Ergod. Th. & Dynam. Sys. (1985), **5**, 107–122
Printed in Great Britain

Characterization of Morse–Smale isotopy classes on surfaces

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(*Received* 1 *December* 1983)

Abstract. In this paper we use Thurston's work on the dynamics of diffeomorphisms on surfaces to show that a diffeomorphism f on a surface is isotopic to a Morse-Smale one if and only if the growth rate of the length of the words representing elements of the fundamental group under iteration by f is one. Morse-Smale isotopy classes are also shown to be the same as Nielsen's algebraically finite type.

Introduction

This paper deals with isotopy classes and Morse-Smale diffeomorphisms on M^2, a two dimensional compact manifold possibly with boundary. We work throughout in the C^∞ category. An isotopy on M^2 is a one parameter family of diffeomorphisms, $F_t : M^2 \to M^2$, $t \in I = [0, 1]$, such that $(p, t) \in M^2 \times I \mapsto F_t(p) \in M^2$ is differentiable. Given $f, g \in \mathrm{Diff}\,(M^2)$, the space of diffeomorphisms of M^2, we say that f and g are *isotopic*, and denote it $f \sim g$, if there is an isotopy $(F_t)_{t\in I}$ on M^2 such that $F_0 = f$ and $F_1 = g$. Clearly $\sim$ is an equivalence relation whose classes are the above mentioned isotopy classes. On the other hand Morse-Smale diffeomorphisms are the simplest, in the sense of having finite non-wandering sets, structurally stable diffeomorphisms. Given $f \in \mathrm{Diff}\,(M^2)$, M^2 boundaryless, we say that f is Morse-Smale, M.S. from now on, if

(1) $\Omega(f)$ is finite (so $\Omega(f) = \mathrm{Per}\,(f)$) and hyperbolic;

(2) $W^s(p) \pitchfork W^u(q)$ for any p and $q \in \Omega(f)$,

where $\Omega(f)$ is the non-wandering set of f, $\Omega(f) = \{p \in M^2$: for every open set containing p, A, there is a positive n such that $f^n(A) \cap A \neq \emptyset\}$. Per (f) is the set of f-periodic points i.e. points $p \in M^2$ such that $f^m(p) = p$ for some $m > 0$; the minimum of such positive numbers is the f-period of p, $\pi(p)$; the unstable and stable manifolds of p, $W^u(p)$ and $W^s(p)$, are defined by $\{q \in M^2 : f^m(q) \to p$ as $m \to \alpha\}$ as $\alpha = -\infty$ or $\alpha = +\infty$, respectively.

The purpose of this paper is to answer, in the two-dimensional setting, the following question raised by Smale [7] in his survey of 1967: which isotopy classes have an M.S. diffeomorphism?

A characterization of such classes, which we call Morse–Smale isotopy classes, is given in theorem A below in terms of the growth rate of the length of words of the fundamental group under iteration by the action induced by the isotopy class.

This growth rate is the same as the growth rate of the length of simple closed curves under iteration by the class; the lengths are measured with any Riemannian metric on the surface. For an isotopy class φ and a Riemannian metric l we denote by $c(\varphi)$ and $C(\varphi)$ the respective growth rates mentioned above. If α is a class of simple closed curves $i(l, \varphi^n(\alpha))$ indicates the minimum of the l-lengths of curves in $\varphi^n(\alpha)$. Morse-Smale isotopy classes also turn out to be the same as Nielsen's algebraically finite classes, i.e. classes with a representative which is periodic outside a finite family of disjoint embedded cylinders. Definitions and preliminary results as well as a discussion of necessary conditions for an isotopy class to have an M.S. representative are given in the first part of the paper. We conclude there that the action of the homology is not enough to detect an M.S. in an isotopy class. In the second part we prove:

THEOREM A. *If M^2 is a two dimensional compact connected boundaryless manifold with negative Euler-Poincaré characteristic then each one of the following conditions is necessary and sufficient for an isotopy class of M^2, φ, to be Morse-Smale*:

(1) $C(\varphi) = c(\varphi) = 1$;

(2) *for every Riemannian metric l and for every class α of simple closed curves we have*

$$\lim_{n\to\infty} i(l, \varphi^n(\alpha))^{1/n} = 1;$$

(3) *φ is of algebraically finite type.*

In the case $\chi(M^2) \geq 0$ we have the following situation:

(a) if M^2 is the sphere, the projective plane or the Klein bottle then every isotopy class is Morse-Smale;

(b) if M^2 is the two dimensional torus an isotopy class is Morse-Smale if and only if:

$$H_1(\varphi): H_1(M^2, \mathbb{C}) \to H_1(M^2, \mathbb{C})$$

has eigenvalues in the set $\{\pm 1, \pm i, 1/2 \pm (\sqrt{3/2})i, -1/2 \pm (\sqrt{3/2})i\}$; here $H_1(\varphi)$ is the action induced by φ on $H_1(M, \mathbb{C})$, the first homology group of M with complex coefficients.

This work is my thesis written under the advice of Jacob Palis at *IMPA*. Besides him, I would like also to thank G. Fleitas, R. Mañé, W. Melo and A. Lins Neto for several helpful conversations. This work was done with the support of *IMPA* and *CNPq*, Brazil.

Definitions and preliminary results

Let G be a group and $\mathcal{G} \subseteq G$. We will say that $\mathcal{G}$ is a set of generators for G if given $g \in G$ we can find $n \geq 0$ and $g_1, \ldots, g_n \in \mathcal{G} \cup \mathcal{G}^{-1}$, $\mathcal{G}^{-1} = \{g^{-1}: g \in \mathcal{G}\}$, such that $g = g_1 \cdot g_2 \cdots g_n$; given $\mathcal{G} \subseteq G$, a set of generators, the *$\mathcal{G}$-length* of an element $g \in G$ is defined by

$$C(\mathcal{G}, g) = \min\{n \geq 0: \exists g_1, \ldots, g_n \in \mathcal{G} \cup \mathcal{G}^{-1} \text{ such that } g = g_1 \ldots g_n\}.$$

We can also define the $\mathscr{G}$-length of the conjugacy class of $g \in G$ by

$$CC(\mathscr{G}, g) = \min\{C(\mathscr{G}, h) \mid h \in \operatorname{conj}(g)\},$$

($\operatorname{conj}(g) = \{h \in G \colon \exists k \in G$ such that $h = kgk^{-1}\}$).

Clearly $CC(\mathscr{G}, g) \le C(\mathscr{G}, g)$. A group G is *finitely generated* if it admits a finite set of generators. If this is the case and $h : G \to G$ is an homomorphism, the *$\mathscr{G}$-growth rate* of h, $C(\mathscr{G}, h)$, is the non-negative extended real number given by:

$$C(\mathscr{G}, h) = \sup_{g \in G} \lim_{n \to \infty} \sup\, [C(\mathscr{G}, h^n(g)]^{1/n}.$$

Similarly, we define the $\mathscr{G}$-growth rate of h in the conjugacy of G, $CC(\mathscr{G}, h)$.

Now suppose we have G, a non-trivial finitely generated group, $\mathscr{G} \subseteq G$ a finite set of generators for G and $h : G \to G$ an homomorphism. The growth rates of h have the following properties:

(1) $C(\mathscr{G}, h) = \max_{g \in \mathscr{G}} \lim_{n \to \infty} \sup\, [C(\mathscr{G}, h^n(g))]^{1/n}$.

(2) $CC(\mathscr{G}, h) \le C(\mathscr{G}, h) < \infty$.

(3) $1 \le CC(\mathscr{G}, h)$ if h is an automorphism of G.

(4) If $\mathscr{G}' \subseteq G$ is another set of generators for G we have $C(\mathscr{G}, h) = C(\mathscr{G}', h)$ and $CC(\mathscr{G}, h) = CC(\mathscr{G}', h)$, showing that we can talk of growth rates without reference to a system of generators.

(5) If $k : G \to G$ is an inner automorphism, $c(h) = c(k \circ h)$.

(6) $C(h^n) = C(h)^n$ for $n \ge 0$.

(7) If H is another group, $\pi : G \to H$ and $k : H \to H$ are homomorphisms, π onto, such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{h} & G \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ H & \xrightarrow{k} & H \end{array}$$

commutes, we have that H is finitely generated, $C(h) \ge C(k)$ and $CC(h) \ge CC(k)$. Of course the equality holds if π is an isomorphism.

With this established, if we have an arcwise connected topological space, X, with finitely generated fundamental group and $f : X \to X$ a continuous map we can define the growth rate, $C(f)$, and the growth rate of the conjugacy classes of f, $CC(f)$, in the fundamental group by the following procedure: pick a point $x_0 \in X$ and an arc, γ, connecting x_0 to $f(x_0)$; consider the homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(X, f(x_0))$ and $\hat{\gamma} : \pi_1(X, f(x_0)) \to \pi_1(X, x_0)$ induced by f and γ respectively and set $C(f) = C(\hat{\gamma} \circ f_*)$ and $CC(f) = CC(\hat{\gamma} \circ f_*)$.

The above listed properties of C and CC show that these definitions are independent of the choices. If $f, g : X \to X$ are homotopic maps the same properties show that $C(f) = C(g)$ proving that the growth rates are indeed invariants of the homotopy class of a map. If, in particular, M^n is a compact connected n-dimensional manifold and φ is an isotopy class of M^n, $C(\varphi)$ and $CC(\varphi)$ are well defined non-negative real numbers.

If $P \in M^n$ and l is a Riemannian metric for M^n we can, using l, give another definition for the length of $g \in \pi_1(M^n, P)$ which, as we will see, is closely related to the one given by generators. Given $g \in \pi_1(M^n, P)$ the l-length of g, $c(l, g)$, and the l-length of the conjugacy class of g, $cc(l, g)$ are defined by:

$$c(l, g) = \inf\{l(\xi): \xi: I \to M^n \text{ is differentiable and } \xi \in g\},$$
$$cc(l, g) = \inf\{c(l, g'): g' \in \operatorname{conj}(g)\},$$

respectively; ($l(\xi) = l$-length of $\xi: I \to M$).

Repeating what we have done above for a continuous map $f: M^n \to M^n$ we define $c(f)$, the l-growth rate of f, and $cc(f)$, the l-growth rate of f in the conjugacy classes of $\pi_1(M, P)$. As a corollary of the following lemma due to Milnor $c(f) = C(f)$ and $cc(f) = CC(f)$, so we don't have to worry about the choices made in the process.

LEMMA 1 ([**4**]). *Given M^n a Riemannian manifold as above and $P \in M^n$ there are K and L positive real numbers and $\mathcal{G} \subseteq \pi_1(M^n, P)$, a finite set of generators such that*

$$K \cdot C(\mathcal{G}, g) \geq c(l, g) \geq L \cdot c(\mathcal{G}, g) \qquad \textit{for every } g \in \pi_1(M^n, P).$$

A simple closed curve on M^n, a, is an embedding $a: S^1 \to M$, $S^1 = \{z \in \mathbb{C}: |z| = 1\}$. If a is a simple closed curve, $\bar{a}: I \to M^n$ is the differentiable map given by $\bar{a}(t) = a(e^{2\pi t i})$, $t \in I$. Two simple closed curves on M^n are *isotopic*, notation $a \sim b$, if we can get a differentiable map $H: S^1 \times I \to M$ for which $H(\ \ , 0) = a$ and $H(\ \ , 1) = b$. It is easily seen that $\sim$ is an equivalence relation and that, if φ is an isotopy class of diffeomorphisms and $\mathscr{a}$ a class of simple closed curves, the class defined by $f \circ a$, $f \in \varphi$ and $a \in \mathscr{a}$, is independent of choices, thus defining the action of an isotopy class on the classes of simple closed curves of M^n.

Each class of simple closed curves, $\mathscr{a}$, defines a conjugacy class in $\pi_1(M^n, P)$, $\mathrm{CONJ}_P(\mathscr{a})$, in the following way: let $a \in \mathscr{a}$ and ξ be a differentiable path connecting P to $a(1)$;

$$\mathrm{CONJ}_P(\mathscr{a}) = \operatorname{conj}([\xi a \xi^{-1}]),$$

where $[\xi a \xi^{-1}]$ denotes the element of the fundamental group of M^n defined by the loop $\xi a \xi^{-1}$. $\mathrm{CONJ}_P(\mathscr{a})$ is well defined and the map $\mathscr{a} \mapsto \mathrm{CONJ}_P(\mathscr{a})$ is injective.

From now on, unless otherwise stated, M will denote a two dimensional compact connected differentiable manifold with or without boundary and with negative Euler-Poincaré characteristic. In this situation, following [**8**] (see also [**0**]) we will denote by $\mathscr{S} = \mathscr{S}(M)$ the set of all classes of simple closed curves of M, $\mathscr{a}$, such that $\mathscr{a}$ has at least one (and so every) bilateral element, i.e. an element with a tubular neighbourhood diffeomorphic to a cylinder and none (it is enough that one) of its elements bounds a disc or cylinder in M.

A finite family of simple closed curves on M, $\{a_1, \ldots, a_n\}$, $n \geq 1$, is said to be a *non-trivial family* if the curves are disjoint and represent distinct elements of $\mathscr{S}$.

A diffeomorphism $f: M \to M$ is *decomposable* if there is a non-trivial family of curves on M, $\{a\}$, such that $f(\bigcup a) = \bigcup a$. (We will denote, when no misunderstanding is possible, the image of a map, $g: X \to Y$, by the same symbol g.) In this case we say that the family $\{a\}$ decomposes the diffeomorphism f and that its isotopy class is decomposable.

If $\mathscr{a} \in \mathscr{S}$ and l is a Riemannian metric for M,

$$i(l, \mathscr{a}) = \inf\{l(\bar{a}): a \in \mathscr{a}\}$$

denotes the l-length of α. For every $\alpha \in \mathcal{S}$ and $P \in M$ we have:

$$i(l, \alpha) \leq CC(l, \mathrm{CONJ}_P(\alpha)) \leq Li(l, \alpha),$$

if $L \geq 1 + 2\delta(M)[\inf\{i(l, \beta): \beta \in \mathcal{S}\}]^{-1}$, where $\delta(M)$ is the l-diameter of M. This shows that for every $\alpha \in \mathcal{S}$, l Riemannian metric and φ isotopy classes of M we have:

$$\limsup_{n \geq 1} [i(l, \varphi^n(\alpha))]^{1/n} \leq CC(\varphi) = cc(\varphi).$$

A *measured foliation* on the boundaryless manifold M is a family of local charts for M, $\mathcal{F} = \{f\}$; $f: U \to \mathbb{R}^2$ and $U \subseteq M$ open, such that:

(1) $\bigcup_{f \in \mathcal{F}} \mathrm{Dom}(f)$ covers M except for a finite number of points which are the singularities of the foliation $\mathcal{F}$;

(2) $f_2 \circ f_1^{-1}: f_1(U_1 \cap U_2) \to f_2(U_1 \cap U_2)$ takes (x, y) to $(g(x, y), \pm y + c)$, where $c = c(f_1, f_2) \in \mathbb{R}$ is a constant, f_1 and $f_2 \in \mathcal{F}$;

(3) around each $p \in M \setminus \bigcup_{f \in \mathcal{F}} \mathrm{Dom}(f)$ there is a chart, g, such that $g(p) = 0$ and $g(\mathcal{F})$ is the pull-back by $w = z^{\frac{1}{2}(n+2)}$ of the foliation

$$\mathrm{Im}\, w = \mathrm{const.}, \qquad n = n(p) \geq 1.$$

If $\partial M \neq \emptyset$ the definition is analogous but we require that each component of the boundary has at least one singularity outside which the boundary component is a leaf of $\mathcal{F}$.

If $a: I \to M$ is differentiable, we define $\int_a \mathcal{F}$, the $\mathcal{F}$-length of a, by

$$\int_a \mathcal{F} = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left| \frac{d(y_i \circ a)}{dt}(t) \right| dt,$$

where $f_i = (x_i, y_i): U_i \subseteq M \to \mathbb{R}^2$ are charts of $\mathcal{F}$ for $i = 1, \ldots, n$, such that $a([t_{i-1}, t_i]) \subseteq U_i$ for $0 = t_0 < t_1 < \cdots < t_n = 1$. $\int_a \mathcal{F}$ being independent of the choices made, we define $\int_\alpha \mathcal{F}$, $\alpha \in \mathcal{S}$, by:

$$\int_\alpha \mathcal{F} = \inf\left\{ \int_{\bar{a}} \mathcal{F}: a \in \alpha \right\}.$$

Two measured foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are said to be *transversal* when they have the same singularities and are transversal outside the singularities and $\partial(M)$. The equality $\mathcal{F}_1 = \lambda \mathcal{F}_2$, $\lambda > 0$, denotes that $\mathcal{F}_1$ and $\mathcal{F}_2$ coincide as foliations and $\int_a \mathcal{F}_1 = \lambda \int_a \mathcal{F}_2$ for every differentiable $a: I \to M$.

A diffeomorphism $f: M \to M$ will be called *pseudo-Anosov*, P.A. for short, if there are $\mathcal{F}^u$ and $\mathcal{F}^s$, transversal measured foliations of M, and $\lambda > 1$ such that $f(\mathcal{F}^s) = (1/\lambda)\mathcal{F}^s$ and $f(\mathcal{F}^u) = \lambda \mathcal{F}^u$. The P.A. isotopy classes are those which have a pseudo-Anosov representative. Thurston [**8**] (see also [**0**]) defined and studied several properties of the P.A. diffeomorphisms; we will be concerned mainly with the following theorems.

THEOREM 1. *Every isotopy class of* M, φ, *has a representative* f, *which is*:

(a) *pseudo-Anosov*; *or*

(b) *decomposable*; *or*

(c) *a periodic isometry of a Riemannian metric* l *for* M *which has Gaussian curvature equal to* -1 *and turns the boundary components into closed geodesics.*

THEOREM 2. *Given* φ *an isotopy class of* M, *there are algebraic integers* $\lambda_1, \ldots, \lambda_k \geq 1$, $k \geq 1$, *such that for every* a *in* $\mathcal{S}$ *and every Riemannian metric* l *on* M *we have*:

$$\lim_{n\to\infty} [i(l, \varphi^n(a))]^{1/n} = \lambda_i$$

for some $i = 1, \ldots, k$. *Furthermore,* φ *is* P.A. *if and only if* $k = 1$ *and* $\lambda_1 > 1$.

THEOREM 3. *Given* $f: M \to M$ *a* P.A. *diffeomorphism,* f *has the least number of periodic points of each period in its isotopy class.*

THEOREM 4. *If* M *is orientable of genus greater than one there exists a* P.A. *diffeomorphism* $f: M \to M$ *such that*:

$$H_1(f): H_1(M, \mathbb{Z}) \to H_1(M, \mathbb{Z})$$

is the identity.

By theorem 3 there are no isotopies connecting P.A. to M.S. diffeomorphisms since every P.A. has infinitely many hyperbolic periodic points. Thus, by theorem 4, there are isotopy classes on every manifold of genus greater than one whose action on the homology is the identity which have no M.S. representative.

If M is a compact connected boundaryless differentiable manifold of dimension $m \geq 1$ and $f: M \to M$ is continuous, the *Lefschetz number* of f, $L(f)$, is defined by:

$$L(f) = \sum_{i=0}^{m} (-1)^i \operatorname{tr}(H_i(f): H_i(M, \mathbb{Q}) \to H_i(M, \mathbb{Q})).$$

If f is a diffeomorphism with hyperbolic fixed points we have, by the Lefschetz formula, that:

$$L(f) = \sum_{f(p)=p} L(p, f)$$

where $L(p, f) = (-1)^n \Delta$ and $n = \dim W^u(p) =$ number of eigenvalues of Df_p of modulus greater than one, and $\Delta = \pm 1$ according to whether $Df_p: T_pW^u(p) \to T_pW^u(p)$ preserves or reverses orientation, respectively. In particular, as observed by Smale [7], if f is an M.S. diffeomorphism

$$|L(f^n)| \leq \sum_{f^n(p)=p} |L(p, f^n)| \leq \#\Omega(f)$$

for every $n \in \mathbb{Z}$, $n \neq 0$. In 1971 Shub [6] proved:

THEOREM 5. *An* M.S. *isotopy class of* M *is unipotent on the homology, i.e. the eigenvalues of* $H_i(\varphi): H_i(M, \mathbb{Q}) \to H_i(M, \mathbb{Q})$ *are roots of unity*; $i = 0, \ldots, m$.

In particular for an M.S. isotopy class we have

$$|\operatorname{tr} H_i(\varphi)^n| \leq \dim H_i(M, \mathbb{Q})$$

for every $n \in \mathbb{Z}$ and $i = 0, 1, \ldots, m$.

By the above remarks it is clear that all these conditions are necessary but not sufficient to characterize the M.S. isotopy classes.

A diffeomorphism $f: M \to M$, M a two-dimensional manifold, is said to be *of algebraically finite type* if there is a finite disjoint family of cylinders in M, $\{\sigma\}$, (a cylinder in M is an embedding $\sigma: S^1 \times [-1, 1] \to M$) and $m \geq 1$ such that:

(a) $f(\bigcup \sigma) = \bigcup \sigma$;

(b) $f^m(p) = p$ for every $p \in M \setminus \bigcup \text{Int}(\sigma)$.

An isotopy class is of algebraically finite type if it has a diffeomorphism of algebraically finite type. Nielsen [5], computing directly the characteristic polynomial of $H_1(\varphi)$, showed that a class of algebraically finite type is unipotent on $H_1(M, \mathbb{Q})$ and conjectured that this would characterize such classes. It follows from Thurston's results mentioned above that such a conjecture is false. A true characterization is provided by our main result: M.S. and algebraically finite type classes are the same.

The proof

We will now prove theorem A, for which we need the following proposition:

PROPOSITION 1. *Let $f: N \to N$ be a periodic isometry of period $n \geq 1$ where N is a compact two-dimensional Riemannian manifold with or without boundary such that ∂N is a union of closed geodesics. Let M be a two-dimensional compact boundaryless manifold containing N as a submanifold and such that $M \setminus N$ is a disjoint union of open discs. In this situation we can get a differentiable Morse-Smale vector field without closed orbits, $X: M \to TM$, (i.e. the critical set of X consists of singularities and X_1, the time-one diffeomorphism induced by X, is* M.S.) *satisfying*:

(a) *X/N is f-invariant*;

(b) *X is transverse to ∂N*;

(c) $M \setminus \text{Int}(N) \subseteq \bigcup \{W^s(p)$: *$p$ is a sink of X*$\}$.

Proof. $f: N \to N$ being an isometry we have, for every $p \in N$, an $\varepsilon = \varepsilon(p) > 0$ such that $\exp_p : \exp_p^{-1}(B(p, \varepsilon)) \subseteq T_p(N) \to B(p, \varepsilon) \subseteq N$ is a conjugacy between:

$$f^{\pi(p)}: B(p, \varepsilon) \to B(p, \varepsilon) \quad \text{and} \quad Df_p^{\pi(p)}: T_p(N) \to T_p(N).$$

This shows that the boundary of the points of N with f-period equal to n, which is compact and f-invariant, may be expressed as

$$A + \bigcup B_1 + \bigcup B_2 + \bigcup C$$

where: A is a finite subset of $\text{Int}(N)$, $\{B_1\} + \{B_2\}$ is a finite disjoint family of submanifolds of $\text{Int}(N)$, $B \simeq S^1$, such that B_1 is bilateral and B_2 is unilateral and $\{C\}$ is a finite disjoint family of properly embedded submanifolds of N, $C \simeq [-1, 1]$. Around each element, F, of the family $\mathcal{F} = A + \{B_1\} + \{B_2\} + \{C\}$, which, excluding the case $F \in A$, is, by the local behaviour of f, a simple closed geodesic or a segment of geodesic, we take a closed normal tubular neighbourhood $T = T(F)$ such that $f^\pi(T) = T$, $\pi = \pi(F)$, $f^\pi: T \to T$ has an amenable behaviour and the $T(F)$'s are disjoint.

More precisely, if $p \in A$ take $T(p) = D(p, \varepsilon)$ the disc of radius $\varepsilon > 0$ sufficiently small; if $F = B_1$ take $T(F) = \sigma$ where $\sigma: S' \times [-1, 1] \to \text{Int}(N)$ is an embedding such that $\sigma(\ \ , 0) = B_1$ and

$$\sigma^{-1} \circ f^\pi \circ \sigma(z, t) = (z, -t),$$

$z \in S^1$ and $t \in [-1, 1]$; if $F = B_2$ take $T(F) = \mu$ where $\mu : M_b \to \text{Int}(N)$ is an embedding such that $\mu(\text{class}(x, 0)) \in B_2$, for every $x \in \mathbb{R}$ and

$$\mu^{-1} \circ f^{\pi} \circ \mu(\text{class}(x, t)) = \text{class}(x, -t)$$

where $\text{class}(x, t) \in M_b$ and M_b is the quotient space of $\mathbb{R} \times [-1, 1]$ by the group generated by $(x, t) \mapsto (x+1, -t)$; and, finally, if $F = C$ take $T(C) = \rho$ where $\rho : [-1, 1]^2 \to N$ is an embedding such that $\rho(\ \ , 0) = C$ and $\rho(0,\ \ ) + \rho(1,\ \ ) \subseteq \partial N$ and

$$\rho^{-1} \circ f^{\pi} \circ \rho(s, t) = (s, -t),$$

s and $t \in [-1, 1]$. The space $K = M \setminus \bigcup \text{Int}\, T(F)$ is a compact differentiable manifold with corners i.e. every point has a neighbourhood diffeomorphic to an open subset of $\{(x, y) \in \mathbb{R}^2 \mid x \geq 0 \text{ and } y \geq 0\}$ and every point of K has f-period n. Therefore K/f, the space of f-orbits, is also a compact two-dimensional manifold with corners covered by K via the map $p \in K \mapsto f\text{-orbit}(p) \in K/f$. K/f is therefore diffeomorphic

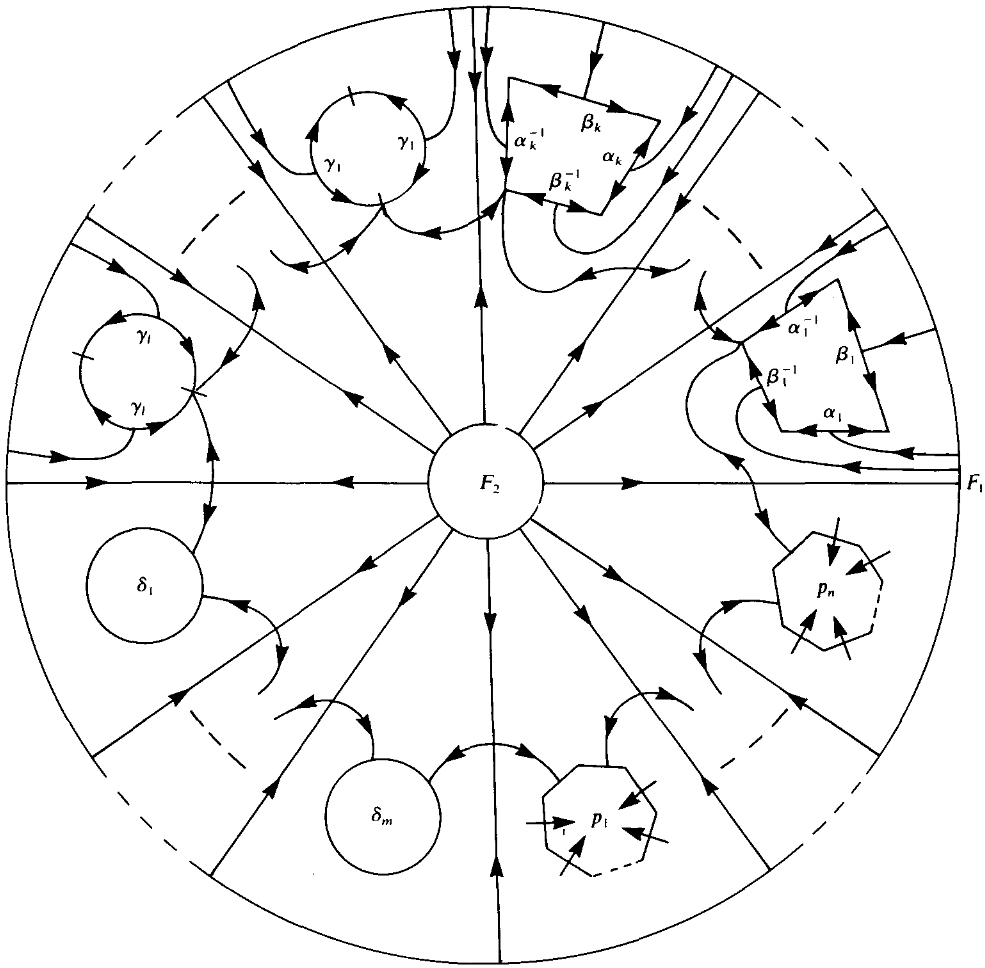

FIGURE 1

to the sphere with k handles and l cross-caps minus the interior of m closed discs and n polygons with r_i (which in this case is ≥ 4) sides, $i = 1, \ldots, n$. On K/f take a vector field with two sources F_1 and F_2, $3k+2l+m+n$ saddles, $k+l$ sinks and transversal to the boundaries and sides of the polygons as shown in figure 1.

Let X'' be the pull-back of this field by π. It is clear then that X'' is f-invariant. Let X' be the extension of X'' to N such that X' in Int $T(F)$ has the symmetric phase-diagram shown in figure 2.

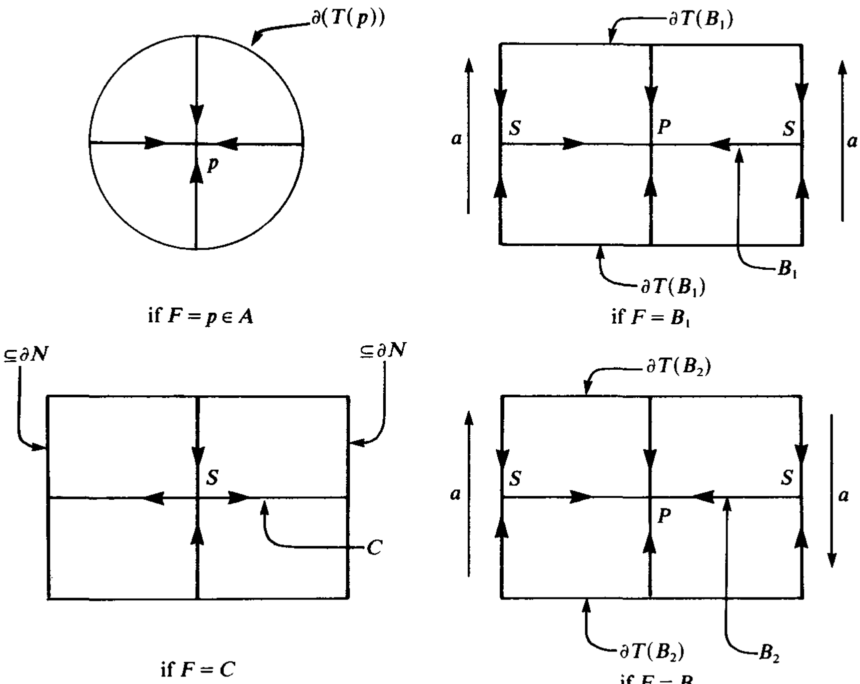

FIGURE 2

It's easy to see that we can do this extension avoiding saddle connections. We can then extend X' to X, a Morse-Smale vector field on M, by putting a sink in the interior of each disc of $M \setminus \text{Int}(N)$ (possible since X' is transverse to ∂N from the interior to the exterior of N) and we are done since X has the required properties.

COROLLARY 1. *If M is a two-dimensional compact boundaryless manifold and $f: M \to M$ a periodic diffeomorphism there exists $X: M \to T(M)$, an f-invariant differentiable Morse-Smale vector field without closed orbits. In particular $f \circ X_t$, $t \neq 0$, is an* M.S. *diffeomorphism isotopic to f. Note that $X_t \circ f = f \circ X_t$.*

Proof. Take l any Riemannian metric for M. The metric $g = \sum_{r=0}^{n-1} (f^i)^* l$, n the period of f, turns f into an isometry. The corollary is then clear from the proposition by taking $M = N$.

Let's now study the problem of M.S. isotopy classes on surfaces of non-negative Euler-Poincaré characteristic. M is then one of the following surfaces: the sphere, the projective plane, the torus or the Klein bottle.

In the case $M \simeq S^2$, M has only two isotopy classes, the orientation preserving and the orientation reserving, both of which have M.S. representatives. In the first case the time-one map of a differentiable M.S. vector field without closed orbits, e.g. the north pole–south pole, will suffice. In the second case we can take, for instance, $f = R \circ X_t = X_t \circ R$, $t \neq 0$, where R is the reflection on the axis ε and X is the north pole–south pole vector field with the necessary symmetry as shown in figure 3:

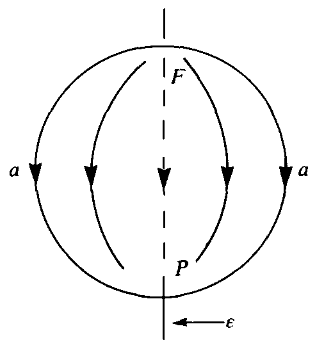

FIGURE 3

If $M = P^2$, the projective plane, we have only one isotopy class, the one of the identity, and again the time-one diffeomorphism induced by a differentiable M.S. vector field without closed orbits suffices, e.g.

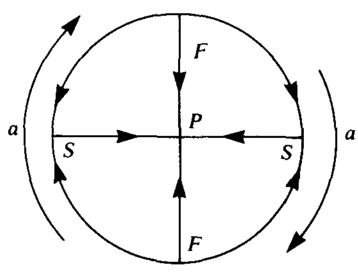

FIGURE 4

Now let $M = T^2$ be the torus. In this case we will show that an isotopy class of T^2, φ, is M.S. if and only if the eigenvalues of $H_1(\varphi): H_1(M, \mathbb{C}) \to H_1(M, \mathbb{C})$ are in the set $\{\pm 1, \pm i, -1/2 \pm (\sqrt{3/2})i, +1/2 \pm (\sqrt{3/2})i\}$. In particular φ has an M.S. representative if and only if $|\mathrm{tr}\, H_1(\varphi)| \leq 2$, a result already obtained by G. Fleitas in an unpublished work.

Suppose there is an M.S. $f \in \varphi$. We have then, by [3], that

$$0 = \text{entropy of } f \geq \log \max \{|\lambda_1|, |\lambda_2|\}$$

where λ_1 and $\lambda_2 \in \mathbb{C}$ are the eigenvalues of $H_1(\varphi)$. Since we have $\det H_1(\varphi) = \pm 1$ it follows that $|\lambda_1| = |\lambda_2| = 1$ and either λ_1 and λ_2 are real and in this situation we have

$\lambda_1, \lambda_2 \in \{1, -1\}$, or λ_1, λ_2 are complex of the form $a+bi$, $b \neq 0$, satisfying $2a \in \mathbb{Z}$ and $a^2+b^2=1$ from which we conclude that the conjugate pair is of the form above. To see the sufficiency, we will consider T^2 as the quotient space of $\mathbb{R}^2$ modulo the group of isometrics of $\mathbb{R}^2$ generated by the translations α and β:

$$\alpha : (x, y) \in \mathbb{R}^2 \mapsto (x+1, y) \in \mathbb{R}^2$$
$$\beta : (x, y) \in \mathbb{R}^2 \mapsto (x, y+1) \in \mathbb{R}^2.$$

This group can be identified with $H_1(T^2, \mathbb{Z}) = \pi_1(T^2)$. With the ordered basis $\{\alpha, \beta\}$ of $H_1(T^2, \mathbb{Z})$, $H_1(\varphi)$ can be expressed by a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$$

and we can get $f \in \varphi$ of the form:

$$\text{class} \begin{pmatrix} x \\ y \end{pmatrix} \in T^2 \mapsto \text{class } A \begin{pmatrix} x \\ y \end{pmatrix} \in T^2.$$

If the eigenvalues of A are complex f is periodic and, by corollary 1, there exists an M.S. diffeomorphism isotopic to f. If the eigenvalues of $H_1(\varphi)$ are real there is an A-invariant one-dimensional subspace of $\mathbb{R}^2$ with rational slope from which we can get $\gamma : S^1 \to T^2$, an embedding which does not bound a disc in T^2 and is f-invariant, i.e. $f(\gamma) = \gamma$. Let σ_1 and σ_2 be embeddings $S^1 \times [-1, 1] \to T^2$ such that $\sigma_1 \cup \sigma_2 = T^2$, $\mathrm{Int}(\sigma_1) \cap \mathrm{Int}(\sigma_2) = \varnothing$; $f' \in \varphi$ such that $f'\sigma_i = \sigma_i$, for $i = 1$ or 2, and $\sigma_i \circ f' \circ \sigma_i^{-1}$ is a diffeomorphism of $S^1 \times [-1, 1]$ which in $S^1 \times [-\frac{1}{2}, \frac{1}{2}]$ is of one of the following four forms:

$$(z, t) \mapsto (z, \pm t) \quad \text{or} \quad (z, t) \mapsto (\bar{z}, \pm t).$$

In $\sigma_i(S^1 \times [-\frac{1}{2}, \frac{1}{2}])$ take the vector field $(-1)^i X'$, for $i = 1$ or 2, where X' is the push-forward, via σ_i, of the vector field with the conveniently symmetric phase diagram given in figure 5:

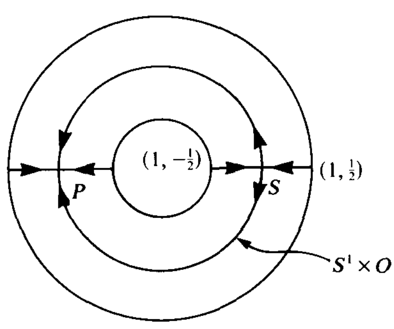

FIGURE 5

Finally, let X be an extension to T^2 of X' such that:

(1) X has no saddle connections;

(2) $X_1(\bigcup_{r=1,2} \sigma_i(S^1 \times [-1, -\frac{1}{2}] + S^1 \times [\frac{1}{2}, 1])) \subseteq \sigma_2(S^1 \times [\frac{1}{2}, \frac{1}{2}])$.

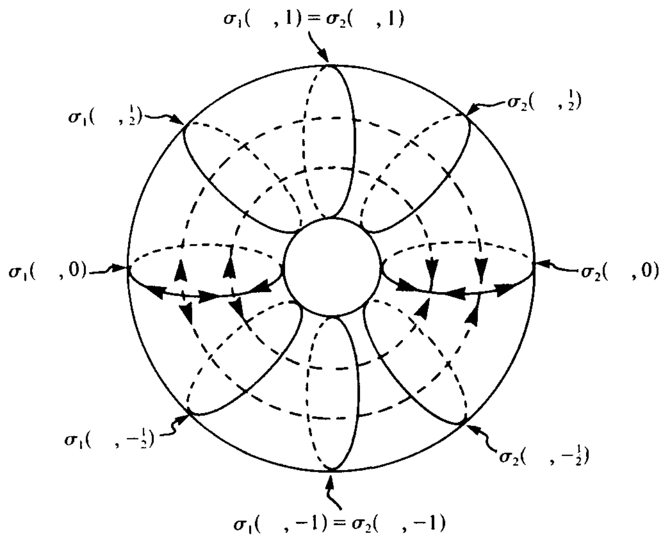

FIGURE 6

With these conditions X is an M.S. vector field and, modifying f' by an arbitrarily small isotopy to avoid tangencies we have that $f'\circ X_1$ is M.S. in φ, thus ending the case $M=T^2$.

If M is the Klein bottle, K^2, we will show that every isotopy class is M.S. To prove this it will be enough to show that given φ, an isotopy class of K^2, we can get $f\in\varphi$ and $\gamma: S^1\to K^2$, a two-sided embedding which does not bound a disc, such that $f(\gamma)=\gamma$, since with a reasoning similar to that in the case $M=T^2$ and $H_1(\varphi)$ with real eigenvalues, we can get the announced result. To see this consider K^2 the orbit space of the group of isometries of $\mathbb{R}^2$ generated by α and β;

$$\alpha:(x,y)\in\mathbb{R}^2\to(x+1,-y)\in\mathbb{R}^2$$
$$\beta:(x,y)\in\mathbb{R}^2=(x,y+1)\in\mathbb{R}^2.$$

This group, which we will denote by G, can be identified with $\pi_1(K^2,P)$, $P\in K^2$. Given $g\in G$, $g=g_k^{\xi_k}\dots g_1^{\xi_1}$ where $k\geq 1$, $g_i\in\{\alpha,\beta\}$ and $\xi_i=\pm 1$, $i=1,\dots,k$; this shows (by induction on $k\geq 1$) that there exists $n=n(g)\in\mathbb{Z}$ such that

$$g(x,y)=\left(x+\sum_{g_i=\alpha}\xi_i,\ (-1)^{\sum_{g_i=\alpha}|\xi_i|}y+n\right)$$

for every $x,y\in\mathbb{R}$. This settled it is easily seen that, given $g\in G$, there are unique m and $n\in\mathbb{Z}$ such that $g=\beta^n\cdot\alpha^m$. Let $f_*:\pi_1(K^2,P)\to\pi_1(K^2,P)$ be the mapping induced by $f\in\varphi$ such that $f(P)=P$. We have $f_*(\alpha)=\beta^c\alpha^a$, $f_*(\beta)=\beta^d\alpha^b$, where a, b, c and $d\in\mathbb{Z}$. Given $F=(f_1,f_2):\mathbb{R}^2\to\mathbb{R}^2$, a covering of f such that $F(\tilde{P})=\tilde{P}$, $\tilde{P}$ above P, we have

(1) $f_1(x+1,-y)=f_1(x,y)+a$;
(2) $f_2(x+1,-y)=(-1)^a f_2(x,y)+c$;
(3) $f_1(x,y+1)=f_1(x,y)+b$;
(4) $f_2(x,y+1)=(-1)^b f_2(x,y)+d$.

From (3) we have that $f_1(x,\ \ )$ is an homotopy on $S^1 \simeq \mathbb{R}/\mathbb{Z}$ of degree b and by (1) we have $f_1(1,\ \ ) \circ R = T \circ f_1(0,\ \ )$ where R is a reflection and T is a rotation in S^1. Thus

$$\operatorname{dgr}(f_1(1,\ \ ) \circ R) = \operatorname{dgr}(T \circ f_1(0,\ \ )) \quad \text{and} \quad -\operatorname{dgr}(f_1(1,\ \ )) = \operatorname{dgr}(f_1(0,\ \ ))$$

from which it follows that $b = \operatorname{dgr} f_1(X_1) = 0$.

Analogously we have $(f^{-1})_* \alpha = \beta^{\bar{c}} \alpha^{\bar{a}}$ and $(f^{-1})_* \beta = \beta^{\bar{d}}$; $\bar{a}, \bar{c}, \bar{d} \in \mathbb{Z}$, from which we have

$$\beta = (f^{-1})_*(f)_* \beta = (\beta^{\bar{d}})^d = \beta^{\bar{d}d}$$

which is possible only if $\bar{d} = \pm 1$. Now $f_*(\beta) = \beta^{\pm 1}$ proving our claim since we can take $\gamma: S^1 \to M$ homotopic to β and change f conveniently without leaving φ.

The case $\chi(M) \geq 0$ being complete we now turn our attention to the case of negative Euler-Poincaré characteristic and the proof of theorem A.

THEOREM A. *Let M be a two-dimensional compact connected boundaryless manifold with Euler-Poincaré characteristic $\chi(M) < 0$ and φ an isotopy class of M. Each one of the following conditions is necessary and sufficient for φ to be Morse-Smale*:

(1) $C(\varphi) = c(\varphi) = 1$;

(2) $\forall a \in \mathcal{S}$, $\forall l$ *Riemannian metric of M*, $\lim_{n\to\infty} i(l, \varphi^n(a))^{1/n} = 1$;

(3) *φ is of algebraically finite type.*

Proof. If there is $f \in \varphi$ M.S. we have by Bowen [**2**] that:

$$0 = \text{entropy}(f) \geq \log C(f) = \log C(\varphi)$$

from which it follows that $C(f) = c(f) = 1$. By the observations above relating the elements of $\mathcal{S}$ to conjugacy classes of $\pi_1(M)$ we have:

$$\limsup_{n\geq 1} [i(l, \varphi^n(a))]^{1/n} \leq CC(\varphi) \leq C(\varphi),$$

for every $a \in \mathcal{S}$ and l a Riemannian metric of M. By theorem 2 the limit of $[i(l, \varphi^n(a))]^{1/n}$ exists showing that (2) is necessary for (1). We will show now that (2) implies (3). As a matter of fact, supposing (2) we will prove (3′) where:

(3′) there are $f \in \varphi$, $\{\sigma\}$ finite disjoint family of embeddings, $\sigma: S' \times [-1, 1] \to M$, and $m \geq 1$ such that:

(a) $f(\bigcup \sigma) = \bigcup \sigma$;

(b) $f^m(p) = p$, for every $p \in M \setminus \bigcup \operatorname{Int}(\sigma)$;

(c) there is a Riemannian metric on M with constant -1 Gaussian curvature on $M \setminus \bigcup \operatorname{Int}(\sigma)$ such that f is an isometry on $M \setminus \bigcup \operatorname{Int}(\sigma)$ and $\sigma(\ \ , -1)$, $\sigma(\ \ , 1)$ are closed geodesics.

By the hypothesis and by theorems 1 and 2 there is an $f \in \varphi$ which is periodic or decomposable. If we can get $f \in \varphi$ periodic we are done so we can suppose that there are decomposable diffeomorphisms in φ. Let $R = R(\varphi)$ be the set of non-trivial families of simple closed curves of M, $\{a\}$, which decomposes elements of φ. On R define the reflexive and transitive relation, $\leq$, given by $\{a_1\} \leq \{a_2\}$ if and only if there is an injective $\gamma: \{a_1\} \to \{a_2\}$ such that a_1 and $\gamma(a_1)$ represent the same element of $\mathcal{S}$, for every a_1. It is easily seen that $\{a_1\} \equiv \{a_2\}$, given by $\{a_1\} \leq \{a_2\}$ and $\{a_2\} \leq \{a_1\}$, is an equivalence relation in R, that $\leq$ induces an order relation on the $\equiv$-classes and, as the cardinality of the elements of these classes are bounded by $K = K(\chi(M)) > 0$, there are maximal elements in $R/\equiv$.

Let $\{a\}$ be maximal in this order and let $f \in \varphi$ be decomposed by $\{a\}$. Taking on M a metric of constant -1 Gaussian curvature, l, we can suppose, by a classical result (see [**1**, theorem 10, p. 243]), that the a's are simple closed geodesics.

Let $\{\sigma_a\}_a$ be a finite disjoint family of cylinders such that $\sigma_a(\ \ ,0) = a$. Modifying f by an isotopy we can suppose that $f(\bigcup_a \sigma_a) = \bigcup \sigma_a$ and $f(\bigcup a) = \bigcup a$. Let $N \subseteq M$ be a connected component of $M \setminus \bigcup \mathrm{Int}(\sigma_a)$, $n \geq 1$ the f-period of N and let $g: N \to N$ be the diffeomorphism induced by f^n. By theorem 1, g is isotopic to a periodic, decomposable or P.A. diffeomorphism, h. Let $F': N \times I \to N$ be an isotopy connecting I_N to $g^{-1} \circ h$. From F' we can get an isotopy on all of M, F, satisfying:

(1) $F(\ \ ,0) = I_M$;

(2) $F(p, t) = F'(p, t)$, for $p \in N$ and $t \in I$;

(3) $F(p, t) = p$ for every $t \in I$ and $p \notin V \subseteq M$, a connected open neighbourhood of N such that $V \cap a = \emptyset$, for every a.

Thus $f \circ F(\ \ ,1)$ is isotopic to f via $f \circ F$. We will now show that the only possibility left by the hypothesis and the construction is h periodic. We will also suppose that N is different from the sphere minus three disjoint open discs where the claim is obvious since every isotopy class has a periodic representative. Now, if h is decomposed by a non-trivial family of simple closed curves of $\mathrm{Int}(N)$, $\{a'\}$, then the non-trivial family

$$\{a\} \cup \{f^i(a') \colon a' \in \{a'\} \text{ and } -n < i \leq 0\}$$

is strictly greater than $\{a\}$ and decomposes $f \circ F(\ \ ,1) \in \varphi$ which contradicts the maximality of $\{a\}$. If h is P.A. we have, for some $\lambda > 1$, that $\lim_{k \to \infty} i(l, h^k(b))^{1/k} = \lambda$ for every simple closed curve $b \subseteq \mathrm{Int}(N)$ which doesn't bound a disc or cylinder. For $k \geq 1$ we have $i(l, h^k(b)) = l(g)$ where g is a simple closed geodesic isotopic in N to $h^k \circ b$. By the same reason $i(l, h^k(b)) = l(g')$ where g' is a simple closed geodesic of M isotopic to $h^k \circ b$. By the Gauss-Bonnet theorem and the negative curvature of l we conclude that $g = g'$ which contradicts the initial assumption since: $[f \circ F(\ \ ,1)]^{kn} = h^k$ in N and

$$\lim_{k \to \infty} [i(l, [f \circ F(\ \ ,1)]^{kn} \circ b)]^{1/kn} = \lambda^{1/n} > 1.$$

Thus h is periodic of period say $\pi \geq 1$ which shows that $f \circ F(\ \ ,1)$ is periodic of period $n\pi$ in $\bigcup_{r \in \mathbb{Z}} f^i(N)$. Theorem 1 also guarantees the existence of the Riemannian metric with the required properties for h and N. Taking this metric to $f^i(N)$, $0 < i < n$, via f^i we have that this metric has constant -1 Gaussian curvature and that the components of $\sigma(\bigcup_{r \in \mathbb{Z}} f^i(N))$ are simple closed geodesics. Working in the same way on the other components of $M \setminus \bigcup \sigma_a$ after a finite number of steps we get a representative diffeomorphism satisfying (3′). Clearly (3′) implies (3) thus showing that (2) implies (3). It is easy to see that (3) implies (1) since $C(f^m) = 1$ and $f^m = I_M$ (m great enough) outside a finite number of disjoint embedded cylinders (on which f^m is a Dehn 'twist') and that $C(f^m) = [C(f)]^m$.

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To finish the proof of the theorem we will now show that if φ satisfies (3′) we are able to exhibit an M.S. diffeomorphism in φ. Given σ and $n=\pi(\sigma)\geq 1$,

$$\sigma^{-1}\circ f^m\circ\sigma : S^1\times[-1,1]\to S^1\times[-1,1]$$

is isotopic to a diffeomorphism h of $S^1\times[-1,1]$ such that on $S^1\times[-\frac{1}{2},\frac{1}{2}]$ h acts as one of the four diffeomorphism:

$$(z,t)\to(z,\pm t)\quad\text{or}\quad(z,t)\to(\bar{z},\pm t)$$

and $h=\sigma^{-1}\circ f^n\circ\sigma$ in a neighbourhood of $S^1\times-1\cup S^1\times 1$.

Let F' be the isotopy of $S^1\times[-1,1]$ connecting the identity of $S^1\times[-1,1]$ to $(\sigma^{-1}\circ f^n\circ\sigma)\circ h$ with the following properties:

(1) $F'(x,0)=I$, $x\in S^1\times[-1,1]$;

(2) $\sigma^{-1}\circ f^n\circ\sigma\circ F'(\quad,1)=h$;

(3) $F'(x,t)=x$, for x in a neighbourhood of $S^1\times-1\cup S'\times 1$, $t\in I$.

Taking, via σ, this isotopy to M we get an isotopy $F:M\times I\to M$ such that

(1) $F(p,t)=p$ for every $t=0$ and $p\in M$ or $p\notin\sigma$ and $t\in I$;

(2) $f\circ F(\quad,1)$ is isotopic to f and $\sigma:S^1\times[-1,1]\to M$ is an embedding such that $\sigma^{-1}\circ[f\circ F(\quad,1)]^n\circ\sigma(z,t)=(z,\pm t)$ or $(\bar{z},\pm t)$, $z\in S^1$ and $t\in[-\frac{1}{2},\frac{1}{2}]$.

Working in the same way in every f-orbit of $\{\sigma\}$ and changing f by an isotopy we get analogous expressions for $f^{\pi(\sigma)}$ in a distinguished cylinder of each orbit. On the submanifold of M,

$$M'=[M\setminus\bigcup\operatorname{Int}(\sigma)]\cup\bigcup\sigma(S'\times[-\tfrac{1}{2},\tfrac{1}{2}])$$

take the f-invariant vector field X' obtained in two steps. First, on each f-orbit

$$\vartheta=(F^n(\sigma(S^1\times[-\tfrac{1}{2},\tfrac{1}{2}])))_{n\in\mathbb{Z}},$$

by pushing forward via the distinguished embedding the vector field of figure 5 above and iterating it through f. Second, on each orbit, $\mathcal{P}$, of the connected components of $M\setminus\bigcup\operatorname{Int}(\sigma)$, N, taking the f-iteration of an $f^{\pi(N)}$-invariant vector field given by proposition 1, where N is chosen and fixed arbitrarily in $\mathcal{P}$.

Let X be the extension of X' to M satisfying

(1) $X_1(\sigma(S^1\times[-1,-\frac{1}{2}])+\sigma(S^1\times[\frac{1}{2},1]))\subseteq\sigma((-\frac{1}{2},\frac{1}{2}))$ for every σ;

(2) X has no saddle connections.

Changing f in $\bigcup\sigma(S^1\times[-1,\frac{1}{2}])+\sigma(S^1\times[\frac{1}{2},1])$ by a small isotopy to avoid tangencies we have that $f\circ X_1$ is M.S. and $f\circ X_1$ is isotopic to f via $(f\circ X_t)_{t\in I}$ which means that $f\circ X_1\in\varphi$, thus showing theorem A, and the proofs of our results are complete.

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