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The Waring problem for upper triangular matrix algebras

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Abstract. Our goal of the paper is to investigate the Waring problem for upper triangular matrix algebras, which gives a complete solution of a conjecture proposed by Panja and Prasad in 2023.

1 Introduction

The classical Waring problem proposed by Edward Waring in 1770 asserted that for every positive integer k there exists a positive integer $g(k)$ such that every positive integer can be expressed as a sum of $g(k)$ kth powers of nonnegative integers. In 1909, David Hilbert solved the problem. Various extensions and variations of this problem have been studied by different groups of mathematicians (see [2–4, 9, 11, 13, 14, 16, 17, 18]).

In 2009, Shalev [18] proved that given a word $w \neq 1$, every element in any finite non-abelian simple group G of sufficiently high order can be written as the product of three elements from $w(G)$, the image of the word map induced by w. In 2011, Larsen, Shalev, and Tiep [14] proved that, under the same assumptions, every element in G is the product of two elements from $w(G)$, which gave a definitive solution of the Waring problem for finite simple groups.

Let $n \geq 2$ be an integer. Let K be a field, and let $K\langle X\rangle$ be the free associative algebra over K, freely generated by the countable set $X = \{x_1, x_2, \dots\}$ of noncommutative variables. We refer to the elements of $K\langle X\rangle$ as polynomials.

Let $p(x_1, \dots, x_m) \in K\langle X\rangle$. Let $\mathcal{A}$ be an algebra over K. The set

$$p(\mathcal{A}) = \{p(a_1, \dots, a_m) \mid a_1, \dots, a_m \in \mathcal{A}\}$$

is called the image of p (on $\mathcal{A}$).

In 2020, Brešar [2] initiated the study of various Waring's problems for matrix algebras. He proved that if $\mathcal{A} = M_n(K)$, where $n \geq 2$ and K is an algebraically closed field with characteristic 0, and f is a noncommutative polynomial which is neither an identity nor a central polynomial of $\mathcal{A}$, then every trace zero matrix in $\mathcal{A}$ is a sum of four matrices from $f(\mathcal{A}) - f(\mathcal{A})$ [2, Corollary 3.19]. In 2023, Brešar and Šemrl [3] proved that any traceless matrix can be written as sum of two matrices from $f(M_n(\mathbb{C})) - f(M_n(\mathbb{C}))$, where $\mathbb{C}$ is the complex field and f is neither an identity nor a

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central polynomial for $M_n(\mathbb{C})$. Recently, they [4] have proved that if $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}\backslash\{0\}$ and $\alpha_1 + \alpha_2 + \alpha_3 = 0$, then any traceless matrix over $\mathbb{C}$ can be written as $\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3$, where $A_i \in f(M_n(\mathbb{C}))$.

By $T_n(K)$, we denote the set of all $n \times n$ upper triangular matrices over K. By $T_n(K)^{(0)}$, we denote the set of all $n \times n$ strictly upper triangular matrices over K. More generally, if $t \geq 0$, the set of all upper triangular matrices whose entries (i, j) are zero, for $j - i \leq t$, will be denoted by $T_n(K)^{(t)}$. It is easy to check that $J^t = T_n(K)^{(t-1)}$, where $t \geq 1$ and J is the Jacobson radical of $T_n(K)$ (see [1, Example 5.58]).

Let $p(x_1, \dots, x_m)$ be a noncommutative polynomial with zero constant term over K. We define its *order* as the least positive integer r such that $p(T_r(K)) = \{0\}$ but $p(T_{r+1}(K)) \neq \{0\}$. Note that $T_1(K) = K$. We say that p has order 0 if $p(K) \neq \{0\}$. We denote the order of p by $\operatorname{ord}(p)$. For a detailed introduction of the order of polynomials, we refer the reader to the book [7, Chapter 5].

In 2023, Panja and Prasad [16] discussed the image of polynomials with zero constant term and Waring-type problems on upper triangular matrix algebras over an algebraically closed field, which generalized two results in [6, 19]. More precisely, they obtained the following main result.

Theorem 1.1 [16, Theorem 5.18] *Let $n \geq 2$ and $m \geq 1$ be integers. Let $p(x_1, \dots, x_m)$ be a polynomial with zero constant term in noncommutative variables over an algebraically closed field K. Set $r = \operatorname{ord}(p)$. Then one of the following statements holds.*

(i) *Suppose that $r = 0$. We have that $p(T_n(K))$ is a dense subset of $T_n(K)$ (with respect to the Zariski topology).*
(ii) *Suppose that $r = 1$. We have that $p(T_n(K)) = T_n(K)^{(0)}$.*
(iii) *Suppose that $1 < r < n - 1$. We have that $p(T_n(K)) \subseteq T_n(K)^{(r-1)}$, and equality might not hold in general. Furthermore, for every n and r, there exists d such that each element of $T_n(K)^{(r-1)}$ can be written as a sum of d many elements from $p(T_n(K))$.*
(iv) *Suppose that $r = n - 1$. We have that $p(T_n(K)) = T_n(K)^{(n-2)}$.*
(v) *Suppose that $r \geq n$. We have that $p(T_n(K)) = \{0\}$.*

They proposed the following conjecture.

Conjecture 1.1 [16, Conjecture] *Let $p(x_1, \dots, x_m)$ be a polynomial with zero constant term in noncommutative variables over an algebraically closed field K. Suppose $\operatorname{ord}(p) = r$, where $1 < r < n - 1$. Then $p(T_n(K)) + p(T_n(K)) = T_n(K)^{(r-1)}$.*

We note that if p is a multilinear polynomial and K is an infinite field, then $p(T_n(K)) = T_n(K)^{(r-1)}$ (see [8, 10, 15]).

In the present paper, we shall prove the following main result of the paper, which gives a complete solution of Conjecture 1.1.

Theorem 1.2 *Let $n \geq 2$ and $m \geq 1$ be integers. Let $p(x_1, \dots, x_m)$ be a polynomial with zero constant term in noncommutative variables over an infinite field K. Suppose $\operatorname{ord}(p) = r$, where $1 < r < n - 1$. We have that $p(T_n(K)) + p(T_n(K)) = T_n(K)^{(r-1)}$. Furthermore, if $r = n - 2$, we have that $p(T_n(K)) = T_n(K)^{(n-3)}$.*

We organize the paper as follows: In Section 2, we shall give some preliminaries. We shall modify some results in [5, 8, 12], which will be used in the proof of Theorem 1.2. In Section 3, we shall give the proof of Theorem 1.2 by using some new arguments (for example, compatible variables in polynomials and recursive polynomials).

2 Preliminaries

Let $\mathcal{N}$ be the set of all positive integers. Let $m \in \mathcal{N}$. Let K be a field. Set $K^* = K \backslash \{0\}$. For any $k \in \mathcal{N}$, we set

$$T_m^k = \{(i_1, \ldots, i_k) \in \mathcal{N}^k \mid 1 \leq i_1, \ldots, i_k \leq m\}.$$

Let $p(x_1, \ldots, x_m)$ be a polynomial with zero constant term in noncommutative variables over K. We can write

$$p(x_1, \ldots, x_m) = \sum_{k=1}^{d} \left(\sum_{(i_1, i_2, \ldots, i_k) \in T_m^k} \lambda_{i_1 i_2 \ldots i_k} x_{i_1} x_{i_2} \ldots x_{i_k} \right), \tag{1}$$

where $\lambda_{i_1 i_2 \ldots i_k} \in K$ and d is the degree of p.

We begin with the following result, which is slightly different from [5, Lemma 3.2]. We give its proof for completeness.

Lemma 2.1 *For any $u_i = (a_{jk}^{(i)}) \in T_n(K)$, $i = 1, \ldots, m$, we set*

$$\bar{a}_{jj} = (a_{jj}^{(1)}, \ldots, a_{jj}^{(m)}),$$

where $j = 1, \ldots, n$. We have that

$$p(u_1, \ldots, u_m) = \begin{pmatrix} p(\bar{a}_{11}) & p_{12} & \ldots & p_{1n} \\ 0 & p(\bar{a}_{22}) & \ldots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & p(\bar{a}_{nn}) \end{pmatrix}, \tag{2}$$

where

$$p_{st} = \sum_{k=1}^{t-s} \left(\sum_{\substack{s=j_1<j_2<\cdots<j_{k+1}=t \\ (i_1,\ldots,i_k) \in T_m^k}} p_{i_1 \ldots i_k}(\bar{a}_{j_1 j_1}, \ldots, \bar{a}_{j_{k+1} j_{k+1}}) a_{j_1 j_2}^{(i_1)} \ldots a_{j_k j_{k+1}}^{(i_k)} \right)$$

for all $1 \leq s < t \leq n$, where $p_{i_1,\ldots,i_k}(z_1, \ldots, z_{m(k+1)})$, $1 \leq i_1, i_2, \ldots, i_k \leq m$, $k = 1, \ldots, n-1$, is a polynomial in commutative variables over K.

Proof Let $u_i = (a_{jk}^{(i)}) \in T_n(K)$, where $i = 1, \ldots, m$. For any $1 \leq i_1, \ldots, i_k \leq m$, we easily check that

$$u_{i_1} \ldots u_{i_k} = \begin{pmatrix} m_{11} & m_{12} & \ldots & m_{1n} \\ 0 & m_{22} & \ldots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & m_{nn} \end{pmatrix},$$

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where

$$m_{st} = \sum_{s=j_1\leq j_2\leq\cdots\leq j_{k+1}=t} a^{(i_1)}_{j_1j_2}\ldots a^{(i_k)}_{j_kj_{k+1}}$$

for all $1\leq s\leq t\leq n$. It follows from (1) that

$$\begin{aligned}
p(u_1,\ldots,u_m) &= \sum_{k=1}^{d}\left(\sum_{(i_1,\ldots,i_k)\in T_m^k}\lambda_{i_1\ldots i_k}u_{i_1}\ldots u_{i_k}\right)\\
&= \sum_{k=1}^{d}\left(\sum_{(i_1,\ldots,i_k)\in T_m^k}\lambda_{i_1\ldots i_k}\begin{pmatrix} m_{11} & m_{12} & \ldots & m_{1n}\\ 0 & m_{22} & \ldots & m_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & m_{nn}\end{pmatrix}\right)\\
&= \begin{pmatrix} p_{11} & p_{12} & \ldots & p_{1n}\\ 0 & p_{22} & \ldots & p_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & p_{nn}\end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
p_{st} &= \sum_{k=1}^{d}\left(\sum_{(i_1,\ldots,i_k)\in T_m^k}\lambda_{i_1\ldots i_k}m_{st}\right)\\
&= \sum_{k=1}^{d}\left(\sum_{(i_1,\ldots,i_k)\in T_m^k}\lambda_{i_1\ldots i_k}\left(\sum_{s=j_1\leq j_2\leq\cdots\leq j_{k+1}=t} a^{(i_1)}_{j_1j_2}\ldots a^{(i_k)}_{j_kj_{k+1}}\right)\right)\\
&= \sum_{k=1}^{d}\left(\sum_{\substack{s=j_1\leq j_2\leq\cdots\leq j_{k+1}=t\\ (i_1,\ldots,i_k)\in T_m^k}}\lambda_{i_1i_2\ldots i_k}a^{(i_1)}_{j_1j_2}\ldots a^{(i_k)}_{j_kj_{k+1}}\right),
\end{aligned}$$

where $1\leq s\leq t\leq n$. In particular,

$$\begin{aligned}
p_{ss} &= \sum_{k=1}^{d}\left(\sum_{(i_1,\ldots,i_k)\in T_m^k}\lambda_{i_1i_2\ldots i_k}a^{(i_1)}_{ss}\ldots a^{(i_k)}_{ss}\right)\\
&= p(\bar{a}_{ss})
\end{aligned}$$

for all $s=1,\ldots,n$, and

$$\begin{aligned}
p_{st} &= \sum_{k=1}^{d}\left(\sum_{\substack{s=j_1\leq j_2\leq\cdots\leq j_{k+1}=t\\ (i_1,\ldots,i_k)\in T_m^k}}\lambda_{i_1i_2\ldots i_k}a^{(i_1)}_{j_1j_2}\ldots a^{(i_k)}_{j_kj_{k+1}}\right)\\
&= \sum_{k=1}^{t-s}\left(\sum_{\substack{s=j_1<j_2<\cdots<j_{k+1}=t\\ (i_1,\ldots,i_k)\in T_m^k}}p_{i_1i_2\ldots i_k}(\bar{a}_{j_1j_1},\ldots,\bar{a}_{j_{k+1}j_{k+1}})a^{(i_1)}_{j_1j_2}\ldots a^{(i_k)}_{j_kj_{k+1}}\right)
\end{aligned}$$

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for all $1 \le s < t \le n$, where $p_{i_1,\ldots,i_k}(z_1,\ldots,z_{m(k+1)})$ is a polynomial in commutative variables over K. This proves the result. ∎

The following result will be used in the proof of our main result.

Lemma 2.2 *Let $m \ge 1$ be an integer. Let $p(x_1,\ldots,x_m)$ be a polynomial with zero constant term in noncommutative variables over K. Let $p_{i_1,\ldots,i_k}(z_1,\ldots,z_{m(k+1)})$ be a polynomial in commutative variables over K in (2), where $1 \le i_1,\ldots,i_k \le m$, $1 \le k \le n-1$. Suppose that $\mathrm{ord}(p) = r$, $1 < r < n-1$. We have that:*

(i) $p(K) = \{0\}$.
(ii) $p_{i_1,\ldots,i_k}(K) = \{0\}$ *for all* $1 \le i_1,\ldots,i_k \le m$, *where* $k = 1,\ldots,r-1$.
(iii) $p_{i'_1,\ldots,i'_r}(K) \ne \{0\}$ *for some* $1 \le i'_1,\ldots,i'_r \le m$.

Proof The statement (i) is clear. We now claim that the statement (ii) holds true. Suppose on the contrary that

$$p_{i'_1\ldots i'_s}(K) \ne \{0\}$$

for some $1 \le i'_1,\ldots,i'_s \le m$, where $1 \le s \le r-1$. Then there exist $\bar{b}_j \in K^m$, where $j = 1,\ldots,s+1$ such that

$$p_{i'_1\ldots i'_s}(\bar{b}_1,\ldots,\bar{b}_{s+1}) \ne 0.$$

We take $u_i = (a_{jk}^{(i)}) \in T_{s+1}(K)$, $i = 1,\ldots,m$, where

$$\begin{cases} \bar{a}_{jj} = \bar{b}_j, & j = 1,\ldots,s+1,\\ a_{k,k+1}^{(i'_k)} = 1, & k = 1,\ldots,s,\\ a_{jk}^{(i)} = 0, & \text{otherwise}. \end{cases}$$

It follows from (2) that

$$p_{1,s+1} = p_{i'_1\ldots i'_s}(\bar{b}_1,\ldots,\bar{b}_{s+1}) \ne 0.$$

This implies that $p(T_{s+1}(K)) \ne \{0\}$, a contradiction. This proves the statement (ii).

We finally claim that the statement (iii) holds true. Note that $p(T_{1+r}(K)) \ne \{0\}$. Thus, we have that there exist $u_i = (a_{jk}^{(i)}) \in T_{1+r}(K)$, $i = 1,\ldots,m$, such that

$$p(u_1,\ldots,u_m) = (p_{st}) \ne 0.$$

In view of the statement (ii), we get that

$$p_{1,r+1} = \sum_{\substack{1=j_1<j_2<\cdots<j_{r+1}=r+1\\ (i_1,\ldots,i_r)\in T_m^r}} p_{i_1 i_2\ldots i_r}(\bar{a}_{j_1 j_1},\ldots,\bar{a}_{j_{r+1}j_{r+1}})a_{j_1 j_2}^{(i_1)}\ldots a_{j_r j_{r+1}}^{(i_r)} \ne 0.$$

This implies that $p_{i'_1,\ldots,i'_r}(K) \ne \{0\}$ for some $1 \le i'_1,\ldots,i'_r \le m$. This proves the statement (iii). The proof of the result is complete. ∎

The following well-known result will be used in the proof of the rest results.

Lemma 2.3 [12, Theorem 2.19] *Let K be an infinite field. Let $f(x_1,\ldots,x_m)$ be a nonzero polynomial in commutative variables over K. Then there exist $a_1,\ldots,a_m \in K$ such that $f(a_1,\ldots,a_m) \ne 0$.*

Lemma 2.4 *Let n, s be integers with $1 \leq s \leq n$. Let $p(x_1, \dots, x_s)$ be a nonzero polynomial in commutative variables over an infinite field K. We have that there exist $a_1, \dots, a_n \in K$ such that*

$$p(a_{i_1}, \dots, a_{i_s}) \neq 0$$

for all $1 \leq i_1 < \cdots < i_s \leq n$.

Proof We set

$$f(x_1, \dots, x_n) = \prod_{1 \leq i_1 < \cdots < i_s \leq n} p(x_{i_1}, \dots, x_{i_s}).$$

It is clear that $f \neq 0$. In view of Lemma 2.3, we have that there exist $a_1, \dots, a_n \in K$ such that

$$f(a_1, \dots, a_n) \neq 0.$$

This implies that

$$p(a_{i_1}, \dots, a_{i_s}) \neq 0$$

for all $1 \leq i_1 < \cdots < i_s \leq n$. This proves the result. ■

The following technical result is a generalized form of [8, Lemma 2.11], which discusses compatible variables in polynomials.

Lemma 2.5 *Let $t \geq 1$. Let $U_i = \{i_1, \dots, i_s\} \subseteq \mathbb{N}$, $i = 1, \dots, t$. Let $p_i(x_{i_1}, \dots, x_{i_s})$ be a nonzero polynomial in commutative variables over an infinite field K, where $i = 1, \dots, t$. Then there exist $a_k \in K$ with $k \in \bigcup_{i=1}^t U_i$ such that*

$$p_i(a_{i_1}, \dots, a_{i_s}) \neq 0$$

for all $i = 1, \dots, t$.

Proof Without loss of generality, we assume that

$$\{1, 2, \dots, n\} = \bigcup_{i=1}^{t} U_i.$$

We set

$$f(x_1, \dots, x_n) = \prod_{i=1}^{t} p_i(x_{i_1}, \dots, x_{i_s}).$$

It is clear that $f \neq 0$. In view of Lemma 2.3, we have that there exist $a_1, \dots, a_n \in K$ such that

$$f(a_1, \dots, a_n) \neq 0.$$

This implies that

$$p_i(a_{i_1}, \dots, a_{i_s}) \neq 0$$

for all $i = 1, \dots, t$. This proves the result. ■

The following technical result will be used in the proof of the main result of the paper.

Lemma 2.6 *Let $s \geq 1$ and $t \geq 2$ be integers. Let K be an infinite field. Let $a_{ij} \in K$, where $1 \leq i \leq t$, $1 \leq j \leq s$ with $a_{11} \in K^*$ and $b \in K^*$. For any $2 \leq i \leq t$, there exists a nonzero element in $\{a_{i1}, \dots, a_{is}\}$. Then there exist $c_i \in K$, $i = 1, \dots, s$, such that*

$$\begin{cases} a_{11}c_1 + \cdots + a_{1s}c_s = b; \\ a_{i1}c_1 + \cdots + a_{is}c_s \neq 0 \end{cases}$$

for all $i = 2, \dots, t$.

Proof Suppose first that $s = 1$. Note that $a_{i1} \in K^*$, $i = 1, \dots, t$. Take $c_1 = a_{11}^{-1}b$. It is clear

$$\begin{cases} a_{11}c_1 = b; \\ a_{i1}c_1 \neq 0 \end{cases}$$

for all $2 \leq i \leq t$. Suppose next that $s \geq 2$. Suppose first that $a_{i1} \neq 0$ for all $i = 2, \dots, t$. We define the following polynomials:

$$\begin{cases} f_1(x_2, \dots, x_s) = b - a_{12}x_2 - \cdots - a_{1s}x_s; \\ f_i(x_2, \dots, x_s) = a_{i1}a_{11}^{-1}b + (a_{i2} - a_{i1}a_{11}^{-1}a_{12})x_2 + \cdots + (a_{is} - a_{i1}a_{11}^{-1}a_{1s})x_s \end{cases}$$

for all $2 \leq i \leq t$. Since $b, a_{i1} \in K^*$, $i = 1, \dots, t$, we note that $f_i \neq 0$ for all $i = 1, \dots, t$. In view of Lemma 2.5, we get that there exist $c_2, \dots, c_s \in K$ such that

$$f_i(c_2, \dots, c_s) \neq 0$$

for all $i = 1, \dots, t$. This implies that

$$\begin{cases} b - a_{12}c_2 - \cdots - a_{1s}c_s \neq 0; \\ a_{i1}a_{11}^{-1}b + (a_{i2} - a_{i1}a_{11}^{-1}a_{12})c_2 + \cdots + (a_{is} - a_{i1}a_{11}^{-1}a_{1s})c_s \neq 0 \end{cases} \tag{3}$$

for all $2 \leq i \leq t$. We set

$$c_1 = a_{11}^{-1}(b - a_{12}c_2 - \cdots - a_{1s}c_s).$$

It follows from (3) that

$$\begin{cases} a_{11}c_1 + \cdots + a_{1s}c_s = b; \\ a_{i1}c_1 + \cdots + a_{is}c_s \neq 0 \end{cases}$$

for all $2 \leq i \leq t$, as desired.

Suppose next that $a_{i1} = 0$, $i = 2, \dots, t$. Note that $a_{il(i)} \neq 0$, for some $2 \leq l(i) \leq s$ for all $i = 2, \dots, t$. We define the following polynomials:

$$\begin{cases} f_1(x_2, \dots, x_s) = a_{12}x_2 + \cdots + a_{1s}x_s - b; \\ f_i(x_2, \dots, x_s) = a_{i2}x_2 + \cdots + a_{is}x_s \end{cases}$$

for all $2 \leq i \leq t$. Note that $f_i \neq 0$ for all $i = 1, \dots, t$. In view of Lemma 2.5, we get that there exist $c_i \in K$, $i = 2, \dots, s$, such that

$$f_i(c_2, \dots, c_s) \neq 0$$

for all $i = 1, \ldots, t$. That is

$$\begin{cases} a_{12}c_2 + \cdots + a_{1s}c_s - b \neq 0; \\ a_{i2}c_2 + \cdots + a_{is}c_s \neq 0 \end{cases}$$

for all $2 \leq i \leq t$. Since $a_{11} \neq 0$ we get that there exists $c_1 \in K$ such that

$$a_{11}c_1 = b - a_{12}c_2 - \cdots - a_{1s}c_s.$$

This implies that

$$\begin{cases} a_{11}c_1 + a_{12}c_2 + \cdots + a_{1s}c_s = b; \\ \qquad\quad a_{i2}c_2 + \cdots + a_{is}c_s \neq 0 \end{cases}$$

for all $2 \leq i \leq t$, as desired.

We finally assume that there exist $a_{i1} \neq 0$ and $a_{j1} = 0$ for some $i, j \in \{2, \ldots, t\}$. Without loss of generality, we assume that $a_{i1} \neq 0$ for all $i = 2, \ldots, t_1$ and $a_{i1} = 0$ for all $i = t_1 + 1, \ldots, t$. We define the following polynomials:

$$\begin{cases} f_1(x_2, \ldots, x_s) = b - a_{12}x_2 - \cdots - a_{1s}x_s; \\ f_i(x_2, \ldots, x_s) = a_{i1}a_{11}^{-1}b + (a_{i2} - a_{i1}a_{11}^{-1}a_{12})x_2 + \cdots + (a_{is} - a_{i1}a_{11}^{-1}a_{1s})x_s; \\ f_j(x_2, \ldots, x_s) = a_{j2}x_2 + \cdots + a_{js}x_s \end{cases}$$

for all $2 \leq i \leq t_1$ and $t_1 + 1 \leq j \leq t$. Note that $b, a_{i1} \in K^*$, $i = 1, \ldots, t_1$, $a_{jl(j)} \neq 0$ where $2 \leq l(j) \leq s$ for all $j = t_1 + 1, \ldots t$. It is clear that $f_i \neq 0$ for all $i = 1, \ldots, t$. In view of Lemma 2.5, we get that there exist $c_i \in K$, $i = 2, \ldots, s$, such that

$$f_i(c_2, \ldots, c_s) \neq 0,$$

where $i = 1, \ldots, t$. This implies that

$$\begin{cases} b - a_{12}c_2 - \cdots - a_{1s}c_s \neq 0; \\ a_{i1}a_{11}^{-1}b + (a_{i2} - a_{i1}a_{11}^{-1}a_{12})c_2 + \cdots + (a_{is} - a_{i1}a_{11}^{-1}a_{1s})c_s \neq 0; \\ a_{j2}c_2 + \cdots + a_{js}c_s \neq 0 \end{cases} \tag{4}$$

for all $2 \leq i \leq t_1$ and $t_1 + 1 \leq j \leq t$. We set

$$c_1 = a_{11}^{-1}(b - a_{12}c_2 - \cdots - a_{1s}c_s).$$

It follows from (4) that

$$\begin{cases} a_{11}c_1 + \cdots + a_{1s}c_s = b; \\ a_{i1}c_1 + \cdots + a_{is}c_s \neq 0; \\ a_{j1}c_2 + \cdots + a_{js}c_s \neq 0 \end{cases}$$

for all $2 \leq i \leq t_1$ and $t_1 + 1 \leq j \leq t$, as desired. The proof of the result is now complete. ∎

3 The proof of Theorem 1.2

Let $n \geq 2$ and $m \geq 1$ be integers. Let $p(x_1, \dots, x_m)$ be a polynomial with zero constant term in noncommutative variables over an infinite field K. Suppose that $1 < r < n-1$, where $r = ord(p)$.

Take any $u_i = (a_{jk}^{(i)}) \in T_n(K)$, $i = 1, \dots, m$. In view of both Lemma 2.1 and Lemma 2.2, we have that

$$p(u_1, \dots, u_m) = (p_{s,r+s+t}), \tag{5}$$

where

$$p_{s,r+s+t} = \sum_{k=r}^{r+t} \left(\sum_{\substack{s=j_1<\cdots<j_{k+1}=r+s+t \\ (i_1,\dots,i_k)\in T_m^k}} p_{i_1\dots i_k}(\bar{a}_{j_1j_1}, \dots, \bar{a}_{j_{k+1}j_{k+1}}) a_{j_1j_2}^{(i_1)} \dots a_{j_kj_{k+1}}^{(i_k)} \right)$$

for all $1 \leq s < r+s+t \leq n$ and

$$p_{i'_1\dots i'_r}(K) \neq \{0\}$$

for some $1 \leq i'_1, \dots, i'_r \leq m$. It follows from Lemma 2.4 that there exist $\bar{c}_1, \dots, \bar{c}_n \in K^m$ such that

$$p_{i'_1\dots i'_r}(\bar{c}_{j_1}, \dots, \bar{c}_{j_{r+1}}) \neq 0 \tag{6}$$

for all $1 \leq j_1 < \cdots < j_{r+1} \leq n$. We set

$$\begin{cases} \bar{a}_{jj} = \bar{c}_j, \quad j = 1, \dots, n; \\ a_{i,i+1}^{(k)} = a_{i,i+1}^{(k)}, \quad i = 1, \dots, r-1 \text{ and } k = 1, \dots, m; \\ a_{r+s-1,r+s+t}^{(i'_k)} = x_{r+s-1,r+s+t}^{(i'_k)}, \quad 1 \leq s < r+s+t \leq n, k = 1, \dots, r; \\ a_{ij}^{(k)} = 0, \quad \text{otherwise}. \end{cases}$$

For any $1 \leq s < r+s+t \leq n$, we set

$$U_{s,r+s+t} = \left\{ (r+u-1, r+u+w, i'_k) \mid x_{r+u-1,r+u+w}^{(i'_k)} \quad \text{in } p_{s,r+s+t} \right\}$$

and

$$\overline{U}_{s,r+s+t} = \{ (r+u-1, r+u, i'_k) \mid (r+u-1, r+u, i'_k) \in U_{s,r+s+t} \}.$$

We define an order on the set

$$\{(s, r+s+t) \mid 1 \leq s < r+s+t \leq n\}$$

as follows:

(i) $(s, r+s+t) < (s_1, r+s_1+t_1)$ if $t < t_1$;
(ii) $(s, r+s+t) < (s_1, r+s_1+t_1)$ if $t = t_1$ and $s < s_1$.

That is,

$$(1, r+1) < \cdots < (n-r, n) < (1, r+2) < \cdots < (n-r-1, n) < \cdots < (1, n). \tag{7}$$

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For any $1 \le s < r+s+t \le n$, we set

$$W_{s,r+s+t} = \bigcup_{(1,r+1)\le(i,r+i+j)\le(s,r+s+t)} U_{i,r+i+j},$$

and

$$\overline{W}_{s,r+s+t} = \bigcup_{(1,r+1)\le(i,r+i+j)\le(s,r+s+t)} \overline{U}_{i,r+i+j}.$$

We begin with the following lemmas, which will be used in the proof of our main result.

Lemma 3.1 *Let* $1 \le s < r+s \le n$. *Suppose that* $(s, r+s) \ne (1, r+1)$. *We claim that*

$$\overline{W}_{s,r+s} \setminus \{(r+s-1, r+s, i'_k) \mid 1 \le k \le r\} = \overline{W}_{s-1,r+s-1}. \tag{8}$$

Proof We first claim that

$$\overline{W}_{s,r+s} \setminus \{(r+s-1, r+s, i'_k) \mid 1 \le k \le r\} \subseteq \overline{W}_{s-1,r+s-1}.$$

Take any $(r+i-1, r+i, i'_k) \in \overline{W}_{s,r+s} \setminus \{(r+s-1, r+s, i'_k) \mid 1 \le k \le r\}$. We have that

$$(r+i-1, r+i, i'_k) \in \overline{U}_{s_2,r+s_2}$$

for some $(1, r+1) \le (s_2, r+s_2) \le (s, r+s)$. This implies that

$$r+i \le r+s_2 \le r+s.$$

We get that $i \le s$. Suppose that $i = s$. It follows that

$$(r+i-1, r+i, i'_k) \in \{(r+s-1, r+s, i'_k) \mid 1 \le k \le r\},$$

a contradiction. Hence $i \le s-1$. It is clear that

$$(r+i-1, r+i, i'_k) \in \overline{U}_{i,r+i},$$

where $(1, r+1) \le (i, r+i) \le (s-1, r+s-1)$. It follows that

$$(r+i-1, r+i, i'_k) \in \overline{W}_{s-1,r+s-1}.$$

We obtain that

$$\overline{W}_{s,r+s} \setminus \{(r+s-1, r+s, i'_k) \mid 1 \le k \le r\} \subseteq \overline{W}_{s-1,r+s-1},$$

as desired. We next claim that

$$\overline{W}_{s-1,r+s-1} \subseteq \overline{W}_{s,r+s} \setminus \{(r+s-1, r+s, i'_k) \mid 1 \le k \le r\}.$$

If $(r+s-1, r+s, i'_k) \in \overline{W}_{s-1,r+s-1}$ for $1 \le k \le r$, we have that

$$r+s \le r+s-1,$$

a contradiction. Hence

$$\{(r+s-1, r+s, i'_k) \mid 1 \le k \le r\} \bigcap \overline{W}_{s-1,r+s-1} = \varnothing.$$

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Since $\overline{W}_{s-1,r+s-1} \subseteq \overline{W}_{s,r+s}$ we get that

$$\overline{W}_{s-1,r+s-1} \subseteq \overline{W}_{s,r+s} \setminus \{(r+s-1, r+s, i'_k) \mid 1 \le k \le r\},$$

as desired. We obtain that

$$\overline{W}_{s-1,r+s-1} = \overline{W}_{s,r+s} \setminus \{(r+s-1, r+s, i'_k) \mid 1 \le k \le r\}.$$

This proves the result. ■

Lemma 3.2 *Let* $1 \le s < r+s+t \le n$. *Suppose that* $t > 0$. *We claim that*

$$\overline{W}_{s_1,r+s_1+t_1} = \overline{W}_{s,r+s+t},$$

where

$$(s_1, r+s_1+t_1) = \max\{(i, r+i+j) \mid (1, r+1) \le (i, r+i+j) < (s, r+s+t)\}.$$

Proof We first claim that

$$\overline{W}_{s,r+s+t} = \overline{W}_{n-r,n}.$$

Since $t > 0$, we note that

$$(s, r+s+t) > (n-r, n).$$

This implies that $\overline{W}_{s,r+s+t} \supseteq \overline{W}_{n-r,n}$. Take any $(r+u-1, r+u, i'_k) \in \overline{W}_{s,r+s+t}$. It is clear that

$$(r+u-1, r+u, i'_k) \in \overline{U}_{u,r+u} \subseteq \overline{W}_{n-r,n}.$$

This implies that $\overline{W}_{s,r+s+t} \subseteq \overline{W}_{n-r,n}$. Hence, $\overline{W}_{s,r+s+t} = \overline{W}_{n-r,n}$ as desired.

Since $(n-r, n) < (s, r+s+t)$ we get that

$$(n-r, n) \le (s_1, r+s_1+t_1) < (s, r+s+t).$$

This implies that

$$\overline{W}_{n-r,n} \subseteq \overline{W}_{s_1,r+s_1+t_1} \subseteq \overline{W}_{s,r+s+t}.$$

Since $\overline{W}_{s,r+s+t} = \overline{W}_{n-r,n}$ we obtain that $\overline{W}_{s_1,r+s_1+t_1} = \overline{W}_{s,r+s+t}$. This proves the result. ■

The following technical result will be used in the proof of the next result.

Lemma 3.3 *Let* $1 \le s < r+s+t \le n$. *If* $(r+i-1, r+i+j, i'_k) \in U_{s,r+s+t}$, *we have that* $j \le t$.

Proof Suppose that $(r+i-1, r+i+j, i'_k) \in U_{s,r+s+t}$. That is, $x^{(i'_k)}_{r+i-1,r+i+j}$ appears in $p_{s,r+s+t}$. In view of (5), we note that every monomial in $p_{s,r+s+t}$ is made up of at least r elements multiplied together. This implies that

$$((r+s+t)-s) - ((r+i+j)-(r+i-1)) \ge r-1.$$

We obtain that $j \le t$. This proves the result. ■

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Lemma 3.4 *Let* $1 \le s < r + s + t \le n$ *and* $t > 0$. *We claim that*

$$W_{s_1,r+s_1+t_1} = W_{s,r+s+t} \backslash \{(r+s-1, r+s+t, i'_k) \mid 1 \le k \le r\},$$

where

$$(s_1, r+s_1+t_1) = \max\{(i, r+i+j) \mid (1, r+1) \le (i, r+i+j) < (s, r+s+t)\}.$$

Proof We first claim that

$$W_{s_1,r+s_1+t_1} \subseteq W_{s,r+s+t} \backslash \{(r+s-1, r+s+t, i'_k) \mid 1 \le k \le r\}.$$

If $(r+s-1, r+s+t, i'_k) \in W_{s_1,r+s_1+t_1}$ for some $1 \le k \le r$, we get that

$$(r+s-1, r+s+t, i'_k) \in U_{s_2,r+s_2+t_2} \tag{9}$$

for some $(1, r+1) \le (s_2, r+s_2+t_2) \le (s_1, r+s_1+t_1)$. It is clear that

$$t_2 \le t_1 \le t.$$

In view of Lemma 3.3, we get that $t \le t_2$. It follows that

$$t_1 = t_2 = t.$$

Since $(s_1, r+s_1+t_1) < (s, r+s+t)$ we get that $s_1 < s$. Since $(s_2, r+s_2+t_2) \le (s_1, r+s_1+t_1)$ we get that $s_2 \le s_1$. Thus, we obtain that $s_2 < s$. It follows from (9) that

$$r+s+t \le r+s_2+t_2.$$

This implies that $s \le s_2$, a contradiction. Hence, we have that

$$(r+s-1, r+s+t, i'_k) \notin W_{s_1,r+s_1+t_1}$$

for all $1 \le k \le r$. It is clear that $W_{s_1,r+s_1+t_1} \subseteq W_{s,r+s+t}$. We obtain that

$$W_{s_1,r+s_1+t_1} \subseteq W_{s,r+s+t} \backslash \{(r+s-1, r+s+t, i'_k) \mid 1 \le k \le r\},$$

as desired. We next claim that

$$W_{s,r+s+t} \backslash \{(r+s-1, r+s+t, i'_k) \mid 1 \le k \le r\} \subseteq W_{s_1,r+s_1+t_1}.$$

For any $(r+i-1, r+i+j, i'_k) \in W_{s,r+s+t} \backslash \{(r+s-1, r+s+t, i'_k) \mid 1 \le k \le r\}$, we have

$$(r+i-1, r+i+j, i'_k) \in U_{s_2,r+s_2+t_2}$$

for some $(1, r+1) \le (s_2, r+s_2+t_2) \le (s, r+s+t)$. This implies that $t_2 \le t$. In view of Lemma 3.3, we note that $j \le t_2$. We have that $j \le t$. It is clear that

$$(r+i-1, r+i+j, i'_k) \in U_{i,r+i+j},$$

where $(1, r+1) \le (i, r+i+j) \le (s, r+s+t)$. Note that

$$(r+i-1, r+i+j, i'_k) \notin \{(r+s-1, r+s+t, i'_k) \mid 1 \le k \le r\}.$$

We get that

$$(i, r+i+j) \neq (s, r+s+t).$$

This implies that

$$(1, r+1) \leq (i, r+i+j) \leq (s_1, r+s_1+t_1) \leq (s, r+s+t).$$

It follows that $U_{i,r+i+j} \subseteq W_{s_1,r+s_1+t_1}$. We have that

$$(r+i-1, r+i+j, i'_k) \in W_{s_1,r+s_1+t_1}.$$

We obtain that

$$W_{s,r+s+t} \backslash \{(r+s-1, r+s+t, i'_k) \mid 1 \leq k \leq r\} \subseteq W_{s_1,r+s_1+t_1},$$

as desired. Thus, we obtain that

$$W_{s_1,r+s_1+t_1} = W_{s,r+s+t} \backslash \{(r+s-1, r+s+t, i'_k) \mid 1 \leq k \leq r\}.$$

This proves the result. ∎

We set

$$\hat{c}_{s,t} = (\bar{c}_s, \bar{c}_{s+1}, \ldots, \bar{c}_{r+s-1}, \bar{c}_{r+s+t}).$$

It follows from (6) that

$$p_{i'_1 \ldots i'_r}(\hat{c}_{s,t}) \neq 0. \tag{10}$$

For any $1 \leq s < r+s \leq n$ and $s \leq r-1$, we set

$$f_{s,r} = \sum_{(i_1,\ldots,i_{r-s}) \in T_m^{r-s}} p_{i_1 \ldots i_{r-s} i'_{r-s+1} \ldots i'_r}(\hat{c}_{s,t}) a_{s,s+1}^{(i_1)} \ldots a_{r-1,r}^{(i_{r-s})}.$$

We set

$$V_{s,r} = \{(i, i+1, k) \mid i = s, \ldots, r-1, \quad k = 1, \ldots, m\},$$

where $1 \leq s < r+s \leq n$ and $s \leq r-1$. It is clear that $f_{s,r}$ is a polynomial on commutative variables indexed by elements from $V_{s,r}$.

For any $1 \leq s < r+s \leq n$ and $s \geq r$, we set

$$f_{s,r} = p_{i'_1 \ldots i'_r}(\hat{c}_{s,t}).$$

We claim that $f_{s,r}(K) \neq \{0\}$ for all $1 \leq s < r+s \leq n$. In view of (10), it suffices to prove that $f_{s,r}(K) \neq 0$, where $1 \leq s < r+s \leq n$ and $s \leq r-1$.

We take $a_{i,i+1}^{(k)} \in K$, $(i, i+1, k) \in V_{s,r}$ such that

$$\begin{cases} a_{s+i,s+i+1}^{(i'_{i+1})} = 1, & i = 0, \ldots, r-s-1; \\ a_{i,i+1}^{(k)} = 0, & \text{otherwise.} \end{cases}$$

It follows from (10) that

$$f_{s,r}(a_{i,i+1}^{(k)}) = p_{i'_1 \ldots i'_r}(\hat{c}_{s,t}) \neq 0,$$

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as desired. In view of Lemma 2.5, we get that there exist $a_{i,i+1}^{(k)} \in K$, $(i, i+1, k) \in \bigcup_{s=1}^{\min\{n-r,r-1\}} V_{s,r}$ such that

$$f_{s,r}(a_{i,i+1}^{(k)}) \neq 0$$

for all $1 \leq s < r+s \leq n$ and $s \leq r-1$.

For any $2 \leq s \leq r+s \leq n$, we define

$$f_{s,r+s-i} = \sum_{(i_1,\dots,i_{r-i}) \in T_m^{r-i}} p_{i_1 \dots i_{r-i} i'_{r-i+1} \dots i'_r}(\hat{c}_{s,t}) a_{s,s+1}^{(i_1)} \dots a_{r+s-i-1,r+s-i}^{(i_{r-i})} \tag{11}$$

for all $1 \leq i \leq \min\{s-1, r-1\}$. It is clear that $f_{s,r+s-i}$ is a polynomial over K on commutative variables indexed by elements from $\overline{W}_{s-i,r+s-i}$, where $1 \leq i \leq \min\{s-1, r-1\}$.

The following result implies that $f_{s,r+s-i}$, where $1 \leq i \leq \min\{s-1, r-1\}$, is a recursive polynomial.

Lemma 3.5 *For any $2 \leq s < r+s \leq n$, we claim that*

$$f_{s,r+s-i} = f_{s,r+s-i-1} x_{r+s-i-1,r+s-i}^{(i'_{r-i})} + \sum_{\substack{1 \leq k \leq r \\ i'_k \neq i'_{r-i}}} \alpha_{s,r+s-i-1,k} x_{r+s-i-1,r+s-i}^{(i'_k)}$$

for all $1 \leq i \leq \min\{s-1, r-1\}$, where both $f_{s,r+s-i-1}$ and $\alpha_{s,r+s-i-1,k}$ are polynomials over K on commutative variables indexed by elements from $\overline{W}_{s-i-1,r+s-i-1}$.

Proof We get from (11) that

(12)

$$f_{s,r+s-i} = \left(\sum_{(i_1,\dots,i_{r-i-1}) \in T_m^{r-i-1}} p_{i_1 \dots i_{r-i-1} i'_{r-i} \dots i'_r}(\hat{c}_{s,t}) a_{s,s+1}^{(i_1)} \dots a_{r+s-i-2,r+s-i-1}^{(i_{r-i-1})} \right) x_{r+s-i-1,r+s-i}^{(i'_{r-i})}$$
$$+ \sum_{\substack{1 \leq k \leq r \\ i'_k \neq i'_{r-i}}} \left(\sum_{(i_1,\dots,i_{r-i-1}) \in T_m^{r-i-1}} p_{i_1 \dots i_{r-i-1} i'_k i'_{r-i+1} \dots i'_r}(\hat{c}_{s,t}) a_{s,s+1}^{(i_1)} \dots a_{r+s-i-2,r+s-i-1}^{(i_{r-i-1})} \right) x_{r+s-i-1,r+s-i}^{(i'_k)}$$

for all $1 \leq i \leq \min\{s-1, r-1\}$. It follows from (11) that

$$f_{s,r+s-i-1} = \sum_{(i_1,\dots,i_{r-i-1}) \in T_m^{r-i-1}} p_{i_1 \dots i_{r-i-1} i'_{r-i} \dots i'_r}(\hat{c}_{s,t}) a_{s,s+1}^{(i_1)} \dots a_{r+s-i-2,r+s-i-1}^{(i_{r-i-1})}.$$

We set

$$\alpha_{s,r+s-i-1,k} = \sum_{(i_1,\dots,i_{r-i-1}) \in T_m^{r-i-1}} p_{i_1 \dots i_{r-i-1} i'_k i'_{r-i+1} \dots i'_r}(\hat{c}_{s,t}) a_{s,s+1}^{(i_1)} \dots a_{r+s-i-2,r+s-i-1}^{(i_{r-i-1})}$$

for all $1 \leq i \leq \min\{s-1, r-1\}$ and $k = 1, \dots, r$. It follows from both (11) and (12) that

$$f_{s,r+s-i} = f_{s,r+s-i-1} x_{r+s-i-1,r+s-i}^{(i'_{r-i})} + \sum_{\substack{1 \leq k \leq r \\ i'_k \neq i'_{r-i}}} \alpha_{s,r+s-i-1,k} x_{r+s-i-1,r+s-i}^{(i'_k)}$$

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for all $1 \le i \le \min\{s-1, r-1\}$. It is clear that both $f_{s,r+s-i-1}$ and $\alpha_{s,r+s-i-1,k}$ are polynomials over K on commutative variables indexed by elements from

$$\overline{W}_{s-i,r+s-i} \backslash \{(r+s-i-1, r+s-i, i'_k) \mid k = 1, \dots r\}.$$

In view of Lemma 3.1, we note that

$$\overline{W}_{s-i-1,r+s-i-1} = \overline{W}_{s-i,r+s-i} \backslash \{(r+s-i-1, r+s-i, i'_k) \mid k = 1, \dots r\}.$$

We have that both $f_{s,r+s-i-1}$ and $\alpha_{s,r+s-i-1,k}$ are polynomials over K on commutative variables indexed by elements from $\overline{W}_{s-i-1,r+s-i-1}$. This proves the result. ∎

Lemma 3.6 *For any $1 \le s < r+s \le n$, we have that*

$$p_{s,r+s+t} = f_{s,r+s-1} x^{(i'_r)}_{r+s-1,r+s+t} + \sum_{\substack{1 \le k \le r \\ i'_k \ne i'_r}} \beta_{s,r+s-1,k} x^{(i'_k)}_{r+s-1,r+s+t} + \beta_{s,r+s+t},$$

where $f_{1,r} \in K^*$, $\beta_{1,r,k} \in K$, $k = 1, \dots, r$ with $i'_k \ne i'_r$, $f_{s,r+s-1}, \beta_{s,r+s-1,k}$, $s \ge 2$, $1 \le k \le r$ with $i'_k \ne i'_r$ are polynomials on some commutative variables in $\overline{W}_{s_1,r+s_1+t_1}$ and $\beta_{s,r+s+t}$, where $t > 0$, is a polynomial over K in some commutative variables in $W_{s_1,r+s_1+t_1}$, where

$$(s_1, r+s_1+t_1) = \max\{(i, r+i+j) \mid (1, r+1) \le (i, r+i+j) < (s, r+s+t)\}.$$

Moreover, $\beta_{s,r+s} = 0$.

Proof It follows from (5) that

$$\begin{aligned} p_{s,r+s+t} &= \left(\sum_{(i_1,\dots,i_{r-1}) \in T^{r-1}_m} p_{i_1 \dots i_{r-1} i'_r}(\hat{c}_{s,t}) a^{(i_1)}_{s,s+1} \cdots a^{(i_{r-1})}_{r+s-2,r+s-1} \right) x^{(i'_r)}_{r+s-1,r+s+t} \\ &+ \sum_{\substack{1 \le k \le r \\ i'_k \ne i'_r}} \left(\sum_{(i_1,\dots,i_{r-1}) \in T^{r-1}_m} p_{i_1 \dots i_{r-1} i'_k}(\hat{c}_{s,t}) a^{(i_1)}_{s,s+1} \cdots a^{(i_{r-1})}_{r+s-2,r+s-1} \right) x^{(i'_k)}_{r+s-1,r+s+t} \\ &+ \sum_{k=r}^{r+t} \left(\sum_{\substack{s = j_1 < \cdots < j_{k+1} = r+s+t \\ (j_k, j_{k+1}) \ne (r+s-1, r+s+t) \\ (i_1,\dots,i_k) \in T^k_m}} p_{i_1 \dots i_k}(\bar{c}_{j_1}, \dots, \bar{c}_{j_{k+1}}) a^{(i_1)}_{j_1 j_2} \cdots a^{(i_k)}_{j_k j_{k+1}} \right). \end{aligned} \tag{13}$$

It follows from (11) that

$$f_{s,r+s-1} = \sum_{(i_1,\dots,i_{r-1}) \in T^{r-1}_m} p_{i_1 \dots i_{r-1} i'_r}(\hat{c}_{s,t}) a^{(i_1)}_{s,s+1} \cdots a^{(i_{r-1})}_{r+s-2,r+s-1}.$$

We set

$$\beta_{s,r+s-1,k} = \sum_{(i_1,\dots,i_{r-1}) \in T^{r-1}_m} p_{i_1 \dots i_{r-1} i'_k}(\hat{c}_{s,t}) a^{(i_1)}_{s,s+1} \cdots a^{(i_{r-1})}_{r+s-2,r+s-1}$$

for $k=1,\dots,r$ with $i'_k \neq i'_r$, and

$$\beta_{s,r+s+t} = \sum_{k=r}^{r+t}\left(\sum_{\substack{s=j_1<\cdots<j_{k+1}=r+s+t\\(j_k,j_{k+1})\neq(r+s-1,r+s+t)\\(i_1,\dots,i_k)\in T_m^k}} p_{i_1\dots i_k}(\bar{c}_{j_1},\dots,\bar{c}_{j_{k+1}})a_{j_1j_2}^{(i_1)}\dots a_{j_kj_{k+1}}^{(i_k)}\right).$$

It follows from (13) that

$$p_{s,r+s+t} = f_{s,r+s-1}x_{r+s-1,r+s+t}^{(i'_r)} + \sum_{\substack{1\leq k\leq r\\ i'_k\neq i'_r}} \beta_{s,r+s-1,k}x_{r+s-1,r+s+t}^{(i'_k)} + \beta_{s,r+s+t}, \tag{14}$$

where $f_{1,r}\in K^*, \beta_{1,r,k}\in K$, $k=1,\dots,r$ with $i'_k\neq i'_r$, $f_{s,r+s-1}, \beta_{s,r+s+t,k}$, where $s\geq 2$, $1\leq k\leq r$ with $i'_k\neq i'_r$, are polynomials on some commutative variables indexed by elements from

$$\overline{W}_{s,r+s+t}\setminus\{(r+s-1,r+s+t,i'_k),\quad k=1,\dots,r\} \tag{15}$$

and $\beta_{s,r+s+t}$, where $t>0$, is a polynomial over K in some commutative variables indexed by elements from

$$W_{s,r+s+t}\setminus\{(r+s-1,r+s+t,i'_k),\quad k=1,\dots,r\}. \tag{16}$$

Suppose first that $t=0$. In view of Lemma 3.1, we note that

$$\overline{W}_{s-1,r+s-1} = \overline{W}_{s,r+s+t}\setminus\{(r+s-1,r+s,i'_k),\quad k=1,\dots,r\}.$$

We get from (15) that $f_{s,r+s-1}, \beta_{s,r+s+t,k}$, where $s\geq 2, 1\leq k\leq r$ with $i'_k\neq i'_r$, are polynomials on some commutative variables indexed by elements from $\overline{W}_{s-1,r+s-1}$. It is clear that $\beta_{s,r+s}=0$. Suppose next that $t>0$. In view of Lemma 3.2, we note that

$$\overline{W}_{s_1,r+s_1+t_1} = \overline{W}_{s,r+s+t}.$$

We get from (15) that $f_{s,r+s-1}, \beta_{s,r+s+t,k}$, where $s\geq 2, 1\leq k\leq r$ with $i'_k\neq i'_r$, are polynomials on some commutative variables indexed by elements from $\overline{W}_{s_1,r+s_1+t_1}$. In view of Lemma 3.4, we note that

$$W_{s_1,r+s_1+t_1} = W_{s,r+s+t}\setminus\{(r+s-1,r+s+t,i'_k),\quad k=1,\dots,r\}.$$

We get from (16) that $\beta_{s,r+s+t}$ is a polynomial over K in some commutative variables indexed by elements from $W_{s_1,r+s_1+t_1}$. This proves the result. ∎

The following result is crucial for the proof of the main result.

Lemma 3.7 *Let $p(x_1,\dots,x_m)$ be a polynomial with zero constant term in noncommutative variables over an infinite field K. Suppose ord$(p)=r$, where $1<r<n-1$. For any $A'=(a'_{s,r+s+t})\in T_n(K)^{(r-1)}$, where $a'_{s,r+s}\neq 0$ for all $1\leq s<r+s+t\leq n$, we have that $A'\in p(T_n(K))$.*

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Proof Take any $A' = (a'_{s,r+s+t}) \in T_n(K)^{(r-1)}$, where $a'_{s,r+s} \neq 0$ for all $1 \leq s < r+s \leq n$. For any $1 \leq s < r+s+t \leq n$, we claim that there exist $c^{(i'_k)}_{r+u-1,r+u+w} \in K$ with

$$(r+u-1, r+u+w, k) \in W_{s,r+s+t}$$

such that

$$p_{i,r+i+j}(c^{(i'_k)}_{r+u-1,r+u+w}) = a_{i,r+i+j}$$

for all $(1, r+1) \leq (i, r+i+j) \leq (s, r+s+t)$ and

$$f_{s',r+s'-v}(c^{(i'_k)}_{r+u-1,r+u}) \neq 0$$

for all $f_{s',r+s'-v}$ on commutative variables in $\overline{W}_{s,r+s+t}$, where $s' \geq 2$ and $1 \leq v \leq \min\{s'-1, r-1\}$.

We prove the claim by induction on $(s, r+s+t)$. Suppose first that $(s, r+s+t) = (1, r+1)$. Note that

$$W_{1,r+1} = \overline{W}_{1,r+1} = \{(r, r+1, i'_k) \mid k = 1, \ldots, r\}.$$

In view of Lemma 3.6, we get that

$$p_{1,r+1} = f_{1,r} x^{(i'_r)}_{r,r+1} + \sum_{\substack{1 \leq k \leq r \\ i_k \neq i'_r}} \beta_{1,r,k} x^{(i'_k)}_{r,r+1}, \tag{17}$$

where $f_{1,r} \in K^*$, $\beta_{1,r,k} \in K$, $k = 1, \ldots, r$ with $i'_k \neq i'_r$.

Take any $f_{s',r+s'-v}$ on $x^{(i'_k)}_{r,r+1}$, where $k = 1, \ldots, r$, $s' \geq 2$, and $1 \leq v \leq \min\{s'-1, r-1\}$, we get from Lemma 3.5 that

$$r + s' - v - 1 = r$$

and so $v = s' - 1$. It follows that

$$f_{s',r+s'-v} = f_{s',r} x^{(i'_{r-v})}_{r,r+1} + \sum_{\substack{1 \leq k \leq r \\ i'_k \neq i'_{r-v}}} \alpha_{s',r,k} x^{(i'_k)}_{r,r+1}. \tag{18}$$

Note that $f_{s',r} \in K^*$ and $\alpha_{s',r,k} \in K$, $k = 1, \ldots, r$ with $i'_k \neq i_{r-v}$. Note that $a'_{1,r+1} \in K^*$. In view of Lemma 2.6, we get from both (17) and (18) that there exist $c^{(i'_k)}_{r,r+1} \in K$, $k = 1, \ldots, r$, such that

$$\begin{cases} p_{1,r+1}(c^{(i'_k)}_{r,r+1}) = a'_{1,r+1}, \\ f_{s',r+s'-v}(c^{(i'_k)}_{r,r+1}) \neq 0, \end{cases}$$

where $2 \leq s' \leq r$ and $v = s' - 1$, as desired.

Suppose next that $(s, r+s+t) \neq (1, r+1)$. We rewrite (7) as follows:

$$(1, r+1) < \cdots < (s_1, r+s_1+t_1) < (s, r+s+t) < \cdots < (1, n),$$

where

$$(s_1, r+s_1+t_1) = \max\{(i, r+i+j) \mid (1, r+1) \leq (i, r+i+j) < (s, r+s+t)\}.$$

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By induction on $(s_1, r+s_1+t_1)$, we have that there exist $c^{(i'_k)}_{r+u-1,r+u+w} \in K$ with

$$(r+u-1, r+u+w, k) \in W_{s_1,r+s_1+t_1}$$

such that

$$p_{i,r+i+j}(c^{(i'_k)}_{r+u-1,r+u+w}) = a'_{i,r+i+j}$$

for all $(1, r+1) \leq (i, r+i+j) \leq (s_1, r+s_1+t_1)$ and

$$f_{s',r+s'-v}(c^{(i'_k)}_{r+u-1,r+u}) \neq 0$$

for any $f_{s',r+s'-v}$ with commutative variables in $\overline{W}_{s_1,r+s_1+t_1}$, where $s' \geq 2$, and $1 \leq v \leq \min\{s'-1, r-1\}$. We now divide the proof into the following two cases.

Suppose first that $t = 0$. Note that

$$(s_1, r+s_1+t_1) = (s-1, r+s-1).$$

That is, $s_1 = s-1$ and $t_1 = 0$. In view of Lemma 3.6, we get that

$$p_{s,r+s} = f_{s,r+s-1}x^{(i'_r)}_{r+s-1,r+s} + \sum_{\substack{1\leq k\leq r \\ i'_k \neq i'_r}} \beta_{s,r+s-1,k}x^{(i'_k)}_{r+s-1,r+s}, \tag{19}$$

where $f_{s,r+s-1}, \beta_{s,r+s-1,k}$, where $k = 1, \ldots, r$ with $i'_k \neq i'_r$, are polynomials in commutative variables in $\overline{W}_{s_1,r+s_1}$. By induction hypothesis, we get that $f_{s,r+s-1} \in K^*$ and $\beta_{s,r+s-1,k} \in K$.

Take any $f_{s',r+s'-v}$ on commutative variables indexed by elements from $\overline{W}_{s,r+s}$, where $s' \geq 2$ and $1 \leq v \leq \min\{s'-1, r-1\}$. Suppose first that $f_{s',r+s'-v}$ is a polynomial on commutative variables indexed by elements from $\overline{W}_{s_1,r+s_1}$. By induction hypothesis we have that $f_{s',r+s'-v} \in K^*$. Suppose next that $f_{s',r+s'-v}$ is not a polynomial on commutative variables indexed by elements from $\overline{W}_{s_1,r+s_1}$. In view of Lemma 3.1, we note that

$$\overline{W}_{s,r+s} \backslash \overline{W}_{s-1,r+s-1} = \{(r+s-1, r+s, i'_k) \mid k = 1, \ldots, r\}.$$

This implies that $x^{(i'_k)}_{r+s-1,r+s}$ appears in $f_{s',r+s'-v}$ for $k = 1, \ldots, r$. In view of Lemma 3.5 we get that

$$(r+s'-v-1, r+s'-v) = (r+s-1, r+s)$$

and so $v = s'-s$. We get that

$$f_{s',r+s'-v} = f_{s',r+s'-v-1}x^{(i'_{r-v})}_{r+s-1,r+s} + \sum_{\substack{1\leq k\leq r \\ i'_k \neq i'_{r-v}}} \alpha_{s',r+s'-v-1,k}x^{(i'_k)}_{r+s-1,r+s}, \tag{20}$$

where $f_{s',r+s'-v-1}$ and $\alpha_{s',r+s'-v-1,k}$, $k = 1, \ldots, r$ with $i'_k \neq i'_{r-v}$, are polynomials over K on commutative variables indexed by elements from $\overline{W}_{s_1,r+s_1}$. By induction hypothesis, we have that $f_{s',r+s'-v-1} \in K^*$ and $\alpha_{s',r+s'-v-1,k} \in K$, where $k = 1, \ldots, r$ with $i'_k \neq i'_{r-v}$.

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Note that $a'_{s,r+s} \in K^*$. In view of Lemma 2.6, we get from both (19) and (20) that there exist $c^{(i'_k)}_{r+s-1,r+s} \in K$, $k = 1, \dots, r$, such that

$$\begin{cases} p_{s,r+s}(c^{(i'_k)}_{r+s-1,r+s}) = a'_{s,r+s}; \\ f_{s',r+s'-v}(c^{(i'_k)}_{r+s-1,r+s}) \neq 0, \end{cases}$$

as desired.

Suppose next that $t > 0$. It follows from Lemma 3.6 that

$$p_{s,r+s+t} = f_{s,r+s-1}x^{(i'_r)}_{r+s-1,r+s+t} + \sum_{\substack{1\le k\le r\\ i'_k\neq i'_r}} \beta_{s,r+s-1,k}x^{(i'_k)}_{r+s-1,r+s+t} + \beta_{s,r+s+t}, \tag{21}$$

where $f_{s,r+s-1}, \beta_{s,r+s-1,k}$, where $k = 1, \dots, r$ with $i'_k \neq i'_r$, are polynomials over K in commutative variables indexed by elements from $\overline{W}_{r+s_1+t_1}$, and $\beta_{s,r+s+t}$ is a polynomial over K in commutative variables indexed by elements from $W_{s_1,r+s_1+t_1}$. By induction hypothesis, we have that $f_{s,r+s-1} \in K^*$, $\beta_{s,r+s-1,k} \in K$ for all $k = 1, \dots, r$ with $i'_k \neq i'_r$, and $\beta_{s,r+s+t} \in K$.

Take $c^{(i'_k)}_{r+s-1,r+s+t} \in K$, where $k = 1, \dots, r$ in (21) such that

$$\begin{cases} c^{(i'_r)}_{r+s-1,r+s+t} = f^{-1}_{s,r+s-1}(a'_{s,r+s+t} - \beta_{s,r+s+t}); \\ c^{(i'_k)}_{r+s-1,r+s+t} = 0, \quad \text{for all } 1 \le k \le r \text{ with } i'_k \neq i'_r. \end{cases}$$

We get that

$$p_{s,r+s+t}(c^{(i'_k)}_{r+s-1,r+s+t}) = a'_{s,r+s+t}.$$

Take any $f_{s',r+s'-v}$ on commutative variables indexed by elements from $\overline{W}_{s,r+s+t}$, where $s' \ge 2$ and $1 \le v \le \min\{s'-1, r-1\}$. In view of Lemma 3.2, we note that

$$\overline{W}_{s,r+s+t} = \overline{W}_{s_1,r+s_1+t_1}.$$

This implies that $f_{s',r+s'-v}$ is a commutative polynomial over K on some commutative variables indexed by elements from $\overline{W}_{s_1,r+s_1+t_1}$. By induction hypothesis, we get that

$$f_{s',r+s'-v} \in K^*,$$

where $s' \ge 2$ and $1 \le v \le \min\{s'-1, r-1\}$, as desired. This proves the claim.

Let $(s, r+s+t) = (1, n)$. We have that there exist $c^{(i'_k)}_{r+u-1,r+u+w} \in K$, $k = 1, \dots, r$, with

$$(r+u-1, r+u+w, k) \in W_{1,n},$$

such that

$$p_{i,r+i+j}(c^{(i'_k)}_{r+u-1,r+u+w}) = a'_{i,r+i+j} \tag{22}$$

for all $(1, r+1) \le (i, r+i+j) \le (1, n)$ and

$$f_{s',r+s'-v}(c^{(i'_k)}_{r+u-1,r+u}) \neq 0$$

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for all $f_{s',r+s'-v}$ on commutative variables indexed by elements from $\overline{W}_{1,n}$, where $s' \geq 2$ and $1 \leq v \leq \min\{s'-1, r-1\}$. It follows from both (5) and (22) that

$$p(u_1,\dots,u_m) = (p_{s,r+s+t}) = (a'_{s,r+s+t}) = A'.$$

This implies that $A' \in p(T_n(K))$. The proof of the result is complete. ∎

Lemma 3.8 *Let $n \geq 4$ and $m \geq 1$ be integers. Let $p(x_1,\dots,x_m)$ be a polynomial with zero constant term in noncommutative variables over an infinite field K. Suppose that $ord(p) = n-2$. We have that $p(T_n(K)) = T_n(K)^{(n-3)}$.*

Proof In view of Lemma 2.2(ii), we note that $p(T_n(K)) \subseteq T_n(K)^{(n-3)}$. It suffices to prove that $T_n(K)^{(n-3)} \subseteq p(T_n(K))$.

For any $u_i = (a_{jk}^{(i)}) \in T_n(K)$, $i = 1,\dots,m$, in view of Lemma 2.2(ii), we get from (2) that

$$p(u_1,\dots,u_m) = \begin{pmatrix} 0 & 0 & \dots & p_{1,n-1} & p_{1n} \\ 0 & 0 & \dots & 0 & p_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \tag{23}$$

where

$$\begin{cases} p_{1,n-1} = \displaystyle\sum_{(i_1,\dots,i_{n-2})\in T_m^{n-2}} p_{i_1\dots i_{n-2}}(\bar{a}_{11},\dots,\bar{a}_{n-1,n-1})a_{12}^{(i_1)}\dots a_{n-2,n-1}^{(i_{n-2})}; \\ p_{2n} = \displaystyle\sum_{(i_1,\dots,i_{n-2})\in T_m^{n-2}} p_{i_1\dots i_{n-2}}(\bar{a}_{22},\dots,\bar{a}_{n,n})a_{23}^{(i_1)}\dots a_{n-1,n}^{(i_{n-2})}; \\ p_{1n} = \displaystyle\sum_{(i_1,\dots,i_{n-1})\in T_m^{n-1}} p_{i_1\dots i_{n-1}}(\bar{a}_{11},\dots,\bar{a}_{nn})a_{12}^{(i_1)}\dots a_{n-1,n}^{(i_{n-1})} \\ \qquad + \displaystyle\sum_{\substack{1=j_1<\dots<j_{n-1}=n \\ (i_1,\dots,i_{n-2})\in T_m^{n-2}}} p_{i_1\dots i_{n-2}}(\bar{a}_{j_1j_1},\dots,\bar{a}_{j_{n-1}j_{n-1}})a_{j_1j_2}^{(i_1)}\dots a_{j_{n-2}j_{n-1}}^{(i_{n-2})}. \end{cases}$$

In view of Lemma 2.2(iii), we have that

$$p_{i'_1,\dots,i'_{n-2}}(K) \neq \{0\},$$

for some $i'_1,\dots,i'_{n-2} \in \{1,\dots,m\}$. It follows from Lemma 2.4 that there exist $\bar{b}_1,\dots,\bar{b}_n \in K^m$ such that

$$p_{i'_1,\dots,i'_{n-2}}(\bar{b}_{j_1},\dots,\bar{b}_{j_{n-1}}) \neq 0$$

for all $1 \leq j_1 < \dots < j_{n-1} \leq n$.

For any $A' = (a'_{s,n-2+s+t}) \in T_n(K)^{(n-3)}$, where $1 \leq s < n-2+s+t \leq n$, we claim that there exist $u_i = (a_{jk}^{(i)}) \in T_n(K)$, $i = 1,\dots,m$, such that

$$p(u_1,\dots,u_m) = (p_{s,n-2+s+t}) = A'.$$

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That is,

$$\begin{cases} p_{1,n-1} = a'_{1,n-1}; \\ p_{2n} = a'_{2n}; \\ p_{1n} = a'_{1n}. \end{cases}$$

We prove the claim by the following two cases:

Case 1. Suppose that $a'_{1,n-1} \neq 0$. We take

$$\begin{cases} \bar{a}_{jj} = \bar{b}_j, \quad \text{for all } j = 1, \dots, n; \\ a_{12}^{(i'_1)} = x_{12}^{(i'_1)}; \\ a_{12}^{(k)} = 0 \quad \text{for all } k = 1, \dots, m \text{ with } k \neq i'_1; \\ a_{n-1,n}^{(i'_{n-2})} = x_{n-1,n}^{(i'_{n-2})}; \\ a_{n-1,n}^{(k)} = 0 \quad \text{for all } k = 1, \dots, m \text{ with } k \neq i'_{n-2}; \\ a_{n-2,n}^{(i'_{n-2})} = x_{n-2,n}^{(i'_{n-2})}; \\ a_{j,j+2}^{(i)} = 0 \quad \text{for all } 1 \leq i \leq m, 3 \leq j+2 \leq n \text{ with } (j, j+2, i) \neq (n-2, n, i'_{n-2}). \end{cases}$$

It follows from (23) that

(24)

$$\begin{cases} p_{1,n-1} = \left(\sum_{(i_2,\dots,i_{n-2}) \in T_m^{n-3}} p_{i'_1 i_2 \dots i_{n-2}}(\bar{b}_1, \dots, \bar{b}_{n-1}) a_{23}^{(i_2)} \dots a_{n-2,n-1}^{(i_{n-2})} \right) x_{12}^{(i'_1)}; \\ p_{2n} = \left(\sum_{(i_1,\dots,i_{n-3}) \in T_m^{n-3}} p_{i_1 \dots i_{n-3} i'_{n-2}}(\bar{b}_2, \dots, \bar{b}_n) a_{23}^{(i_1)} \dots a_{n-2,n-1}^{(i_{n-3})} \right) x_{n-1,n}^{(i'_{n-2})}; \\ p_{1n} = \left(\sum_{(i_2,\dots,i_{n-2}) \in T_m^{n-3}} p_{i'_1 i_2 \dots i_{n-2} i'_{n-2}}(\bar{b}_1, \dots, \bar{b}_n) a_{23}^{(i_2)} \dots a_{n-2,n-1}^{(i_{n-2})} \right) x_{12}^{(i'_1)} x_{n-1,n}^{(i'_{n-2})} \\ \qquad \left(\sum_{(i_2,\dots,i_{n-3}) \in T_m^{n-4}} p_{i'_1 i_2 \dots i_{n-3} i'_{n-2}}(\bar{b}_1, \dots, \bar{b}_{n-2}, \bar{b}_n) a_{23}^{(i_2)} \dots a_{n-3,n-2}^{(i_{n-3})} \right) x_{12}^{(i'_1)} x_{n-2,n}^{(i'_{n-2})}. \end{cases}$$

We set

$$(25) \qquad \begin{cases} f_{1,n-1} = \sum_{(i_2,\dots,i_{n-2}) \in T_m^{n-3}} p_{i'_1 i_2 \dots i_{n-2}}(\bar{b}_1, \dots, \bar{b}_{n-1}) a_{23}^{(i_2)} \dots a_{n-2,n-1}^{(i_{n-2})}, \\ f_{2n} = \sum_{(i_1,\dots,i_{n-3}) \in T_m^{n-3}} p_{i_1 \dots i_{n-3} i'_{n-2}}(\bar{b}_2, \dots, \bar{b}_n) a_{23}^{(i_1)} \dots a_{n-2,n-1}^{(i_{n-3})}, \\ f_{1n} = \sum_{(i_2,\dots,i_{n-3}) \in T_m^{n-4}} p_{i'_1 i_2 \dots i_{n-3} i'_{n-2}}(\bar{b}_1, \dots, \bar{b}_{n-2}, \bar{b}_n) a_{23}^{(i_2)} \dots a_{n-3,n-2}^{(i_{n-3})}, \end{cases}$$

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and

$$\begin{aligned} V_{1,n-1} &= \{(i, i+1, k) \mid i = 2, \dots, n-2, k = 1, \dots, m\}; \\ V_{2n} &= V_{1,n-1}; \\ V_{1n} &= \{(i, i+1, k) \mid i = 2, \dots, n-3, k = 1, \dots, m\}. \end{aligned}$$

Note that $f_{1,n-1}, f_{2n}, f_{1n}$ are polynomials over K on commutative variables indexed by elements from $V_{1,n-1}, V_{2n}, V_{1n}$, respectively.

We claim that $f_{1,n-1}, f_{2n}, f_{1n} \neq 0$. Indeed, we take $a_{jk}^{(i)} \in K$, $(j, k, i) \in V_{1,n-1}$ such that

$$\begin{cases} a_{s,s+1}^{(i'_s)} = 1, & \text{for all } s = 2, \dots, n-2; \\ a_{jk}^{(i)} = 0, & \text{otherwise.} \end{cases}$$

It follows from (25) that

$$f_{1,n-1}(a_{jk}^{(i)}) = p_{i'_1 \dots i'_{n-2}}(\bar{b}_1, \dots, \bar{b}_{n-1}) \neq 0,$$

as desired. Next, we take $a_{jk}^{(i)} \in K$, $(j, k, i) \in V_{2n}$ such that

$$\begin{cases} a_{s,s+1}^{(i'_{s-1})} = 1, & \text{for all } s = 2, \dots, n-2; \\ a_{jk}^{(i)} = 0, & \text{otherwise.} \end{cases}$$

It follows from (25) that

$$f_{2n}(a_{jk}^{(i)}) = p_{i'_1 \dots i'_{n-2}}(\bar{b}_2, \dots, \bar{b}_n) \neq 0,$$

as desired. Finally, we take $a_{jk}^{(i)} \in K$, $(j, k, i) \in V_{1n}$ such that

$$\begin{cases} a_{s,s+1}^{(i'_s)} = 1, & \text{for all } s = 2, \dots, n-3; \\ a_{jk}^{(i)} = 0, & \text{otherwise.} \end{cases}$$

It follows from (25) that

$$f_{1n}(a_{jk}^{(i)}) = p_{i'_1 \dots i'_{n-2}}(\bar{b}_1, \dots, \bar{b}_{n-2}, \bar{b}_n) \neq 0,$$

as desired. In view of Lemma 2.5, we get that there exist $a_{jk}^{(i)} \in K$, where $(j, k, i) \in V_{1,n-1} \cup V_{2n} \cup V_{1n}$ such that

$$\begin{cases} f_{1,n-1}(a_{jk}^{(i)}) \neq 0; \\ f_{2n}(a_{jk}^{(i)}) \neq 0; \\ f_{1n}(a_{jk}^{(i)}) \neq 0. \end{cases}$$

We set

$$\alpha = \sum_{(i_2, \dots, i_{n-2}) \in T_{n-3}} p_{i'_1 i_2 \dots i_{n-2} i'_{n-2}}(\bar{b}_1, \dots, \bar{b}_n) a_{23}^{(i_2)} \dots a_{n-2,n-1}^{(i_{n-2})}.$$

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It follows from (24) that

$$(26)\qquad \begin{cases} p_{1,n-1} = f_{1,n-1} x_{12}^{(i_1')}; \\ p_{2n} = f_{2n} x_{n-1,n}^{(i_{n-2}')}; \\ p_{1n} = f_{1n} x_{12}^{(i_1')} x_{n-2,n}^{(i_{n-2}')} + \alpha x_{12}^{(i_1')} x_{n-1,n}^{(i_{n-2}')}. \end{cases}$$

We take

$$\begin{cases} x_{12}^{(i_1')} = f_{1,n-1}^{-1} a_{1,n-1}'; \\ x_{n-1,n}^{(i_{n-2}')} = f_{2n}^{-1} a_{2n}'; \\ x_{n-2,n}^{(i_{n-2}')} = f_{1n}^{-1} f_{1,n-1} (a_{1,n-1}')^{-1} \left(a_{1n}' - \alpha f_{1,n-1}^{-1} a_{1,n-1}' f_{2n}^{-1} a_{2n}' \right). \end{cases}$$

It follows from (26) that

$$\begin{cases} p_{1,n-1} = a_{1,n-1}'; \\ p_{2n} = a_{2n}'; \\ p_{1n} = a_{1n}', \end{cases}$$

as desired.

Case 2. Suppose that $a_{1,n-1}' = 0$. We take

$$\begin{cases} \bar{a}_{jj} = \bar{b}_j, & \text{for all } j = 1, \dots, n; \\ a_{12}^{(k)} = 0, & \text{for all } k = 1, \dots, m; \\ a_{23}^{(i_1')} = x_{23}^{(i_1')}; & \\ a_{23}^{(k)} = 0, & \text{for all } k = 1, \dots, m \text{ with } k \neq i_1'; \\ a_{13}^{(i_1')} = x_{13}^{(i_1')}; & \\ a_{j,j+2}^{(k)} = 0, & \text{for all } 1 \leq j < j+2 \leq n \text{ with } (j, j+2, k) \neq (1, 3, i_1'). \end{cases}$$

It follows from (23) that

$$(27)\qquad \begin{cases} p_{1,n-1} = 0; \\ p_{2n} = \left(\sum_{(i_2,\dots,i_{n-2}) \in T_m^{n-3}} p_{i_1' i_2 \dots i_{n-2}} (\bar{b}_2, \dots, \bar{b}_n) a_{34}^{(i_2)} \dots a_{n-1,n}^{(i_{n-2})} \right) x_{23}^{(i_1')}; \\ p_{1n} = \left(\sum_{(i_2,\dots,i_{n-2}) \in T_m^{n-3}} p_{i_1' i_2 \dots i_{n-2}} (\bar{b}_1, \bar{b}_3, \dots, \bar{b}_n) a_{34}^{(i_2)} \dots a_{n-1,n}^{(i_{n-2})} \right) x_{13}^{(i_1')}. \end{cases}$$

We set

$$(28)\qquad \begin{cases} g_{2n} = \sum_{(i_2,\dots,i_{n-2}) \in T_m^{n-3}} p_{i_1' i_2 \dots i_{n-2}} (\bar{b}_2, \dots, \bar{b}_n) a_{34}^{(i_2)} \dots a_{n-1,n}^{(i_{n-2})}; \\ g_{1n} = \sum_{(i_2,\dots,i_{n-2}) \in T_m^{n-3}} p_{i_1' i_2 \dots i_{n-2}} (\bar{b}_1, \bar{b}_3, \dots, \bar{b}_n) a_{34}^{(i_2)} \dots a_{n-1,n}^{(i_{n-2})}; \end{cases}$$

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and

$$V = \{(i, i+1, k) \mid i = 3, \ldots, n-1, k = 1, \ldots, m\}.$$

Note that both g_{2n} and g_{1n} are polynomials over K on some commutative variables indexed by elements from V. We claim that $g_{2n}, g_{1n} \neq 0$. Indeed, we take $a_{jk}^{(i)} \in K$, $(j, k, i) \in V$ such that

$$\begin{cases} a_{s,s+1}^{(i'_{s-1})} = 1, & \text{for all } s = 3, \ldots, n-1; \\ a_{jk}^{(i)} = 0, & \text{otherwise.} \end{cases}$$

It follows from (28) that

$$g_{2n} = p_{i'_1 \ldots i'_{n-2}}(\bar{b}_2, \ldots, \bar{b}_n) \neq 0;$$
$$g_{1n} = p_{i'_1 \ldots i'_{n-2}}(\bar{b}_1, \bar{b}_3, \ldots, \bar{b}_n) \neq 0,$$

as desired. It follows from (27) that

$$(29) \qquad \begin{cases} p_{1,n-1} = 0; \\ p_{2n} = g_{2n} x_{23}^{(i'_1)}; \\ p_{1n} = g_{1n} x_{13}^{(i'_1)}. \end{cases}$$

We take

$$\begin{cases} x_{23}^{(i'_1)} = g_{2n}^{-1} a'_{2n}; \\ x_{13}^{(i'_1)} = g_{1n}^{-1} a'_{1n}. \end{cases}$$

It follows from (29) that

$$\begin{cases} p_{1,n-1} = 0; \\ p_{2n} = a'_{2,n}; \\ p_{1n} = a'_{1n}, \end{cases}$$

as desired. We obtain that

$$p(u_1, \ldots, u_m) = (p_{s,n-2+s+t}) = (a'_{s,n-2+s+t}) = A'.$$

This implies that $T_n(K)^{(n-3)} \subseteq p(T_n(K))$. Hence $p(T_n(K)) = T_n(K)^{(n-3)}$. ■

We are ready to give the proof of the main result of the paper.

The proof of Theorem 1.2 For any $A = (a_{s,r+s+t}) \in T_n(K)^{(r-1)}$, we set

$$\begin{cases} f_{s,r+s}(x_{s,r+s}) = a_{s,r+s} - x_{s,r+s}; \\ g_{s,r+s}(x_{s,r+s}) = x_{s,r+s} \end{cases}$$

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for all $1 \le s < r+s \le n$. It is clear that both $f_{s,r+s}$ and $g_{s,r+s}$ are nonzero polynomials in commutative variables over K, where $1 \le s < r+s \le n$. It follows from Lemma 2.5 that there exist $b_{s,r+s} \in K$, $1 \le s < r+s \le n$, such that

$$\begin{cases} f_{s,r+s}(b_{s,r+s}) \neq 0; \\ g_{s,r+s}(b_{s,r+s}) \neq 0 \end{cases}$$

for all $1 \le s < r+s \le n$. That is,

$$\begin{cases} a_{s,r+s} - b_{s,r+s} \neq 0; \\ \qquad\qquad b_{s,r+s} \neq 0 \end{cases}$$

for all $1 \le s < r+s \le n$. We set

$$b_{s,r+s+t} = a_{s,r+s+t}$$

for all $1 \le s < r+s+t \le n$ and $t > 0$ and

$$\begin{cases} c_{s,r+s} = a_{s,r+s} - b_{s,r+s}, & \text{for all } 1 \le s < r+s \le n; \\ c_{s,r+s+t} = 0, & \text{for all } 1 \le s < r+s+t \le n \text{ and } t > 0. \end{cases}$$

We set

$$B = (b_{s,r+s+t}) \quad \text{and} \quad C = (c_{s,r+s+t}).$$

It is clear that

$$A = B + C,$$

where $B, C \in T_n(K)^{(r-1)}$ with $b_{s,r+s}, c_{s,r+s} \in K^*$ for all $1 \le s < r+s \le n$. In view of Lemma 3.7, we get that there exist $u_i, v_i \in T_n(K)$, $i = 1, \ldots, m$, such that

$$p(u_1, \ldots, u_m) = B \quad \text{and} \quad p(v_1, \ldots, v_m) = C.$$

It follows that

$$p(u_1, \ldots, u_m) + p(v_1, \ldots, v_m) = A.$$

This implies that

$$T_n(K)^{(r-1)} \subseteq p(T_n(K)) + p(T_n(K)).$$

In view of Lemma 2.2(ii), we note that $p(T_n(K)) \subseteq T_n(K)^{(r-1)}$. Since $T_n(K)^{(r-1)}$ is a subspace of $T_n(K)$, we get that

$$p(T_n(K)) + p(T_n(K)) \subseteq T_n(K)^{(r-1)}.$$

We obtain that

$$p(T_n(K)) + p(T_n(K)) = T_n(K)^{(r-1)}.$$

In particular, if $r = n-2$, we get from Lemma 3.8 that

$$p(T_n(K)) = T_n(K)^{(n-3)}.$$

The proof of the result is complete. ∎

We conclude the paper with following example.

Example 3.1 Let $n \geq 5$ and $1 < r < n-2$ be integers. Let K be an infinite field. Let

$$p(x,y) = [x,y]^r.$$

We have that $\operatorname{ord}(p) = r$ and $p(T_n(K)) \neq T_n(K)^{(r-1)}$.

Proof It is easy to check that $p(T_r(K)) = \{0\}$. Set

$$f(x,y) = [x,y].$$

Note that f is a multilinear polynomial over K. It is clear that $\operatorname{ord}(f) = 1$. In view of [10, Theorem 4.3] or [15, Theorem 1.1], we have that

$$f(T_{r+1}(K)) = T_{r+1}(K)^{(0)}.$$

It implies that there exist $A, B \in T_{r+1}(K)$ such that

$$[A,B] = e_{12} + e_{23} + \cdots + e_{r,r+1}.$$

We get that

$$p(A,B) = [A,B]^r = e_{1,r+1} \neq 0.$$

This implies that $p(T_{r+1}(K)) \neq \{0\}$. We obtain that $\operatorname{ord}(p) = r$.

Suppose on contrary that $p(T_n(K)) = T_n(K)^{(r-1)}$ for some $n \geq 5$ and $1 < r < n-2$. For $e_{1,r+1} + e_{3,r+3} \in T_n(K)^{(r-1)}$, we get that there exists $B, C \in T_n(K)$ such that

$$p(B,C) = [B,C]^r = e_{1,r+1} + e_{3,r+3}.$$

It is clear that $[B,C] \in T_n(K)^{(0)}$. We set

$$[B,C] = (a_{s,1+s+t}).$$

It follows that

$$[B,C]^r = e_{1,r+1} + e_{3,r+3}.$$

We get from the last relation that

$$\begin{cases} (a_{12}a_{23}\ldots a_{r,r+1})e_{1,r+1} = e_{1,r+1}; \\ (a_{23}a_{34}\ldots a_{r+1,r+2})e_{2,r+2} = 0; \\ (a_{34}a_{45}\ldots a_{r+2,r+3})e_{3,r+3} = e_{3,r+3}. \end{cases}$$

This is a contradiction. We obtain that $p(T_n(K)) \neq T_n(K)^{(r-1)}$ for all $n \geq 5$ and $1 < r < n-2$. This proves the result. ■

We remark that [16, Example 5.7] is a special case of Example 3.1 ($r = 2$ and $n = 5$).

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