A GENERALIZATION OF A THEOREM OF ZASSENHAUS

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A near-ring is a triple $(R, +, \cdot)$ such that $(R, +)$ is a group, $(R, \cdot)$ is a semigroup and $\cdot$ is left distributive over $+$; i.e. $w(x + z) = wx + wz$ for each w, x, z in R. A near-field is a near-ring such that the nonzero elements form a group under multiplication. Zassenhaus [3] showed that if R is a finite near-field, then $(R, +)$ is abelian. B.H. Neumann [1] extended this result to all near-fields. Recently another proof of this important result was given by Zemmer [4]. The purpose of this note is to give another generalization of the Zassenhaus theorem. In fact, we shall prove the following.

THEOREM. Let R be a finite near-ring with an identity 1 such that $(-1)x = x$ implies that $x = 0$. Then $(R, +)$ is abelian.

The proof of the theorem is based on the following result in group theory (cf. [2, page 357]).

LEMMA. Let $(G, +)$ be a finite group with an automorphism α such that $\alpha^2 = I$ and such that 0 is the only fixed point for α. Then G is abelian.

Proof of Theorem. Consider the map $\alpha : (R, +) \to (R, +)$ defined by $(y)\alpha = (-1)y$ for all y in R. Then it is routine to check that α has the properties stated in the lemma.

COROLLARY (Zassenhaus). Let $(R, +, \cdot)$ be a finite near-field with identity 1. Then $(R, +)$ is abelian.

Proof. Suppose $(-1)x = x$ for some x in R. If $x \neq 0$, then there exists a y in R such that $xy = 1$. Thus $(-1)xy = xy = 1$ and hence $1 + 1 = 0$. It follows that $w + w = w(1 + 1) = 0$ for each w in R. Consequently $(R, +)$ is abelian. If $x = 0$, then the conclusion follows from the theorem.

In order to see that the theorem is indeed a generalization of the Zassenhaus result, we now exhibit an example of a near-ring which satisfies the hypotheses of the theorem and yet is not a near-field.

Example. Let R be a finite near-field such that $(R, +)$ has no elements of order two. Consider

$$G = R \times R = \{(r, s) : r \in R, \ s \in R\}.$$

Define $+$ and $\cdot$ as follows:

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$$(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2) \text{ and}$$

$$(r_1, s_1) \cdot (r_2, s_2) = (r_1 \cdot r_2, s_1 \cdot s_2).$$

Then it is easily verified that $(G, +, \cdot)$ is a near-ring which satisfies the hypotheses of the theorem. Since G contains zero divisors we conclude that G is not a near-field.

REFERENCES

1. B.H. Neumann, On the commutativity of addition. J. London Math. Soc. 15 (1940) 203-208.

2. W.R. Scott, Group theory. (Prentice Hall, 1964).

3. H. Zassenhaus, Über endlich Fastkörper. Abh. Math. Sem., Univ. Hamburg 11 (1936) 187-220.

4. J.L. Zemmer, The additive group of an infinite near-field is abelian. J. London Math. Soc. 44 (1969) 65-67.

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