chosen to have as many vertices as possible.

Suppose first that N is one of the triangulations of Figure V. For the first three J' is non-planar by (3.3). The required quadrilateral Q = abcd is marked in each diagram.

For the fourth diagram let A_i, (i = 1, 2, 3), be the connected subgraph of J' defined by a_i and c_i, with the edge joining them. Let B_i consist of the single vertex b_i. Then condition (i) of (3.1) holds for J', whence J' is non-planar.

Next suppose M is one of the triangulations of Figure IV. Then N is formed from M by subdividing one face, the order of the subdivision being 1. J' is non-planar by (3.3). The required quadrilateral abcd is again marked in each diagram.

Next suppose M is the triangulation shown in Figure III. Two new vertices are introduced by the subdivision. Suppose first that they are in two different faces of M not incident with the same edge of the pentagon abcde. Then they can be separated by a quadrilateral such as dxby, and J' is non-planar by (3.3).

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In the remaining alternative with Figure III the two new vertices may be supposed to be in triangles xab and yab, possibly in the same one of these triangles. Let the new vertices be v and w. We may suppose the notation adjusted so that v and x are not joined in J. It is clear that w is not joined to both x and y in J. Let A_1 be the connected subgraph of J' defined by the edges ad and bd. Let A_2 be the one defined by the edges xy and vx. Let A_3, A_4 and A_5 consist of the single vertices c, e and w respectively. Then J' satisfies condition (ii) of (3.1) and is thus non-planar.

Next we suppose that M is the triangulation shown in Figure II. Three new vertices b_1, b_2 and b_3 are introduced by the subdivision. Let A_i, $(i = 1, 2, 3)$, be the connected subgraph of J' defined by the edge a_ic_i, and let B_i be the one consisting of the single vertex b_i. Then J' satisfies condition (i) of (3.1) and is thus non-planar.

In the remaining case M is the triangulation of Figure I. Suppose the face xyz has a subdivision of order s. Then the orders of the subdivisions of the other three faces sum to $5 - s$. Given a separating triangle T with all its vertices inside or on the boundary of the face xyz, and which is not xyz itself, we can consider the operation of fusing all the faces, edges and vertices inside T so that they become a single face. Repeating this procedure sufficiently often we obtain a simple triangulation M_1 with at least four vertices, one of its faces being the exterior of the face xyz of M. Evidently N is a subdivision of M_1 and the associated subdivision of the face xyz of M_1 is of order $6 - s$.

But by the choice of M the triangulation M_1 still has only four vertices. It too can be represented by Figure 1.

If $s = 5$ then $6 - s = 1$, and no face of M_1 has a subdivision of order higher than 4.

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If $s = 4$ then $6 - s = 2$, and no face of M_1 has a subdivision of order higher than 3.

We can therefore reduce to the case in which no face of M has more than three new vertices of N in its interior.

If now some face of M contains three new vertices of N the graph J' is non-planar by (3.2).

In the remaining case let the 5 new vertices of N be a, b, c, d and e. If two belong to the same face of M distinguish one as the leading vertex of that face. If a is the leading vertex of a face F of M define A as the connected subgraph of J' given by the edge ax, where x is the vertex of M not incident with F. Otherwise let A consist of the single vertex a. Define B, C, D and E similarly. Then J' satisfies condition (ii) of (3.1) and is thus non-planar.

This completes the proof that every triangulation of 9 vertices has a non-planar complementary graph. It now follows from (2.1) that the complete 9-graph is not biplanar.

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University of Waterloo

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