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A NEW PROOF OF THE WIENER–HOPF FACTORIZATION VIA BASU'S THEOREM

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Abstract

We illustrate how Basu's theorem can be used to derive the spatial version of the Wiener–Hopf factorization for a specific class of piecewise-deterministic Markov processes. The classical factorization results for both random walks and Lévy processes follow immediately from our result. The approach is particularly elegant when used to establish the factorization for spectrally one-sided Lévy processes.

Keywords: Wiener–Hopf factorization; Basu's theorem; Wiener's Tauberian theorem; Lévy process

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1. Introduction

Lévy processes are widely used for modeling purposes in many areas of applied probability, such as queueing theory, insurance risk theory, and mathematical finance. One result that has proven to be useful in such settings is known as the Wiener–Hopf factorization; here we state the spatial version only. Suppose that $X := \{X(t);\, t \geq 0\}$ represents a Lévy process, with $X(0) = 0$, and let e_q be an exponential random variable with rate $q > 0$, where e_q is independent of X.

Theorem 1. (Wiener–Hopf factorization.) *For a Lévy process* $X := \{X(t);\, t \geq 0\}$*, the random variables* $\inf_{0\leq s\leq e_q} X(s)$ *and* $X(e_q) - \inf_{0\leq s\leq e_q} X(s)$ *are independent. Hence, for each* $\omega \in \mathbb{R}$*,*

$$\mathrm{E}[\mathrm{e}^{\mathrm{i}\omega X(e_q)}] = \mathrm{E}[\mathrm{e}^{\mathrm{i}\omega \inf_{0\leq s\leq e_q} X(s)}]\mathrm{E}[\mathrm{e}^{\mathrm{i}\omega (X(e_q) - \inf_{0\leq s\leq e_q} X(s))}].$$

We see that the distribution of X at time e_q can be expressed as an independent sum of two random variables. Note that one of these random variables is equal in distribution to the reflected Lévy process at time e_q, which has a tractable characteristic function if $\inf_{0\leq s\leq e_q} X(s)$ has one as well.

Two relatively simple proofs of the factorization have recently appeared in the literature: see [13] and [15]. Both of these papers make use of arguments from martingale theory. In [13] the Kella–Whitt martingale [10] is used to derive the factorization for a spectrally one-sided Lévy process, and the authors of [15] showed how a more elaborate martingale approach can be used to establish the factorization for an arbitrary Lévy process.

We take a different approach. Here we illustrate how Basu's theorem can be used to establish the factorization for a particular subclass of piecewise-deterministic Markov processes (PDMPs). This argument does not make use of any concepts from stochastic calculus: only

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Basu's theorem and Wiener's Tauberian theorem are needed. A well-known approximation approach can then be used to derive the factorization for both random walks and Lévy processes. Readers will also find that in some cases such an approximation argument is not needed: we will show how Basu's theorem gives a very quick proof of the Wiener–Hopf factorization for spectrally one-sided Lévy processes, without the need of an approximation step.

This paper is organized as follows. Section 2 contains a brief review of the necessary concepts needed from mathematical statistics. In Section 3 we present the proof of the main result, and we follow in Section 4 by considering the special setting of spectrally one-sided Lévy processes.

2. Preliminaries

We will make use of the following standard concepts from mathematical statistics: a simple introduction can be found in [2].

Suppose that we have a random sample $\{Q_i\}_{i=1}^n$, and let U and V be statistics with respect to the sample, i.e. both U and V are measurable functions of the sample. We are interested in estimating an unknown parameter $\theta \in \Theta$ (the parameter space), which appears in the distribution of the random sample. We use the measure P_θ to emphasize that, under P_θ, the unknown parameter of the data is θ.

Definition 1. We say that U is a sufficient statistic with respect to θ if the conditional joint distribution of the data $\{Q_i\}_{i=1}^n$, given U, is independent of θ.

Definition 2. We say that V is an ancillary statistic if its distribution does not depend on θ.

Definition 3. We say that U is a boundedly complete statistic if it satisfies the following criterion: if g is a bounded measurable function which satisfies $\mathrm{E}_\theta[g(U)] = 0$ for all $\theta \in \Theta$ then $\mathrm{P}_\theta(g(U) = 0) = 1$ for each $\theta \in \Theta$.

We are now ready to state Basu's theorem.

Theorem 2. (Basu's theorem.) *Suppose that U is a boundedly complete, sufficient statistic with respect to $\theta \in \Theta$, and suppose that V is an ancillary statistic with respect to $\theta \in \Theta$. Then U and V are independent under P_θ for each $\theta \in \Theta$.*

The proof of Basu's theorem is simple, and can be found in many textbooks on mathematical statistics: see, for example, [2]. Basu's theorem is also featured in the recent survey articles of Boos and Hughes-Oliver [1] and Ghosh [6], where classical and new applications are discussed in both papers.

The applications of Basu's theorem in our study will always involve a given stochastic process $\{X(t); t \geq 0\}$ and an independent exponential random variable e_q with rate $q > 0$. Our data will always be $(\inf_{0\leq s\leq e_q} X(s), X(e_q) - \inf_{0\leq s\leq e_q} X(s))$, with the two statistics of interest being the two coordinates of the data. The unknown parameter is always the initial condition $X(0) = \theta$, and the measure P_θ will always represent conditioning on $X(0) = \theta$, as is typically done in the theory of Markov processes.

3. Our main result

Our process of interest in this section is a PDMP $X := \{X(t); t \geq 0\}$, which evolves as follows. The process X has jumps at the random locations $\{T_k\}_{k\geq 1}$, which also form the points of an ordinary renewal process. The size of the jump at location T_k is B_k, $k \geq 1$, and we assume that the $\{B_k\}_{k\geq 1}$ sequence is independent and identically distributed (i.i.d.), and independent of the jump locations. Between jumps, X moves in a deterministic manner according to a nonnegative function h, defined on the real line.

A more specific description of the evolution of X, starting at a deterministic $X(0) = \theta$, is as follows: for $0 \leq t < T_1$, $X(t) = X(0) + h(t)$ and $X(T_1) = X(T_1-) + B_1$ with $X(t-) = \lim_{s\uparrow t} X(s)$ for each $t \geq 0$. Similarly, for $T_1 \leq t < T_2$, $X(t) = X(T_1) + h(t - T_1)$, $X(T_2) = X(T_2-) + B_2$, and, for an arbitrary $n \geq 1$, $X(t) = X(T_n) + h(t - T_n)$ for $T_n \leq t < T_{n+1}$, with $X(T_{n+1}) = X(T_{(n+1)}-) + B_{n+1}$.

We further assume that there exists a sequence of left-continuous functions $\{h_n\}_{n\geq 0}$ such that (i) $h_n\colon [0,\infty) \to \{k/2^n : k \geq 0\}$, and (ii) the sequence $\{h_n\}_{n\geq 1}$ converges uniformly to h on compact sets as $n \to \infty$. These functions are used to define a collection of approximating PDMPs $\{X^n\}_{n\geq 0}$. For each $n \geq 0$, X^n is a PDMP that is governed by the renewal process N, the function h_n, and the set of jump sizes $\{B_{n,k}\}_{k\geq 1}$, where, for each $k \geq 1$,

$$B_{n,k} = \frac{\inf\{l \in \mathbb{Z}\colon l/2^n \geq B_k\}}{2^n}.$$

We also assume that the initial value $X^n(0) = \theta_n \in \{k/2^n; k \in \mathbb{Z}\}$ is such that $\theta_n \to \theta$ as $n \to \infty$. Note that at each time instant $t \geq 0$, $X^n(t)$ takes values in the space $\{k/2^n; k \in \mathbb{Z}\}$.

We now establish that $\inf_{0\leq s\leq e_q} X^0(s)$ is both sufficient and boundedly complete.

Lemma 1. $\inf_{0\leq s\leq e_q} X^0(s)$ *is a sufficient statistic with respect to its unknown initial condition* $\theta_0 \in \mathbb{Z}$.

Proof. Let A and B be two subsets of the integers, and let x be an arbitrary integer. Setting P_{θ_0} as a probability measure, under which the law of X^0 is as the integer-valued PDMP described above with initial value $X^0(0) = \theta_0$, we see that, for $x \leq \theta_0$,

$$\begin{aligned}\mathrm{P}_{\theta_0}\Big(X^0(e_q) - \inf_{0\leq s\leq e_q} X^0(s) \in A,\ \inf_{0\leq s\leq e_q} X^0(s) \in B \;\Big|\; \inf_{0\leq s\leq e_q} X^0(s) = x\Big)\\ = \mathbf{1}(x \in B)\frac{\mathrm{P}_{\theta_0}(X^0(e_q) \in A + x,\ \inf_{0\leq s\leq e_q} X^0(s) = x)}{\mathrm{P}_{\theta_0}(\inf_{0\leq s\leq e_q} X^0(s) = x)}.\end{aligned} \tag{1}$$

Set $\tau_x = \inf\{t \geq 0\colon X^0(t) \leq x\}$. A conditioning argument gives

$$\begin{aligned}&\mathrm{P}_{\theta_0}\Big(X^0(e_q) \in A + x,\ \inf_{0\leq s\leq e_q} X^0(s) = x\Big)\\ &\quad= \mathrm{P}_{\theta_0}\Big(X^0(e_q) \in A + x,\ \inf_{0\leq s\leq e_q} X^0(s) = x \;\Big|\; X^0(e_q \wedge \tau_x) = x\Big)\mathrm{P}_{\theta_0}(X^0(e_q \wedge \tau_x) = x)\\ &\quad= \mathrm{P}_x\Big(X^0(e_q) \in A + x,\ \inf_{0\leq s\leq e_q} X^0(s) = x\Big)\mathrm{P}_{\theta_0}(X^0(\tau_x \wedge e_q) = x).\end{aligned} \tag{2}$$

Note that we are making use of the fact that the process regenerates at the points of our renewal process N; moreover, since $h^0 \geq 0$, the infimum of X^0 over any interval must be attained at a renewal instant within that interval.

A similar argument shows that

$$\mathrm{P}_{\theta_0}\Big(\inf_{0\leq s\leq e_q} X^0(s) = x\Big) = \mathrm{P}_x\Big(\inf_{0\leq s\leq e_q} X^0(s) = x\Big)\mathrm{P}_{\theta_0}(X^0(\tau_x \wedge e_q) = x). \tag{3}$$

Substituting both (2) and (3) into (1) yields

$$\begin{aligned}
&\mathrm{P}_{\theta_0}\Big(X^0(e_q) - \inf_{0\le s\le e_q} X^0(s) \in A,\ \inf_{0\le s\le e_q} X^0(s) \in B \,\Big|\, \inf_{0\le s\le e_q} X^0(s) = x\Big) \\
&\quad = \mathbf{1}(x\in B)\frac{\mathrm{P}_x(X^0(e_q)\in A+x,\ \inf_{0\le s\le e_q} X^0(s)=x)\,\mathrm{P}_{\theta_0}(X^0(\tau_x\wedge e_q)=x)}{\mathrm{P}_x(\inf_{0\le s\le e_q} X^0(s)=x)\,\mathrm{P}_{\theta_0}(X^0(\tau_x\wedge e_q)=x)} \\
&\quad = \mathbf{1}(x\in B)\frac{\mathrm{P}_x(X^0(e_q)\in A+x,\ \inf_{0\le s\le e_q} X^0(s)=x)}{\mathrm{P}_x(\inf_{0\le s\le e_q} X^0(s)=x)},
\end{aligned}$$

and this conditional probability does not depend on θ_0. Hence, $\inf_{0\le s\le e_q} X^0(s)$ is a sufficient statistic.

Remark 1. Note how a discrete state space setting makes proving sufficiency of $\inf_{0\le s\le e_q} X(s)$ a simple matter. An analogous proof in a general state space setting appears to be difficult to produce, although it seems obvious that it must hold.

Our next task is to realize that $\inf_{0\le s\le e_q} X_0^0(s)$ is infinitely divisible, where X_0^0 represents the PDMP X^0, with $\theta_0 = 0$. Clearly, $X^0(t) = X_0^0(t) + \theta_0$ under P_{θ_0} for each $t \ge 0$.

Lemma 2. $\inf_{0\le s\le e_q} X_0^0(s)$ *is an infinitely divisible random variable.*

Proof. The proof of this statement can be found in [5].

We now check to see if $\inf_{0\le s\le e_q} X^0(s)$ is boundedly complete. To show this, we make use of Wiener's Tauberian theorem. Such an argument has been used before to verify that location families are boundedly complete; see, for example, Theorem 2.4 of [7]. In fact, the version of Wiener's Tauberian theorem used here is also applied in Example 2.6(d) of [14].

Lemma 3. $\inf_{0\le s\le e_q} X^0(s)$ *is boundedly complete.*

Proof. The proof follows from a discrete version of Wiener's Tauberian theorem; see [8, p. 71]. This theorem tells us that, since the characteristic function of $\inf_{0\le s\le e_q} X_0^0(s)$ does not vanish (because of infinite divisibility), we may conclude that any absolutely summable sequence $x = \{x(k)\}_{k\in\mathbb{Z}} \in \ell_1$ can be approximated under the ℓ_1-norm by elements $\mathcal{A} \in \ell_1$ of the form

$$\mathcal{A}(j) = \sum_{k=1}^{m} a_k p_0(j + c_k), \tag{4}$$

where p_0 represents the probability mass function of $\inf_{0\le s\le e_q} X_0^0(s)$, m is an arbitrary positive integer, $\{a_k\}_{k=1}^m$ is a sequence of complex numbers, and $\{c_k\}_{k=1}^m$ is a collection of integers.

Fix a bounded, measurable function $g\colon \mathbb{Z}\to\mathbb{R}$ satisfying $\mathrm{E}_\theta[g(\inf_{0\le s\le e_q} X^0(s))] = 0$ for all $\theta\in\mathbb{Z}$, and let $F\colon \ell_1\to\mathbb{R}$ be a function satisfying, for each $x\in\ell_1$,

$$F(x) = \sum_{k\in\mathbb{Z}} g(k)x(k).$$

Continuity of F under the ℓ_1-norm yields $F(x) = 0$ for each $x\in\ell_1$, since we can find a sequence $\{x_n\}_{n\ge1}$ of linear combinations of the form (4), satisfying $F(x_n) = 0$ for all n, that converge to x. Thus, $g(n) = 0$ for each integer n, which completes the proof.

We conclude the proof of the factorization by showing that $X^0(e_q) - \inf_{0\le s\le e_q} X^0(s)$ is an ancillary statistic.

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Lemma 4. $X^0(e_q) - \inf_{0\le s\le e_q} X^0(s)$ *is an ancillary statistic.*

Proof. The proof of this result is straightforward: note that

$$X^0(e_q) - \inf_{0\le s\le e_q} X^0(s) = \theta_0 + X_0^0(e_q) - \inf_{0\le s\le e_q} (\theta_0 + X_0^0(s)) = X_0^0(e_q) - \inf_{0\le s\le e_q} X_0^0(s),$$

which does not depend on θ_0.

Theorem 3. $\inf_{0\le s\le e_q} X(s)$ *is independent of* $X(e_q) - \inf_{0\le s\le e_q} X(s)$*, where* e_q *is an exponential random variable with rate* $q > 0$*, independent of* X^0.

Proof. Combining Basu's theorem with our previous lemmas shows that $\inf_{0\le s\le e_q} X^0(s)$ is independent of $X^0(e_q) - \inf_{0\le s\le e_q} X^0(s)$. From our proof technique, it is clear that, for each $n \ge 1$, $2^n \inf_{0\le s\le e_q} X^n(s)$ is both sufficient and boundedly complete, and so $\inf_{0\le s\le e_q} X^n(s)$ is independent of $X^n(e_q) - \inf_{0\le s\le e_q} X^n(s)$. Finally, since f_n converges uniformly on compact sets to f as n gets large, $\inf_{0\le s\le e_q} X(s)$ is independent of $X(e_q) - \inf_{0\le s\le e_q} X(s)$, because, with probability 1, the sample paths of X^n converge uniformly on compact sets to the sample paths of X.

Remark 2. It was shown in [5] that $\inf_{0\le s\le e_q} X_0^0(e_q)$ is infinitely divisible, by making use of the fact that this random variable can be expressed as a geometric sum of i.i.d. undershoot variables. Some readers may find this argument unsatisfactory, as the main idea used to verify that this is an i.i.d. geometric sum seems to be quite close to the decomposition approach used in the random walk proof of the Wiener–Hopf factorization found in, for example, the proof of Theorem 1 of [12], which was inspired by the work of Greenwood and Pitman [9]. This decomposition approach, in the random walk setting, can also be used to establish the required independence property.

Nevertheless, we still feel that our approach should be of interest to the applied probability community, as we can use *any* proof we like to show that the characteristic function of $\inf_{0\le s\le e_q} X_0^0(s)$ does not vanish, and, for some models, this may be much easier to do by other methods; one example will be given in the next section.

We conclude this section with a brief explanation of how this result carries over to the Lévy setting.

Theorem 4. *Suppose that* $X := \{X(t); t \ge 0\}$ *represents a Lévy process with characteristic triplet* (a, σ^2, π)*, which we assume to be known. Then* $\inf_{0\le s\le e_q} X(s)$ *is independent of* $X(e_q) - \inf_{0\le s\le e_q} X(s)$.

Proof. Any continuous-time Markov chain that is also a Lévy process satisfies the Wiener–Hopf factorization, as these processes are within our subclass of PDMPs. A two-step scaling procedure can be used to show how this result extends to all types of Lévy process; details can be found in Section 4 of [5].

4. Spectrally one-sided Lévy processes

In some instances, it is possible to use Basu's theorem to establish the factorization without having to use an approximation argument, undershoot variables, or a Tauberian theorem.

Suppose that $X := \{X(t); t \ge 0\}$ is a spectrally positive Lévy process, with characteristic triplet (a, σ^2, π) and unknown (deterministic) initial condition $X(0) = \theta$: saying that X is

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spectrally positive means that the support of the Lévy jump measure π is contained in $[0, \infty)$, i.e. the process has no negative jumps. It is well known that in this setting, under P_θ,

$$\inf_{0\le s\le e_q} X(s) \stackrel{D}{=} \theta - E,$$

where E is an exponential random variable with rate $\Phi(q)$, with Φ being a suitable inverse function; see Section 3.3 of [11] for details. This leads to the following proposition.

Proposition 1. $\inf_{0\le s\le e_q} X(s)$ *is a boundedly complete, sufficient statistic with respect to* θ.

Proof. Proving that $\inf_{0\le s\le e_q} X(s)$ is a sufficient statistic with respect to θ is relatively simple, since the process has no negative jumps: here, we can make use of the strong Markov property and the memoryless property of e_q to conclude that, under P_θ, for $x < \theta$,

$$\begin{aligned} &P_\theta\Big(X(e_q) - \inf_{0\le s\le e_q} \in A,\ \inf_{0\le s\le e_q} X(s) \in B \,\Big|\, \inf_{0\le s\le e_q} X(s) = x\Big) \\ &\quad = \mathbf{1}(x\in B)\lim_{h\downarrow 0} \frac{P_\theta(X(e_q) - \inf_{0\le s\le e_q} X(s) \in A,\ \inf_{0\le s\le e_q} X(s) \in [x, x+h])}{P_\theta(\inf_{0\le s\le e_q} X(s) \in [x, x+h])} \\ &\quad = \mathbf{1}(x\in B)\lim_{h\downarrow 0} \frac{P_{x+h}(X(e_q) - \inf_{0\le s\le e_q} X(s) \in A,\ \inf_{0\le s\le e_q} X(s) \in [x, x+h])}{P_{x+h}(\inf_{0\le s\le e_q} X(s) \in [x, x+h])}, \end{aligned}$$

and this conditional probability does not depend on θ. Hence, $\inf_{0\le s\le e_q} X(s)$ is a sufficient statistic of $(X(e_q) - \inf_{0\le s\le e_q} X(s), \inf_{0\le s\le e_q} X(s))$ with respect to θ. Readers wishing to see a reference on regular conditional distributions that explains the limiting step given above are referred to [4].

We now show that $\inf_{0\le s\le e_q} X(s)$ is boundedly complete. Suppose that g is a bounded Borel measurable function, satisfying $E_\theta[g(\inf_{0\le s\le e_q} X(s))] = 0$ for all $\theta \in \mathbb{R}$. Then, for each θ,

$$0 = E_\theta\Big[g\Big(\inf_{0\le s\le e_q} X(s)\Big)\Big] = \int_{-\infty}^{\theta} g(u)\Phi(q)e^{-\Phi(q)(\theta-u)}\,du.$$

However, $e^{-\Phi(q)\theta} > 0$ for each θ, which gives

$$0 = \int_{-\infty}^{\theta} g(u)\Phi(q)e^{\Phi(q)u}\,du,$$

and so we conclude that $g(u) = 0$ for almost all u, which proves the claim.

Finally, it is now obvious that $X(e_q) - \inf_{0\le s\le e_q} X(s)$ is an ancillary statistic, so we may conclude that the Wiener–Hopf factorization holds.

5. Final remarks

Recently, DasGupta [3] showed how Basu's theorem can be used to prove that a fairly large subclass of random variables is infinitely divisible. Interestingly, he commented that 'the possible horizon of applications of Basu's theorem is probably much wider than has been understood so far.' Our approach to the Wiener–Hopf factorization seems to further support this claim, and we believe that a proper understanding of this theorem could possibly lead to a greater understanding of other concepts in applied probability, both classical and new.

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