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SOFT C^*-ALGEBRAS

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Abstract In this paper we consider soft group and crossed product C^*-algebras. In particular we show that soft crossed product C^*-algebras are isomorphic to classical crossed product C^*-algebras. We also prove that large classes of soft C^*-algebras have stable rank equal to infinity.

Keywords: soft C^*-algebras; soft crossed products; soft relations

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1. Introduction

Soft C^*-algebras, first introduced by Blackadar in [**1**], arise naturally from well-known examples of classical C^*-algebras. More precisely we have the following definition.

Definition 1.1. For given $\varepsilon \in [0,2]$, $\ell, k \in \mathbb{N}$, and a set of monomials $\{r_p\}_{p=1,\dots,k}$ in ℓ variables, the universal C^*-algebra $A_\varepsilon(\ell, \{r_p\}_{p=1,\dots,k})$ generated by unitaries $a_1, \dots, a_\ell$ satisfying the conditions $\|r_p(a_1,\dots,a_\ell) - 1\| \leqslant \varepsilon$ for all $p = 1,\dots,k$ is called a soft C^*-algebra.

The two-dimensional soft torus $C^*_\varepsilon(\mathbb{Z}^2)$, $\varepsilon \in \mathbb{R}$, $0 \leqslant \varepsilon < 2$, was later introduced in [**3**] by Exel, who showed that $K_j(C^*_\varepsilon(\mathbb{Z}^2))$ is naturally isomorphic to $K_j(C^*(\mathbb{Z}^2))$, $j = 0,1$. $C^*_\varepsilon(\mathbb{Z}^2)$ is the universal C^*-algebra generated by two unitaries u_ε and v_ε subject to the relation $\|u_\varepsilon v_\varepsilon - v_\varepsilon u_\varepsilon\| \leqslant \varepsilon$. Elliott, Exel and Loring [**2**] considered $C^*_\varepsilon(\mathbb{Z}^2) \rtimes_\sigma \mathbb{Z}_2$ (where σ denotes the flip automorphism), determined its K-theory, and expressed it in terms of soft crossed products. In this paper we will look at some examples of soft C^*-algebras and crossed products and study some of their properties. Soft group C^*-algebras are, roughly speaking, universal C^*-algebras obtained by 'softening' classical group relations, and they are defined in the following way.

Definition 1.2. Let Γ be a finitely generated and finitely presented group given in terms of generators and relations by

$$\Gamma = \langle g_i,\ i = 0,\dots,n-1 \mid r_p(g_0,\dots,g_{n-1}) = 1,\ p = 1,\dots,P\rangle,$$

where the r_p are monomials in $g_0,\dots,g_{n-1}$ and their inverses. Then the (parametrized) soft group C^*-algebra $C^*_{\varepsilon,\Theta}(\Gamma)$, $\Theta = \{\rho_p\}_{p=1,\dots,P}$, $\rho_p \in \mathbb{T}$, $\varepsilon = (\varepsilon_1,\dots,\varepsilon_P) \in \mathbb{R}^P$,

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$0 \leqslant \varepsilon_p \leqslant 2$, $\forall p = 1, \dots, P$, is defined to be the universal C^*-algebra generated by unitaries $u_0, \dots, u_{n-1}$ subject to the 'softened' relations $\|r_p(u_0, \dots, u_{n-1}) - \rho_p\| \leqslant \varepsilon_p$, $p = 1, \dots, P$. To simplify the notation we will drop Θ when $\Theta = \{1\}_{p=1,\dots,P}$ and ε when $\varepsilon = 0$.

Note that in general one needs to show the existence of a representation realizing the given relations to deduce the existence of $C^*_{\varepsilon,\Theta}(\Gamma)$. Throughout this paper, we will implicitly assume that all the Γ and Θ are chosen so that the existence of a representation satisfying the given relations is guaranteed. Note also that different C^*-algebras could arise from 'softening' different presentations of Γ.

Roughly speaking, soft crossed products are universal C^*-algebras obtained by 'softening' classical crossed product relations (see Definition 2.2). One of our results, Theorem 3.1, is a characterization of soft crossed products in terms of classical crossed products. Our proof is constructive. However, it is true that for any stable C^*-algebra D and for any separable group Γ, D is isomorphic to $A \rtimes_{\mathcal{G}} \Gamma$ for some action of Γ on some C^*-algebra A. (For example, take $A = D \otimes C_0(\Gamma)$, and $\mathcal{G}$ be the product of the trivial action of Γ on D, to obtain $D \cong D \otimes \mathcal{K}(L^2(\Gamma)) \cong A \rtimes_{\mathcal{G}} \Gamma$.) But we prove that large classes of soft C^*-algebras have stable rank equal to infinity (Corollary 4.6). Hence soft C^*-algebras are in general not stable (Corollary 4.7). Therefore Theorem 3.1 offers new insights on the structure of soft crossed product C^*-algebras.

In more detail the contents of this paper are as follows. In § 2 we define soft crossed products and look at some examples. In § 3 we prove Theorem 3.1, our crossed product characterization. In § 4 we derive some properties of soft C^*-algebras (cf. Propositions 4.1 and 4.2 and Theorem 4.3). Although our results are stated for the finitely generated case, they can be easily extended to countable generated groups and C^*-algebras. In the sequel, all the (universal and non-universal) C^*-algebras are assumed to be unital, unless obviously otherwise.

2. Soft crossed products: examples

Definition 2.1. Let Γ be a finitely generated group, and let $\bar{\Gamma}$ be a finite set of generators for Γ. A $(\Gamma, \bar{\Gamma})$-representation $\mathcal{A}_{\bar{\Gamma}}$ of Γ on a C^*-algebra A is, by definition, the restriction to $\bar{\Gamma}$ of an action $\mathcal{A}$ of Γ on A.

When there is no danger of confusion, we will call a $(\Gamma, \bar{\Gamma})$-representation, a representation.

The following is a slight generalization of the definition of soft crossed products given in [**2**].

Definition 2.2. Let A be a unital C^*-algebra generated by a set $\{a_i\}_{i \in I}$, I finite, and Γ a discrete group generated as a group by $\bar{\Gamma} = \{g_j\}_{j \in J}$, J finite, acting on A via the representation $\mathcal{A}_{\bar{\Gamma}}$. For any $\varepsilon = (\varepsilon_{i,j})_{i \in I,\ j \in J}$, $0 \leqslant \varepsilon_{i,j} \leqslant 2$, $\forall i \in I$, $\forall j \in J$, and $\Theta = \{\rho_{i,j}\}_{i \in I,\ j \in J}$, $\rho_{i,j} \in \mathbb{T}$, the parametrized soft crossed product C^*-algebra $A \rtimes^{\varepsilon,\Theta}_{\mathcal{A}_{\bar{\Gamma}}} \Gamma$ associated with the representation $\mathcal{A}_{\bar{\Gamma}}$, is the universal C^*-algebra generated by a copy of A and a unitary element u_g for each g in Γ subject to the following relations:

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(i) $\|u_{g_j} a_i u_{g_j}^* - \rho_{i,j} \mathcal{A}_{\bar{\Gamma}}(g_j)(a_i)\| \leqslant \varepsilon_{i,j}$, $\forall i \in I$, $j \in J$; and

(ii) $u_g u_h = u_{gh}$, $\forall g, h \in \Gamma$.

To simplify the notation, we will drop Θ when $\Theta = \{1\}$, and ε when $\varepsilon = 0$.

As for $C^*_{\varepsilon,\Theta}(\Gamma)$, parametrized crossed products exist only whenever there is a concrete realization of the given relations.

In the remainder of this section we will look at some examples. The three-dimensional non-commutative torus $C^*_\varepsilon(\mathbb{Z}^3)$ is the universal C^*-algebra generated by three unitaries u_ε, v_ε and z_ε subject to the relations

$$\|u_\varepsilon v_\varepsilon - v_\varepsilon u_\varepsilon\| \leqslant \varepsilon_1, \qquad \|u_\varepsilon z_\varepsilon - z_\varepsilon u_\varepsilon\| \leqslant \varepsilon_2, \qquad \|v_\varepsilon z_\varepsilon - z_\varepsilon v_\varepsilon\| \leqslant \varepsilon_3.$$

Proposition 2.3. *$C^*_\varepsilon(\mathbb{Z}^3)$ is isomorphic to $W_\varepsilon \rtimes_{\mathcal{T}} \mathbb{Z}$, where W_ε is the universal C^*-algebra generated by unitaries u_ℓ, v_ℓ, $\ell \in \mathbb{Z}$, subject to the relations*

$$\|u_\ell v_\ell - v_\ell u_\ell\| \leqslant \varepsilon_1, \quad \|u_{\ell+1} - u_\ell\| \leqslant \varepsilon_2, \quad \|v_{\ell+1} - v_\ell\| \leqslant \varepsilon_3, \quad \forall \ell \in \mathbb{Z},$$

and $\mathcal{T} : W_\varepsilon \to W_\varepsilon$ is defined by $T(u_\ell) = u_{\ell+1}$, $T(v_\ell) = v_{\ell+1}$, $\forall \ell \in \mathbb{Z}$. (We denote by T the automorphism associated with the generator 1 of $\mathbb{Z}$ in $\mathcal{T}$.)

Proof. In $C^*_\varepsilon(\mathbb{Z}^3)$ put $U_\ell = z_\varepsilon^\ell u_\varepsilon z_\varepsilon^{-\ell}$, and $V_\ell = z_\varepsilon^\ell v_\varepsilon z_\varepsilon^{-\ell}$, $\forall \ell \in \mathbb{Z}$. Note that $\|U_\ell V_\ell - V_\ell U_\ell\| \leqslant \varepsilon_1$ and also $\|U_{\ell+1} - U_\ell\| \leqslant \varepsilon_2$, $\|V_{\ell+1} - V_\ell\| \leqslant \varepsilon_3$, $\forall \ell \in \mathbb{Z}$. Therefore there exists a (unique) morphism $\zeta : W_\varepsilon \to C^*_\varepsilon(\mathbb{Z}^3)$ such that $\zeta(u_\ell) = U_\ell$, and $\zeta(v_\ell) = V_\ell$, $\forall \ell \in \mathbb{Z}$. ζ can be extended to $W_\varepsilon \rtimes_{\mathcal{T}} \mathbb{Z}$ by setting $\zeta(W) = z_\varepsilon$, where W is the unitary implementing the automorphism T associated with the generator 1 of $\mathbb{Z}$. Conversely, in $W_\varepsilon \rtimes_{\mathcal{T}} \mathbb{Z}$, the unitaries u_0, v_0 and W satisfy the same relations as u_ε, v_ε and z_ε in $C^*_\varepsilon(\mathbb{Z}^3)$. Hence there exists a (unique) morphism $\eta : C^*_\varepsilon(\mathbb{Z}^3) \to W_\varepsilon \rtimes_{\mathcal{T}} \mathbb{Z}$ such that $\eta(u_\varepsilon) = u_0$, $\eta(v_\varepsilon) = v_0$ and $\eta(z_\varepsilon) = W$. Clearly ζ and η are each other's inverses. □

More generally we can define the $n+1$-dimensional non-commutative torus $C^*_\varepsilon(\mathbb{Z}^{n+1})$ as the universal C^*-algebra generated by unitaries u_j, $j = 1, \dots, n$, and z subject to the relations

$$\|u_j u_k - u_k u_j\| \leqslant \varepsilon_{j,k}, \quad \|z u_j - u_j z\| \leqslant \varepsilon_{0,j}, \quad \forall j, k \in \{1, \dots, n\}.$$

Proposition 2.4. *The C^*-algebra $C^*_\varepsilon(\mathbb{Z}^{n+1})$ is isomorphic to the C^*-algebra $W_{\varepsilon,n} \rtimes_{\mathcal{T}} \mathbb{Z}$, where $W_{\varepsilon,n}$ is defined to be the universal C^*-algebra generated by unitaries $u_{j,\ell}$, $j = 1, \dots, n$, and $\ell \in \mathbb{Z}$, subject to the relations $\|u_{j,\ell+1} - u_{j,\ell}\| \leqslant \varepsilon_{0,j}$, $\forall j, \ell$, and $\|u_{j,\ell} u_{k,\ell} - u_{k,\ell} u_{j,\ell}\| \leqslant \varepsilon_{j,k}$, $\forall j, k, \ell$, and $\mathcal{T} : W_{\varepsilon,n} \to W_{\varepsilon,n}$ is given by $T(u_{j,\ell}) = u_{j,\ell+1}$. (We denote by T the automorphism corresponding to the generator 1 of $\mathbb{Z}$ in $\mathcal{T}$.)*

Proof. In $C^*_\varepsilon(\mathbb{Z}^{n+1})$ put $U_{j,\ell} = z^\ell u_j z^{-\ell}$, $j = 1, \dots, n$, $\ell \in \mathbb{Z}$. Note that $\|U_{j,\ell} U_{k,\ell} - U_{k,\ell} U_{j,\ell}\| \leqslant \varepsilon_{j,k}$ and also $\|U_{j,\ell+1} - U_{j,\ell}\| \leqslant \varepsilon_{0,j}$. Therefore there exists a (unique) morphism $\zeta : W_{\varepsilon,n} \to C^*_\varepsilon(\mathbb{Z}^{n+1})$ such that $\zeta(u_{j,\ell}) = U_{j,\ell}$, $\forall j, \ell$. ζ can be extended to $W_{\varepsilon,n} \rtimes_{\mathcal{T}} \mathbb{Z}$

by setting $\zeta(W) = z$, where W is the unitary implementing T. On the other hand, in $W_{\varepsilon,n} \rtimes_{\mathcal{T}} \mathbb{Z}$, the elements $u_{j,0}$, and W satisfy the relations satisfied by u_j and z, so there exists a (unique) morphism $\psi : C^*_\varepsilon(\mathbb{Z}^{n+1}) \to W_{\varepsilon,n} \rtimes_{\mathcal{T}} \mathbb{Z}$ such that $\psi(u_j) = u_{j,0}$ and $\psi(z) = W$. Clearly ζ and ψ are each other's inverses. □

Proposition 2.4 can be easily modified to give a characterization of the parametrized soft C^*-algebras $C^*_{\varepsilon,\Theta}(\mathbb{Z}^{n+1})$.

3. Soft crossed products

We will now state and prove our crossed product characterization result.

Theorem 3.1. *Let B be a finitely generated and finitely polynomially presented C^*-algebra. Let Γ be a finitely generated and finitely presented group and $\mathcal{D}_{\bar{\Gamma}}$ be a representation of Γ on B by monomial automorphisms ($\bar{\Gamma}$ is a finite set of generators for Γ). Then there is an action $\mathcal{G}$ of Γ on a C^*-algebra A such that*

$$B \underset{\mathcal{D}_{\bar{\Gamma}}}{\overset{\varepsilon}{\rtimes}} \Gamma \cong A \rtimes_{\mathcal{G}} \Gamma.$$

Proof. Suppose that B is generated by the unitaries b_j, $j = 1, \dots, m$, subject to the polynomial relations r_k, $k \in \mathbb{F}$, $\mathbb{F} \subseteq \mathbb{N}$, $\mathbb{F}$ finite. Let Γ be given in multiplicative notation by $\Gamma = \langle g_0, \dots, g_{n-1} \mid z_p(g_0, \dots, g_{n-1}) = 1,\ \forall p \in \mathbb{P} \rangle$, $\mathbb{P} \subseteq \mathbb{N}$ finite, $\bar{\Gamma} = \{g_0, \dots, g_{n-1}\}$.

Also assume that $\mathcal{D}_{\bar{\Gamma}}$ is given on the generators by (for simplicity of notation let $\mathcal{D}_{\bar{\Gamma},\ell} := \mathcal{D}_{\bar{\Gamma}}(g_\ell)$) $\mathcal{D}_{\bar{\Gamma},\ell}(b_j) = P_{j,\ell}(b_1, \dots, b_m)$ and $r_k(\mathcal{D}^s_{\bar{\Gamma},\ell}(b_1), \dots, \mathcal{D}^s_{\bar{\Gamma},\ell}(b_m)) = 0$, $\forall \ell = 0, \dots, n-1$, $j = 1, \dots, m$, $k \in \mathbb{F}$, $\forall s \in \mathbb{Z}$. Define A to be the universal C^*-algebra generated by unitaries $a_{j,g}$ and $a^{\{\ell\}}_{j,g}$, $j = 1, \dots, m$, $\ell = 0, \dots, n-1$, $g \in \Gamma$, subject to the relations $r_k(a_{1,g}, \dots, a_{m,g}) = 0$, $r_k(a^{\{\ell\}}_{1,g}, \dots, a^{\{\ell\}}_{m,g}) = 0$, and $\|a_{j,gg_\ell} - a^{\{\ell\}}_{j,g}\| \leqslant \varepsilon_{j,\ell}$, $a^{\{\ell\}}_{j,g} = P_{j,\ell}(a_{1,g}, \dots, a_{m,g})$, for all $g \in \Gamma$, $\forall \ell = 0, \dots, n-1$, and $\forall j = 1, \dots, m$. Note that $B \rtimes^\varepsilon_{\mathcal{D}_{\bar{\Gamma}}} \Gamma$ is the universal C^*-algebra generated by a copy of B and unitaries $\omega_0, \dots, \omega_{n-1}$ subject to the relations $z_p(\omega_0, \dots, \omega_{n-1}) = 1$, $\|\omega_\ell b_j \omega_\ell^* - \mathcal{D}_{\bar{\Gamma},\ell}(b_j)\| \leqslant \varepsilon_{j,\ell}$, $\forall p, j, \ell$. In $B \rtimes^\varepsilon_{\mathcal{D}_{\bar{\Gamma}}} \Gamma$ define the following elements $A_{j,g}$ and $A^{\{\ell\}}_{j,g}$:

$$A_{j,g} = \omega_g b_j \omega_g^*, \quad A^{\{\ell\}}_{j,g} = \omega_g \mathcal{D}_{\bar{\Gamma},\ell}(b_j) \omega_g^*, \quad \forall j = 1, \dots, m, \quad g \in \Gamma, \quad \ell = 0, \dots, n-1,$$

where, if $g = g_{k_1}^{Z_1} \dots g_{k_q}^{Z_q} \in \Gamma$, $Z_j \in \mathbb{Z}$, $j = 1, \dots, q$, in terms of the canonical generators of $\mathcal{A}_{\bar{\Gamma}}$, we put $\omega_g = \omega_{k_1}^{Z_1} \dots \omega_{k_q}^{Z_q}$. Then

$$\begin{aligned}\|A_{j,gg_\ell} - A^{\{\ell\}}_{j,g}\| &= \|\omega_{gg_\ell} b_j \omega_{gg_\ell}^* - \omega_g \mathcal{D}_{\bar{\Gamma},\ell}(b_j) \omega_g^*\| \\ &\leqslant \|\omega_\ell b_j \omega_\ell^* - \mathcal{D}_{\bar{\Gamma},\ell}(b_j)\| \leqslant \varepsilon_{j,\ell}, \quad \forall g, j, \ell.\end{aligned}$$

Moreover, $r_k(b_1,\dots,b_m)=0$ and $r_k(\mathcal{D}_{\bar{\Gamma},\ell}(b_1),\dots,\mathcal{D}_{\bar{\Gamma},\ell}(b_m))=0$ imply that

$$r_k(A_{1,g},\dots,A_{m,g})=0,\qquad r_k(A_{1,g}^{\{\ell\}},\dots,A_{m,g}^{\{\ell\}})=0,$$

$\forall k\in\mathbb{F}$, $\forall\ell=0,\dots,n-1$ and $\forall g\in\Gamma$. Notice also that $A_{j,g}^{\{\ell\}}=P_{j,\ell}(A_{1,g},\dots,A_{m,g})$, $\forall\ell,j,g$.

Hence there exists a (unique) morphism $\phi : A\to B\rtimes^{\varepsilon}_{\mathcal{D}_{\bar{\Gamma}}}\Gamma$ such that $\phi(a_{j,g})=A_{j,g}$ and $\phi(a_{j,g}^{\{\ell\}})=A_{j,g}^{\{\ell\}}$, $\forall\ell=0,\dots,n-1$, $\forall j=1,\dots,m$, $\forall g\in\Gamma$. Notice that this ensures the existence of a concrete representation for A, and hence its existence. Now define the following automorphisms $G_t : A\to A$, $t=0,\dots,n-1$, by $G_t(a_{j,g})=a_{j,g_tg}$ and $G_t(a_{j,g}^{\{\ell\}})=a_{j,g_tg}^{\{\ell\}}$, for any j,g,ℓ. The automorphisms G_t, $t=0,\dots,n-1$, satisfy $z_p(G_0,\dots,G_{n-1})=1$. Hence they determine an action $\mathcal{G}$ of Γ on A. Since $A_{j,g_tg}=\omega_tA_{j,g}\omega_t^*$ and $A_{j,g_tg}^{\{\ell\}}=\omega_tA_{j,g}^{\{\ell\}}\omega_t^*$, we can extend ϕ to $A\rtimes_{\mathcal{G}}\Gamma$ by setting $\phi(W_t)=\omega_t$, where we denote by W_t the unitary implementing G_t, $t=0,\dots,n-1$. On the other hand, in $A\rtimes_{\mathcal{G}}\Gamma$, the elements $a_{j,1}$, $a_{j,1}^{\{\ell\}}$ and W_ℓ, $j=1,\dots,m$, $\ell=0,\dots,n-1$, satisfy the following relations $\|W_\ell a_{j,1}W_\ell^*-a_{j,1}^{\{\ell\}}\|\leqslant\varepsilon_{j,\ell}$ and

$$r_k(a_{1,1},\dots,a_{m,1})=0,\qquad r_k(a_{1,1}^{\{\ell\}},\dots,a_{m,1}^{\{\ell\}})=0,\qquad a_{j,1}^{\ell}=P_{j,\ell}(a_{1,1},\dots,a_{m,1}),$$

$\forall j=1,\dots,m$, $\forall\ell=0,\dots,n-1$ and $\forall k\in\mathbb{F}$. Hence there exists a (unique) homomorphism $\psi : B\rtimes^{\varepsilon}_{\mathcal{D}_{\bar{\Gamma}}}\Gamma\to A\rtimes_{\mathcal{G}}\Gamma$ such that $\psi(b_j)=a_{j,1}$, $\psi(\mathcal{D}_{\bar{\Gamma},\ell}(b_j))=a_{j,1}^{\{\ell\}}$ and $\psi(\omega_\ell)=W_\ell$, $\ell=0,\dots,n-1$, $j=1,\dots,m$. Clearly ζ and ψ are each other's inverses. □

Theorem 3.1 can of course be extended to parametrized soft crossed products C^*-algebras.

4. Applications

In this section we will describe some additional properties of soft C^*-algebras. Firstly we will prove that soft C^*-algebras form right continuous fields. Additionally we will show that such fields are continuous for large classes of soft C^*-algebras. We will also prove that many soft C^*-algebras have infinite stable rank.

Proposition 4.1. *For given $\ell,k\in\mathbb{N}$ and a set of monomials $\{r_p\}_{p=1,\dots,k}$, the soft C^*-algebras $\{A_\varepsilon(\ell,\{r_p\})\}_{\varepsilon\in[0,2]}$ form a right continuous field of C^*-algebras over $[0,2]$.*

Proof. This proof is a generalization of the proof of Proposition 1.2 of [**4**]. Assume that all the norm inequalities defining A_ε are of type $\|a-b\|\leqslant\varepsilon$, with a and b unitaries. We will show that the field F of C^*-algebras having fibres A_ε, $\varepsilon\in[0,2]$, is right continuous. Let $\phi_\varepsilon : C^*(\mathbb{F}_\ell)\to A_\varepsilon$ be the canonical homomorphism (ℓ is the number of generators of A_ε) and $J_\varepsilon=\ker\phi_\varepsilon$. Right continuity amounts to showing that

$$J_\varepsilon=J_\varepsilon^+,\quad \text{for } \varepsilon\in[0,2)\quad \text{(cf. [4])},$$

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where J_ε^+ is the ideal $\overline{\cup_{\alpha>\varepsilon} J_\alpha}$. By universality of A_ε, there is a homomorphism

$$A_\varepsilon = C^*(\mathbb{F}_\ell)/J_\varepsilon \to C^*(\mathbb{F}_\ell)/J_\varepsilon^+$$

sending generators to generators. Therefore $J_\varepsilon \subseteq J_\varepsilon^+$. As the other inclusion is trivial, we are done. □

Note that the soft crossed products we consider in the proposition below are soft C^*-algebras as in Definition 1.1.

Proposition 4.2. *The soft C^*-algebras $C^*(\mathbb{F}_n) \rtimes_{\mathcal{A}_{\bar{\Gamma}}}^{\varepsilon,\Theta} \mathbb{Z}$, $\varepsilon \in [0,2]$, where $\mathcal{A}_{\bar{\Gamma}}$ is the identity representation of $\mathbb{Z}$ on $C^*(\mathbb{F}_n)$, form continuous fields of C^*-algebras over $[0,2]$.*

Proof. For simplicity take $\Theta = \{1\}$. By Theorem 3.1, $C^*(\mathbb{F}_n) \rtimes_{\mathcal{A}_{\bar{\Gamma}}}^{\varepsilon} \mathbb{Z}$ is isomorphic to the crossed product $A \rtimes_{\mathcal{G}} \mathbb{Z}$. Here A denotes the universal C^*-algebra generated by unitaries w_j^i, $i = 1, \dots, n$, $j \in \mathbb{Z}$, subject to the relations $\|w_j^i - w_{j+1}^i\| \leqslant \varepsilon$. If w denotes the unitary implementing $\mathcal{G}(1)$, $A \rtimes_{\mathcal{G}} \mathbb{Z}$ is then the universal C^*-algebra generated by unitaries w_j^i, $i = 1, \dots, n$, $j \in \mathbb{Z}$, and w subject to the relations $\|w_j^i - w_{j+1}^i\| \leqslant \varepsilon$, $w w_j^i w^* = w_{j+1}^i$. By using the methods of [**4**], the conclusion follows. □

In a similar way, one can show that soft parametrized rotation algebras [**5**] form continuous fields of C^*-algebras over $[0,2]$.

Now we will show that the stable rank of large classes of soft C^*-algebras is equal to infinity. Stable rank is defined in [**7**], for example. We will start by considering Exel's non-commutative torus.

Theorem 4.3. *The soft non-commutative torus $C_\varepsilon^*(\mathbb{Z}^2)$, $0 < \varepsilon < 2$, has stable rank equal to infinity.*

Proof. For any $N \in \mathbb{N}$, there exists a unital surjective homomorphism ψ from $C^*(\mathbb{F}_2)$ to $C([0,1]^{N^2}) \otimes M_{N+1}(\mathbb{C})$ (Theorem 1 in [**6**]). This is sufficient to ensure that the stable rank of $C^*(\mathbb{F}_2)$ is infinity as surjective homomorphisms do not increase stable rank (Theorem 4.3 in [**7**]). To show that the stable rank of the soft torus $C_\varepsilon^*(\mathbb{Z}^2)$ is also infinity, we only need to show that ψ factors through $C_\varepsilon^*(\mathbb{Z}^2)$. To do so, first note that ψ sends the two generators of $C^*(\mathbb{F}_2)$ to the unitaries u and v in the proof of Theorem 1 of [**6**]. By comparing the proof of Theorem 1 of [**6**] and that of Lemma 3 of [**6**], we see that we can take $u = \exp(2\pi \mathrm{i} X)$ and $v = Y$. As noted by the author, for any $\delta > 0$, we can choose in the proof of Lemma 3 of [**6**] self–adjoint generators $\{a_1, \dots, a_n\}$ for $A = C([0,1]^{N^2})$ such that the norm of the element $X_0 = (x_{i,j})$ in $M_N(A)$ (where $X = X_0 \oplus 1 \in M_{N+1}(A)$) is smaller than δ and thus $\|\exp(2\pi \mathrm{i} X_0) - 1\| \leqslant |\exp(2\pi\delta) - 1|$. Hence, for any $\varepsilon > 0$, there exist self-adjoint generators for A such that $\|u - 1\| \leqslant \varepsilon/2$ (note that $u = \exp(2\pi \mathrm{i} X_0) \oplus 1$) and so $\|uv - vu\| \leqslant \varepsilon$. Therefore ψ factors through $C_\varepsilon^*(\mathbb{Z}^2)$. □

Corollary 4.4. *Non-commutative tori (with at least one parameter $\varepsilon > 0$) have stable rank equal to infinity.*

Proof. Apply Theorem 4.3 of [**7**] and Theorem 4.3. □

In the spirit of Theorem 3.1, the soft non-commutative torus $C^*_\varepsilon(\mathbb{Z}^2)$ is also isomorphic $D_\varepsilon \rtimes_{\mathcal{S}} \mathbb{Z}$, with D_ε the universal C^*-algebra generated by unitaries w_j, $j \in \mathbb{Z}$, subject to $\|w_{j+1} - w_j\| \leqslant \varepsilon$, and $\mathcal{S}$ shifts j by one (cf. [**3**, **4**]). Now, by Theorem 4.3 and [**7**], D_ε also has stable rank equal to infinity. An independent proof of this last fact is also given below.

Proposition 4.5. *The C^*-algebra D_ε, $0 < \varepsilon < 2$, has stable rank equal to infinity.*

Proof. D_ε can be characterized, by taking logarithms, as the C^*-algebra generated by a unitary v and self-adjoint operators h_j, $j \in \mathbb{Z}$, subject to $\|h_j\| \leqslant 2\cos(\varepsilon/2)$. Then, by using this characterization, it is easily seen that D_ε admits $C[0, \cos(\varepsilon/2)]^N$ as a quotient (for any $N \in \mathbb{N}$), which has stable rank N [**7**]. By Theorem 4.3 of [**7**], we are done. □

Corollary 4.6. *Any (soft) C^*-algebra surjecting onto a C^*-algebra having stable rank equal to infinity has stable rank equal to infinity.*

Proof. Apply Theorem 4.3 of [**7**]. □

Corollary 4.7. *Any (soft) C^*-algebra having stable rank equal to infinity is not stable.*

Proof. By [**7**], any stable algebra has stable rank equal to either 1 or 2. □

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