Canad. Math. Bull. Vol. **25** (1), 1982

ANTIDIRECTED SUBTREES OF DIRECTED GRAPHS

BY
STEFAN A. BURR

The purpose of this paper is to prove the following result:

THEOREM. *Let T be a directed tree with k arcs and with no directed path of length 2. Then if G is any directed graph with n points and at least $4kn$ arcs, T is a subgraph of G.*

It would be appropriate to call T an *antidirected* tree or a *source-sink* tree, since every point either has all its arcs directed outward or all inward. As N. G. de Bruijn has noted (personal communication), such a linear bound in n cannot hold if T is replaced by any directed graph other than a union of such trees. The above theorem strengthens one of Graham [1], where an implicit bound of $c(k)n$ is obtained, where $c(k)$ is exponentially large. The proof we give here is also shorter. We first give two simple lemmas. Both are essentially due to Erdös, but it is not clear where either first appeared. Their proofs are easy, so we give them here for completeness; in neither case do we state quite the best possible result.

LEMMA 1. *Any graph has a bipartite subgraph with at least half as many edges.*

Proof. We use induction on p, the number of points. If $p = 1$ or 2, the result is trivial. Now assume the result is true for $p-1$ and let G have p points. If v is a point of G, then by hypothesis $G-v$ contains a bipartitite graph B with at least half as many edges as $G-v$. Let the parts of B be X_1 and X_2, and let E_1 and E_2 be the sets of edges incident to v and to points of X_1 and X_2 respectively. Clearly either E_1 or E_2 has at least half the edges incident with v, so that by adjoining one of them, and v, to B, we have a bipartite graph with at least half as many edges as G. This completes the proof.

LEMMA 2. *Let G have p points and q edges. Then G has a subgraph G' with minimum degree at least q/p.*

Proof. Successively remove points with degree less than q/p until it becomes impossible to proceed. This clearly does not exhaust all the edges of G, so a suitable G' remains.

Proof of the Theorem. First, using Lemma 1, we can find a bipartite $G_1 \subset G$ with n points and at least $2kn$ arcs. Let the parts of G_1 be X and X'. Clearly,

Received by the editors Marh 24, 1980.

https://doi.org/10.4153/CMB-1982-017-4 Published online by Cambridge University Press

some set of at least kn arcs is either directed from X to X' or from X' to X, and therefore has no directed path of length 2. By Lemma 2, there is a graph $G_2 \subset G$ with minimum degree at least k, and of course, no directed path of length 2. Clearly, $T \subset G_2$, completing the proof.

We could have made the above theorem slightly sharper by tightening up various steps, for instance Lemma 1. In fact, Lemma 1 has been sharpened [2, 3]. There seems no point in doing so, however, since the $4kn$ can almost certainly be made rather smaller. The best lower bound we have found is $(k-1)n+1$. To see this bound, take $T = K_{1,k}$ with all arcs directed outward from the center, and take $G = K_{2k-2,2k-2}$, with half the arcs directed in each direction. This directed graph has $4k-4=n$ points and $(2k-2)^2=(k-1)n$ arcs, and no copy of T. It is not hard to extend this example so that n becomes large while T stays fixed.

Bibliography

1. R. L. Graham, *On Subtrees of Directed Graphs with no Path of Length Exceeding One*, Canad. Math. Bull. **13** (1970), 329–332.

2. C. S. Edwards, *Some Extremal Properties of Bipartite Subgraphs*, Canad. J. Math. **25** (1973), 475–485.

3. C. S. Edwards, *An Improved Lower Bound for the Number of Edges in a Largest Bipartite Subgraph*, Recent Advances in Graph Theory, Academia, Prague, 1975, 167–181.

City College, CUNY
New York, N.Y.