MOMENT SEQUENCES AND THE BERNSTEIN POLYNOMIALS*

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(received September 26, 1968)

1. Introduction. The Bernstein polynomials

$$B_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} \tag{1.1}$$

and the Bernstein power series

$$P_n(f,x) = \sum_{k=n}^{\infty} f\left(\frac{k-n}{k}\right) \binom{k}{n} x^{k-n} (1-x)^{n+1} \tag{1.2}$$

have been the subject of much research (e.g. [1; 2; 3; 6; 7; 8]). It is the purpose of this paper to demonstrate the relationship between these linear operators and certain classes of moment sequences defined below.

Let $\{\alpha_n(x)\}$ be a sequence of real-valued functions defined on $[0,1]$. Denote by $(h_{nk}(x))$ and $(q_{nk}(x))$ respectively the Hausdorff and quasi-Hausdorff matrices generated by $\{\alpha_n(x)\}$ [4, Chapter 11]. Then

$$h_{nk}(x) = \begin{cases} \binom{n}{k} \Delta^{n-k} \alpha_k(x), & 0 \le k \le n \\ 0 \quad , & k > n \end{cases} \tag{1.3}$$

and

*This paper is a portion of the author's doctoral dissertation written at Lehigh University in 1967-68 under the direction of Professor J.P. King.

$$(1.4) \qquad q_{nk}(x) = \begin{cases} 0 , & k < n \\ \binom{k}{n} \Delta^{k-n} \alpha_n(x) , & k \geq n , \end{cases}$$

where, for any non-negative integers n and p,

$$(1.5) \qquad \Delta^p \alpha_n(x) = \sum_{j=0}^{p} (-1)^j \binom{p}{j} \alpha_{n+j}(x) .$$

The sequence $\{\alpha_n(x)\}$ is called a generalized moment sequence if there exists a function $\beta(x, t)$, of bounded variation in t for each $x \varepsilon [0, 1]$, such that for all $x \varepsilon [0, 1]$

$$(1.6) \qquad \alpha_n(x) = \int_0^1 t^n d\beta (x, t) , \quad n = 0, 1, 2, \ldots .$$

When $\{\alpha_n(x)\}$ is a sequence of constant functions, (1.6) becomes the usual definition of moment sequence [6, page 57].

The sequence $\{\alpha_n(x)\}$ is called totally monotone if $\Delta^p \alpha_n(x) \geq 0$ for all $x \varepsilon [0, 1]$ and all integers $n, p \geq 0$.

Let $\{\alpha_n(x)\}$ be a generalized moment sequence. For all functions, f, defined on the interval [0, 1] associate the linear operator

$$(1.7) \qquad H_n(f, x) = \sum_{k=0}^{\infty} f(\frac{k}{n}) h_{nk}(x)$$

with the matrix (1.3), and associate the linear operator

$$(1.8) \qquad Q_n(f, x) = \sum_{k=0}^{\infty} f(\frac{k-n}{k}) q_{nk}(x)$$

with the matrix (1.4).

https://doi.org/10.4153/CMB-1969-050-8 Published online by Cambridge University Press

Now we consider the operators H_n and Q_n and demonstrate that they are generalizations of (1.1) and (1.2) respectively. We show that the uniform convergence of $\{H_n(f, x)\}$ to $f(x)$ on $[0, 1]$, for all $f \in C[0, 1]$, characterizes a class of totally monotone generalized moment sequences and we raise the question of how many of these sequences exist. Finally we discuss a similar result for the operator Q_n on the interval $[0, a]$, $0 \leq a < 1$.

In the sequel, let $e_k(x) = x^k$ for $k = 0, 1, \ldots$.

2. The operator H_n. The n-th order Bernstein polynomial (1.1) is a special case of the operator H_n and is obtained when

$$\beta(x, t) = \begin{cases} 0, & 0 \leq t < x \\ 1, & x \leq t \leq 1 . \end{cases}$$

For this choice of the function β we see, from the Stieltjes integral (1.6) and from (1.5), that $\alpha_n = x^n$ and $\Delta^{n-k} \alpha_k(x) = (1-x)^{n-k} x^k$.

To prove the main result of this section (Theorem 2.2), we need the following lemma (the proof of which follows readily from (1.5) and (1.6)).

LEMMA 2.1. Let $\{\alpha_n(x)\}$ be a generalized moment sequence. If $\beta(x, t)$ is the function having $\{\alpha_n(x)\}$ as its moment sequence, then

$$(2.1) \qquad \Delta^p \alpha_n(x) = \int_0^1 (1-t)^p t^n d\beta(x, t)$$

and

$$(2.2) \qquad \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k} \alpha_k(x) = \alpha_0(x)$$

for all non-negative integers n and p.

THEOREM 2.2. If $\{\alpha_n(x)\}$ is a totally monotone generalized

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moment sequence, then a necessary and sufficient condition that $\lim_{n\to\infty} H_n(f, x) = f(x)$ uniformly on $[0, 1]$, for each $f \varepsilon C[0, 1]$, is $\alpha_j(x) = x^j$ for $j = 0, 1, 2$ and $x \varepsilon [0, 1]$.

Proof. Since $\{\alpha_n(x)\}$ is totally monotone, H_n is a positive linear operator (i.e. $f(x) \geq 0$ for all $x \varepsilon [0, 1]$ implies $H_n(f, x) \geq 0$). By a theorem of Korovkin [5, page 14], we need only show that the sequence $\{H_n(e_j; x)\}$ converges to $\alpha_j(x)$ uniformly on $[0, 1]$ for $j = 0, 1, 2$. First we see from (1.7), (1.3), and (2.2) that

$$H_n(e_0, x) = \alpha_0(x). \tag{2.3}$$

Secondly, we have from (1.7), (1.3), and (2.1)

$$\begin{aligned} H_n(e_1, x) &= \sum_{k=0}^{n} \frac{k}{n} \binom{n}{k} \Delta^{n-k} \alpha_k(x) \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} \Delta^{n-1-k} \alpha_{k+1}(x) \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} \int_0^1 (1-t)^{n-1-k} t^{k+1} d\beta(x, t) \\ &= \int_0^1 t d\beta(x, t) \end{aligned}$$

where $\beta(x, t)$ is the function having $\{\alpha_n(x)\}$ as its generalized moment sequence. Hence

$$H_n(e_1, x) = \alpha_1(x). \tag{2.4}$$

In a similar manner, we obtain

(2.5) $$H_n(e_2; x) = \frac{n-1}{n}\,\alpha_2(x) + \frac{1}{n}\,\alpha_1(x).$$

It follows from (2.3), (2.4), and (2.5) that

(2.6) $$\lim_{n\to\infty} H_n(f, x) = f(x) \text{ uniformly on } [0,1] \text{ for each } f \varepsilon C[0,1],$$

if and only if $\alpha_j(x) = x^j$ for $j = 0, 1, 2$.

Theorem 2.2 shows that the convergence properties of the operator H_n depend only on the first three terms of the moment sequence $\{\alpha_n(x)\}$, which must be 1, x, and x^2. Thus the following question should be answered: What totally monotone generalized moment sequences $\{\alpha_n(x)\}$ have $\alpha_0(x) = 1$, $\alpha_1(x) = x$, and $\alpha_2(x) = x^2$ for all $x \varepsilon [0,1]$? It is conjectured that $\{x^n\}$ is the only sequence and that Theorem 2.2 characterizes the Bernstein polynomials. The question is answered in part by the following lemma.

LEMMA 2.3. Let $\{\lambda_n\}$ be a sequence of real numbers such that $\{\lambda_n x^n\}$ is a totally monotone generalized moment sequence. If $\lambda_0 = \lambda_1 = 1$, then $\lambda_n = 1$ for all non-negative integers n.

Proof. Since $\{\lambda_n x^n\}$ is totally monotone, we have

(2.7) $$\Delta(\lambda_n x^n) = x^n(\lambda_n - \lambda_{n+1} x) \geq 0$$

for all $x \varepsilon [0,1]$. It follows from (2.7) that for all non-negative integers n

(2.8) $$0 \leq \lambda_{n+1} \leq \lambda_n \leq 1.$$

Also

(2.9) $$\Delta^2(\lambda_n x^n) = x^n(\lambda_n - 2\lambda_{n+1} x + \lambda_{n+2} x^2) \geq 0.$$

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In particular, at $x = 1$ in (2.9) we see

(2.10) $$\lambda_n - 2\lambda_{n+1} + \lambda_{n+2} \geq 0 .$$

The result follows from (2.8) and (2.10) by mathematical induction.

3. The operator Q_n. When $\{\alpha_n(x)\}$ is a generalized moment sequence, we will denote $\int_0^1 \frac{d\beta(x,t)}{t}$ by $\alpha_{-1}(x)$.

Let $x \in [0, 1]$ and

$$\beta(x,t) = \begin{cases} 0 , & 0 \leq t < 1 - x \\ 1 - x, & 1 - x \leq t \leq 1 . \end{cases}$$

Evaluating the Stieltjes integrals (1.6) and (1.5), we obtain $\alpha_n(x) = (1-x)^{n+1}$ for all $n \geq 0$, $\Delta^{k-n}\alpha_n(x) = (1-x)^{n+1} x^{k-n}$, and $\alpha_{-1}(x) = 1$ for $x \in [0,1]$, $\alpha_{-1}(1) = 0$. Here Q_n is the Bernstein power series (1.2). The main result of this section is the following theorem.

THEOREM 3.1. Let $\{\alpha_j(x)\}$ be a generalized moment sequence, $\beta(x,t)$ the function having $\{\alpha_j(x)\}$ as its moment sequence, and $0 \leq a < 1$. Suppose, for $x \in [0, a]$,

(3.1) $$\alpha_{-1}(x) \text{ is finite},$$

(3.2) $$\beta(x,t) \text{ is increasing in } t,$$

and

(3.3) $$\beta(x,0) = \lim_{t \to 0^+} \beta(x,t) = 0 .$$

A necessary and sufficient condition that $\lim_{n\to\infty} Q_n(f,x) = f(x)$ uniformly on $[0, a]$ for each $f \in C[0,1]$ is $\alpha_j(x) = (1-x)^{j+1}$ for $j = -1, 0, 1$ $(x \in [0, a])$.

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The proof of Theorem 3.1 depends on the following lemmas. The first and third are easy computations. The second requires the first and is discussed in [4, page 282].

LEMMA 3.2. *If* $y \varepsilon [0, 1)$, *then for all non-negative integers* n,

$$\frac{1}{(1-y)^{n+1}} = \sum_{k=n}^{\infty} \binom{k}{n} y^{k-n} . \tag{3.4}$$

LEMMA 3.3. *If conditions* (3.1), (3.2), *and* (3.3) *are satisfied, then*

$$\int_0^1 \frac{d\beta(x,t)}{t} = \sum_{k=n}^{\infty} \binom{k}{n} \Delta^{k-n} \alpha_n(x) \tag{3.5}$$

for all $x \varepsilon [0, a]$.

LEMMA 3.4. *If* k *and* n *are positive integers,* $k \geq 2$, *then*

$$\frac{k^2}{(k+n)^2} \binom{k+n}{k} = \frac{k+n-1}{k+n} \binom{k+n-2}{k-2} + \frac{k}{(k+n)^2} \binom{k+n}{k} . \tag{3.6}$$

Proof of Theorem 3.1. It follows from (3.2) and (2.1) that $\{\alpha_j(x)\}$ is totally monotone. Hence Q_n is a positive linear operator (see (1.8)). By (3.5), (1.4), and (1.8) we have

$$Q_n(e_0, x) = \alpha_{-1}(x) . \tag{3.7}$$

Using the same arguments we have employed before, (3.4), and (3.5), we can easily obtain

$$Q_n(e_1, x) = \alpha_{-1}(x) - \alpha_0(x) . \tag{3.8}$$

Now

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Now

$$Q_n(e_2, x) = \sum_{k=n}^{\infty} \left(\frac{k-n}{k}\right)^2 \binom{k}{n} \Delta^{k-n} \alpha_n(x)$$

$$= \sum_{k=0}^{\infty} \left(\frac{k}{k+n}\right)^2 \binom{k+n}{n} \Delta^{k} \alpha_n(x)$$

$$= \sum_{k=1}^{\infty} \left(\frac{k}{k+n}\right)^2 \binom{k+n}{n} \int_0^1 (1-t)^k t^n \, d\beta(x,t)$$

$$= \int_0^1 t^n \sum_{k=1}^{\infty} \frac{k+1}{k+n+1} \binom{k+n}{n} (1-t)^{k+1} \, d\beta(x,t)$$

$$\geq \int_0^1 t^n \sum_{k=0}^{\infty} \frac{k}{k+n} \binom{k+n}{n} (1-t)^{k+1} \, d\beta(x,t)$$

$$= \int_0^1 t^n \sum_{k=0}^{\infty} \binom{k+n}{n} (1-t)^{k+2} \, d\beta(x,t)$$

$$= \int_0^1 t^n (1-t)^2 \frac{1}{[1-(1-t)]^{n+1}} \, d\beta(x,t)$$

$$= \alpha_{-1}(x) - 2\alpha_0(x) + \alpha_1(x).$$

Hence

(3.9) $$Q_n(e_2, x) \geq \alpha_{-1}(x) - 2\alpha_0(x) + \alpha_1(x).$$

Also, using (3.6) and repeating the above argument, we see

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$$Q_n(e_2, x) = \int_0^1 t^n \sum_{k=0}^{\infty} \left(\frac{k}{k+n}\right)^2 \binom{k+n}{n} (1-t)^k \, d\beta(x,t)$$

$$= \int_0^1 t^n \sum_{k=2}^{\infty} \frac{k+n-1}{k+n} \binom{k+n-2}{k-2} (1-t)^k \, d\beta(x,t)$$

$$+ \int_0^1 t^n \sum_{k=1}^{\infty} \frac{k}{(k+n)^2} \binom{k+n}{k} (1-t)^k \, d\beta(x,t)$$

$$\leq \int_0^1 t^n \sum_{k=0}^{\infty} \binom{k+n}{k} (1-t)^{k+2} \, d\beta(x,t)$$

$$+ \int_0^1 t^n \sum_{k=0}^{\infty} \frac{1}{k+n+1} \binom{k+n}{k} (1-t)^{k+1} \, d\beta(x,t)$$

$$\leq \int_0^1 \frac{(1-t)^2}{t} \, d\beta(x,t) + \frac{1}{n} \int_0^1 \frac{(1-t)}{t} \, d\beta(x,t)$$

$$= \alpha_{-1}(x) - 2\alpha_0(x) + \alpha_1(x) + \frac{1}{n}(\alpha_{-1}(x) - \alpha_0(x)).$$

Thus

(3.10) $$Q_n(e_2, x) \leq \alpha_{-1}(x) - 2\alpha_0(x) + \alpha_1(x) + \frac{1}{n}(\alpha_{-1}(x) - \alpha_0(x)).$$

It follows from (3.9) and (3.10) that

(3.11) $$\lim_{n\to\infty} Q_n(e_2, x) = \alpha_{-1}(x) - 2\alpha_0(x) + \alpha_1(x).$$

Now, if $\alpha_j(x) = (1-x)^{j+1}$ for $j = -1, 0, 1$, we see from (3.7), (3.8), and (3.11) that

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(3.12) $$\lim_{n\to\infty} Q_n(e_k, x) = e_k(x) \text{ for } k = 0, 1, 2.$$

By Korovkin's theorem [5, page 14], (3.12) is sufficient for the convergence of $\{Q_n(f, x)\}$ to $f(x)$ uniformly on $[0, a]$ for all $f \in C[0, 1]$.

Conversely, suppose $\{Q_n(f, x)\}$ converges uniformly to $f(x)$ on $[0, a]$ for all $f \in C[0, 1]$. From (3.7), (3.8), and (3.11) respectively we see

$$\lim_{n\to\infty} Q_n(e_0, x) = 1 = \alpha_{-1}(x),$$

$$\lim_{n\to\infty} Q_n(e_1, x) = x = \alpha_{-1}(x) - \alpha_0(x),$$

and

$$\lim_{n\to\infty} Q_n(e_2, x) = x^2 = \alpha_{-1}(x) - 2\alpha_0(x) + \alpha_1(x).$$

The result now follows.

Since the function $\beta(x, t)$ which generates the Bernstein power series (see the beginning of Section 3) satisfies (3.2) and (3.3) and $\alpha_{-1}(x) = 1$ for all $x \in [0, 1)$, we see from Theorem 3.1 that

(3.13) $$\lim_{n\to\infty} \sum_{k=n}^{\infty} f\left(\frac{k-n}{k}\right) \binom{k}{n} (1-x)^{n+1} x^{k-n} = f(x)$$

uniformly on $[0, a]$ for all $f \in C[0, 1]$, $0 \leq a < 1$. The result (3.13) is a special case of a theorem of Cheney and Sharma [3, page 242].

Meyer-König and Zeller [7] considered the operator

(3.14) $$P_n(f, x) = \sum_{k=n}^{\infty} f\left(\frac{k-n}{k}\right) \binom{k-1}{n-1} (1-x)^n x^{k-n}.$$

But (3.14) is essentially (1.2) and the convergence properties of these two operators are the same (see [7, Theorem 1, page 91]).

We remark that if Theorem 2.2 does in fact characterize the Bernstein polynomials, Theorem 3.1 will characterize the Bernstein power series.

The author is indebted to the referee for many helpful suggestions.

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