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RADICALS OF RINGS AND THEIR SUBRINGS

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It is a fundamental fact in the theory of radicals of associative rings that if S is a radical and I is a two-sided ideal of R then $S(I) \subseteq S(R)$. In view of this result it seems to be interesting to investigate radicals satisfying such or similar connections for other type of subrings. There are many works devoted to similar problems (**2**, **7**, **8**, **10**). In this paper we try to get a uniform description of some facts in this area.

All rings in this paper are associative. For undefined terms and used facts we refer to (**12**).

An ideal of a ring always means a two-sided ideal. For any subring P of a ring R, P^* will denote the ideal of R generated by P and P^{**} the ideal of P^* generated by P.

A hereditary class of rings means a class closed under taking ideals.

1. General remarks

Throughout this paper D will denote a class of pairs (P, R) where P is a subring of a ring R such that the following conditions are satisfied

D1. $(I, R) \in D$ whenever I is an ideal of R;

D2. if $(P, R) \in D$ then $(P, P^*) \in D$;

D3. if $(P, R) \in D$ and f is a homomorphism of R then $(f(P), f(R)) \in D$.

If $(P, R) \in D$ then we say that P is a D-subring of R. A radical class S will be called D-radical if for any ring R, $S(R)$ contains all D-subrings of R which are in S.

We say that a subring P of a ring R is accessible (left accessible) if R contains a sequence of subrings

$$P = P_0 \subseteq \ldots \subseteq P_n = R$$

such that P_i is an ideal (left ideal) of P_{i+1} for $i = 0, \ldots, n-1$.

Taking as D-subrings exactly: ideals, accessible subrings, left ideals, left accessible subrings or subrings we obtain different examples of classes D. Since semisimple classes are hereditary it can be concluded that in the first two cases D-radicals are exactly radicals. Thus any result concerning D-radicals gives information about all radicals. In the remaining examples D-classes and D-radicals will be called left strong (c.f. (**2**)), left stable (c.f. (**3**)) and strict (c.f. (**10**)) respectively.

Let us remark that if a class D is left stable, strict or if D-subrings are exactly accessible subrings, then D satisfies

D4. if $(P_1, R) \in D$ and $(P_2, P_1) \in D$ then $(P_2, R) \in D$.

The following proposition is an immediate consequence of the definition of D-radicals.

Proposition 1. *A radical S is D-radical if and only if any S-semisimple ring does not contain non-zero S-radical D-subrings.*

If D satisfies D1–D4 *then a radical S is D-radical if and only if the semisimple class of S is hereditary with respect to D-subrings.*

Corollary 1. *If D satisfies* D1–D4 *then for any class N of rings there exists the upper D-radical determined by N i.e. the largest D-radical S for which all rings in N are S-semisimple.*

Proof. Let $\mathfrak{R}$ be the family of all semisimple and hereditary with respect to D-subrings classes containing N. Certainly the class of all rings is in $\mathfrak{R}$, so $\mathfrak{R}$ is not empty. Let $N = \bigcap_{K \in \mathfrak{R}} K$. Of course N is a class which is semisimple and hereditary with respect to the D-subrings. Thus by Proposition 1 the upper radical U determined by N is a D-radical. Of course all rings from N are U-semisimple and U is the largest radical with this property.

It has been proved in (**4**) that upper left strong radicals may not exist, so Corollary 1 is not true for all D-classes.

Proposition 2 (c.f. (**4, 9**)). *A radical S is D-radical if and only if the fact that $P \in S$ is a D-subring of a ring R implies that $P^* \in S$.*

Proof. Let S be a D-radical and let $P \in S$ be a D-subring of a ring R. Then by D2, P is a D-subring of P^*. Thus $P \subseteq S(P^*)$. But $S(P^*)$ is an ideal of R, so $P^* = S(P^*)$.

As an obvious consequence of the above Proposition we get

Corollary 2. *The intersection of any family of D-radicals is D-radical as well. In particular for any class M of rings there exists the smallest D-radical LD(M) containing M.*

The radical $LD(M)$ will be called the lower D-radical determined by M.

For any ring R, R^1 will denote the natural extension of R to the ring with unity.

Lemma 1. *If P is a subring of a ring R then the ring $P^*/(P^*)^2$ is a sum of ideals that are homomorphic images of P.*

Proof. Of course $P^* = R^1PR^1$. Since $R^1P^2R^1 \subseteq (P^*)^2$ then for any a, $b \in R^1$ the mapping $f_{a,b}: P \to P^*/(P^*)^2$ defined by $f_{a,b}(p) = apb + (P^*)^2$ is a homomorphism. Now $\operatorname{Im} f_{a,b} = aPb + (P^*)^2/(P^*)^2$ is an ideal of $P^*/(P^*)^2$. These and the equality $P^*/(P^*)^2 = \sum_{a,b \in R^1} aPb + (P^*)^2/(P^*)^2$ end the proof.

By Andrunakievic's Lemma one can get easily

Lemma 2. *Let P be a subring of a ring R. Then*

(a) $(P^*)^3 \subseteq P^{**}$;

(b) *for any $a \in R^1$, $b \in P^*$, $aPb + P^{**}$ is an ideal of P^*;*

(c) *for any* $a \in R^1$, $b \in P^*$ *the mapping* $f_{a,b}: P \to aPb + P^{**}/P^{**}$ *given by* $f_{a,b}(p) = apb + P^{**}$ *is a homomorphism.*

In the sequel M will denote a homomorphically closed class of rings.

Corollary 3 (c.f. (**10**)). *If* $0 \neq P \in M$ *is a* D*-subring in* R *then any non-zero homomorphic image of* P^* *contains a non-zero* D*-subring of* M.

Proof. Let $f: P^* \to A$ be a non-zero epimorphism of rings. By D3, $f(P) \in M$ is a D-subring in A. Thus if $f(P) \neq 0$ then A contains a non-zero D-subring of M. So, let $f(P) = 0$ or, in other words, let $P \subseteq \operatorname{Ker} f = K$. Then $P^{**} \subseteq K$. If $(P^*)^2 \subseteq K$, then the result is a consequence of Lemma 1 and D1. If $(P^*)^2 \nsubseteq K$ then the fact that $(P^*)^2 = R^1PP^*$ implies that there exist $a \in R^1$, $b \in P^*$ such that $aPb \nsubseteq K$. By Lemma 2 and D1, D3, $(aPb + K)/K \in M$ is a non-zero D-ideal of $P^*/K \approx A$. This completes the proof.

Let us define for any class M: $M_1 = M$ and $M_\alpha =$ the class of all rings such that any non-zero homomorphic image of R contains a non-zero D-subring of M_β for some $\beta < \alpha$.

It is obvious that the classes M_α are homomorphically closed and if $\beta < \alpha$ then $M_\beta \subseteq M_\alpha$.

By Corollary 3 we have

Theorem 1. *A class* M *is* D*-radical if and only if* $M = M_2$.

We say that a class N is regular if any non-zero ideal of a ring in N can be homomorphically mapped onto a non-zero ring in N.

Directly by Theorem 1 we have

Corollary 4. *If a class* N *is regular then the upper radical* U_N *determined by* N *is a* D*-radical if and only if any non-zero* D*-subring of a ring in* N *can be homomorphically mapped onto a non-zero ring in* N.

2. Constructions of lower D-radicals

By Theorem 1 we can immediately extend Kurosh's construction of lower radicals ((**12**), §9) to D-radicals.

Corollary 5. *For any class* M, $LD(M) = \bigcup M_\alpha$.

For some classes M and D it is also possible to extend a construction of Baer's lower radical ((**12**), §27, Th. 27.2 and 12.7). For this purpose we define for any ring R ideals $L_\alpha(R)$. Let $L_1(R)$ be the ideal of R generated by all D-subrings of R from M. If ideals $L_\alpha(R)$ are defined for $\alpha < \beta$ and β is a limit ordinal then let $L_\beta(R) = \bigcup_{\alpha<\beta} L_\alpha(R)$. If $\beta = \alpha + 1$ then let $L_\beta(R) \supseteq L_\alpha(R)$ be the ideal of R such that $L_\beta(R)/L_\alpha(R) = L_1(R/L_\alpha(R))$. Let $L(R) = \bigcup L_\alpha(R)$. Of course $L(R)$ is contained in the $LD(M)$-radical

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of R. In addition, $L(R)$ is the $LD(M)$-radical of R for any ring R if and only if the following condition is satisfied.

A. A ring R is $LD(M)$-semisimple if and only if R does not contain non-zero D-subrings of M.

Let us observe that the condition A implies $LD(M)=M_2$.

By Corollary 5 we conclude that the condition A is equivalent to the following:

B. If a ring R does not contain non-zero D-subrings of M_2 then R does not contain non-zero D-subrings of M.

It is easy to see that the condition B issues from the next:

B′. If a ring R contains a D-subring P and P contains a non-zero D-subring of M then R contains a non-zero D-subring of M.

The condition B' is automatically satisfied if D satisfies D1–D4. So we have

Corollary 6. *If D satisfies* D1–D4 *then a ring R is $LD(M)$-semisimple if and only if R does not contain non-zero D-subrings of M.*

If D-subrings are exactly ideals then by Andrunakievic's Lemma it follows that the condition B′ is satisfied if M is a subclass of the class of idempotent rings or M is a hereditary class containing all zero-rings.

From the above remarks we obtain immediately many known characterizations of lower radicals (see (**12**), chapter III). We also obtain that the $LD(M)$-radical of a ring R is equal to $L(R)$ if D is stable or strict.

The condition B is also satisfied for left strong classes D and some M (c.f. (**2**), Theorem 3). The following result will be used in the next section.

Proposition 3. *If M is a hereditary radical and $0\neq L\in M_2$ is a left ideal of a ring R then R contains a non-zero left ideal of M.*

Proof. Let $K\in M$ be a left ideal of L. Then LK is a left ideal of R and an ideal of K. By hereditariness of M, $LK\in M$. This ends the proof if $LK\neq 0$. Now let $LK=0$ for any left ideal $K\in M$ of L. Then $L(K+KL)=0$. In particular $(K+KL)^2=0$. Thus K and Kl for $l\in L$ are ideals of $K+KL$. Now it is easy to see that the map $f\colon K\to Kl$ given by $f(k)=kl$ is a ring homomorphism, so $Kl\in M$. Therefore $K+KL=K+\sum_{l\in L} Kl\in M$ and hence any left ideal $K\in M$ of L is contained in $M(L)$. But then $L/M(L)$ does not contain non-zero left M-ideals. Since $L\in M_2$ then $L=M(L)$. This proves the Proposition.

Now we shall describe a construction of lower D-radicals which generalises the Tangeman–Kreiling construction (**11**) for lower radicals.

Let M be a given class. We define a class M^α for each ordinal α as follows: $M^1=M$; $M^{\alpha+1}=\{A\mid A$ contains a D-subring $P\in M^\alpha$ with $A/P^*\in M^\alpha\}$; $M^\beta=\{A\mid A$ is the union of a chain of ideals from $\bigcup_{\alpha<\beta} M^\alpha\}$ if β is a limit ordinal.

Let $\bar{M}=\bigcup M^\alpha$.

Immediately from the above definition we get:

(a) if a ring A is the union of a chain of ideals of $\bar{M}$ then $A \in \bar{M}$. In particular any ring contains a maximal ideal from $\bar{M}$;

(b) if $I \in \bar{M}$ is an ideal of A and $A/I \in \bar{M}$ then $A \in \bar{M}$. In particular if I is a maximal $\bar{M}$-ideal of a ring R then R/I does not contain non-zero $\bar{M}$-ideals;

(c) if $P \in \bar{M}$ is a D-subring of A and $A/P^* \in \bar{M}$ then $A \in \bar{M}$.

By a simple induction we obtain

Lemma 3. *All classes M^α are homomorphically closed.*

From (a), (b) and Lemma 3 we have

Corollary 7. *$\bar{M}$ is a radical class.*

Lemma 4. *If $P \in \bar{M}$ is a D-subring of R then $P^* \in \bar{M}$.*

Proof. By (c) it follows that it is enough to prove that $P^*/P^{**} \in \bar{M}$. We shall prove first that $(P^*)^2 + P^{**}/P^{**} \in \bar{M}$. The condition (a) implies that $(P^*)^2 + P^{**}/P^{**}$ contains a maximal $\bar{M}$-ideal K/P^{**}. If $(P^*)^2 \nsubseteq K$ then there exist $a \in R^1$ and $b \in P^*$ such that $aPb \nsubseteq K$. Then by Lemmas 2 and 3 $aPb + K/K$ is an $\bar{M}$-ideal in $(P^*)^2/K$, contrary to (b). Thus $(P^*)^2 + P^{**}/P^{**} \in \bar{M}$. Similarly, from Lemmas 1 and 3, we obtain $P^*/(P^*)^2 \in \bar{M}$. This and (b) proves the Lemma.

Corollary 7 and Lemma 4 give

Theorem 2. $LD(M) = \bar{M}$.

3. Other remarks

In this section S will be a hereditary radical, D will satisfy D1–D3 and

D5. If $(P, R) \in D$ and I is an ideal of R then $(P \cap I, R) \in D$.

The condition D5 is weaker than D4. Indeed, $P \cap I$ is an ideal of P, so by D1 $(P \cap I, P) \in D$. Thus by D4, $(P \cap I, R) \in D$.

Let us observe that all examples of classes D in this paper satisfy D5.

To determine whether S is a D-radical we can investigate the class

$U = \{R \mid R/S(R)$ does not contain non-zero D-subrings in $S\}$

Of course $S \subseteq U$ and S is D-radical if and only if U is the class of all rings. It is routine to verify

Proposition 4. (a) *if $I \in U$ is an ideal of R and $R/I \in U$ then $R \in U$;*

(b) *if R is the union of a chain of ideals of U then $R \in U$;*

(c) *the class $U_1 = \{R \mid$ any homomorphic image of R is in $U\}$ is radical.*

If R is a zero-ring then D-subrings of R are exactly ideals of R. Thus a zero-ring R is S-semisimple if and only if R does not contain non-zero D-subrings of S. Therefore U contains all zero-rings. Hence Proposition 4 implies that it also contains Baer's lower radical.

Proposition 5. *If the class U is hereditary then so is U_1.*

Proof. Let I be an ideal of $R \in U_1$, J an ideal of I and J^* the ideal of R generated by J. By assumptions $R/J^* \in U$ and $I/J^* \in U$. Since U contains all nilpotent rings then $J^*/J \in U$. Now from Proposition 4(a) and the fact that $(I/J)/(J^*/J) \approx I/J^*$ we have $I/J \in U$, so $I \in U_1$.

Of course if D satisfies D1–D4 then the class U is hereditary. We also have

Theorem 3. *If D is a left strong class then the class U is hereditary.*

Proof. Let I be an ideal of $R \in U$. We shall prove that $I \in U$. Of course we can assume that $S(I)=0$. If L is a left S-ideal of I then IL is an ideal of L. The hereditariness of S implies that $IL \in S$. Since IL is a left ideal of R then $IL \subseteq S(R)$. But $IL \subseteq I$, so $IL \subseteq S(A) \cap I = S(I) = 0$. Thus $IL = 0$ and, in consequence, $I(L+LI)=0$. Hence $(L+LI)^2 = 0$, so $L+LI \in U$. Therefore $L \subseteq S(L+LI)$ and, since $L+LI$ is an ideal of I, $S(L+LI) \subseteq S(I) = 0$. Thus $L=0$ and the theorem follows.

The class U_1 contains a natural subclass:

$$U_2 = \left\{ R \,\middle|\, \begin{array}{l} \text{any homomorphic image of } R \text{ does not contain} \\ \text{non-zero } D\text{-subrings in } S \end{array} \right\}$$

It is well known (1) that if D-subrings are exactly ideals then U_2 is the radical complementary to S. Let us observe that Corollary 6 implies that if D satisfies D1–D4 and the radical $LD(S)$ is hereditary then U_2 is the radical complementary to $LD(S)$. These conditions are satisfied (6) when a class D is left stable and S is supernilpotent. By Proposition 3 and (5) it follows that these conditions are also satisfied when classes D are left strong and S is supernilpotent. So we have

Corollary 8. *If a class D is left strong or left stable and S is supernilpotent then U_2 is the radical complementary to $LD(S)$.*

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